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CHARACTERIZATION OF DOMAINS THROUGH FAMILIES OF MEASURES

Abstract. Let Ω be a plane domain limited by a regular Jordan curve Γ. For every (L) measurable subset E of Γ and every point $z \in \Omega$, consider the probability $P(E; z)$ that a Brownian particle starting its motion at z hits the boundary Γ (by the first time) in a point belonging to E. Now, let C be a constant such that $0 < C < |\Gamma|$ and consider the optimization problem

(1) $$\sup\{P(E; z) : |E| = C\}$$

($|\cdot|$ denotes the Lebesgue measure on the boundary Γ). What are the domains Ω such that *single arcs* of the boundary are optimal subsets for (1) for every $z \in \Omega$ and every $0 < C < |\Gamma|$?

For a plane domain Ω which is starlike with respect to an interior point O, the internal visual angle $\Theta(O; E)$ of a measurable subset of the boundary $E \subseteq \partial\Omega$ is defined to be the angle under which E is observed from O. Posing the optimization problem

(2) $$\sup\{\Theta(O; E) : |E| = C\},$$

it is asked for the *convex domains* Ω such that *single arcs* of the boundary are optimal subsets for (2) for every $O \in \Omega$ and every $0 < C < |\Gamma|$.

A suitable response to these questions is given in this paper.

1. Introduction and preliminaries

Let Ω be a plane domain bounded by a regular curve γ. For every point $x \in \Omega$, suppose we are given a measure λ_x defined on the Lebesgue measurable subsets of the boundary $\partial\Omega$. Denote by Λ the whole family of these measures indexed by $x \in \Omega$; i.e., $\Lambda = \{\lambda_x : x \in \Omega\}$. To fix ideas, consider a Brownian particle starting its motion at $x \in \Omega$. If Γ is a (measurable) subset of $\partial\Omega$, the particle has a certain probability of hitting the boundary $\partial\Omega$ for the first time at a point of Γ. As it is well known, this probability coincides with the harmonic measure $\omega(x; \Omega, \Gamma)$ of Γ at the point $x \in \Omega$. In this case, $\Lambda = \{\omega(x; \Omega, \cdot) : x \in \Omega\}$ and we expect that a number of special properties to be shared by all the measures in Λ provided that the geometry of Ω has a certain "symmetry". For example, if $\Omega = B_r$ is a circle of radius r, then we easily see that Λ is a family of *coalescent measures* (with respect to the Lebesgue measure on the boundary) in the sense that, for every $0 < C < 2\pi r$, the optimization problem

$$\max\{\omega(x;\Omega,\Gamma) : \Gamma \subseteq \partial B_r,\ |\Gamma| = C\} \tag{3}$$

is solved by a single arc Γ^* with measure $|\Gamma| = C$. This means, in the probabilistic interpretation, that a certain single "window" Γ^* of length C on the circumference ∂B_r maximizes the probability that a Brownian particle starting at any point $x \in B_r$ hits the boundary ∂B_r by the first time at a point of the "windows" Γ with "total length" C.

Our present interest will be focalized on a sort of inverse problem: *what can be said on the geometry of Ω when it is known that a particular property is enjoyed by every member of the family of measures Λ?* Of course, this question is meaningless when formulated in its full generality (what is seen by realizing that certain families Λ of measures are defined disregarding the geometry of the domain Ω; v.g., the family of identically zero measures $\lambda_x \equiv 0,\ x \in \Omega$), but in posing our problem we are implicitly assuming that the family of measures is somewhat related to the geometry of the domain Ω. Indeed, the above question is directed to deepen in the nature of these relationships when these ones are known to exist. For instance, we can ask for the domains Ω such that the family $\{\omega(x;\Omega,\cdot) : x \in \Omega\}$ of harmonic measures with respect to Ω is a coalescent family. As another significative example, we consider in this paper the family of internal visual angles.

By assuming that Ω is starlike with respect to an interior point O, it makes sense to consider the *internal visual angle* $\Theta(O;E)$ under which an arc E of γ is seen by an observer placed at O (see Figure 1). More generally, if E is a measurable subset of γ; then, the quantity $\Theta(O;E)$ represents the total visual angle under which the subset E of the boundary is seen from O. By representing the boundary curve γ in polar coordinates with pole at O, we have

$$\Theta(O;E) = \{\theta : (\theta, \rho(O,\theta)) \in E\} \subseteq [0, 2\pi],$$

where $\rho(O;\theta),\ 0 \leq \theta < 2\pi$, is the polar equation of γ. Of course, $\Theta(O;E)$ is a measurable set and the same notation will be indistinctly used for its measure from now on. When a measurable set $E \subseteq \gamma$ is varying on γ so that its total length $|E|$ is kept equal to a constant $0 < C < |\partial\Omega|$, we can expect the visual angle $\Theta(O;E)$ to attain a maximum value for certain subsets $E^*(O) \subseteq \gamma$; concretely, we are referring to subsets $E^*(O)$ of γ that solve the optimization problem

$$\max\{\Theta(O;E) : |E| = C\}. \tag{4}$$

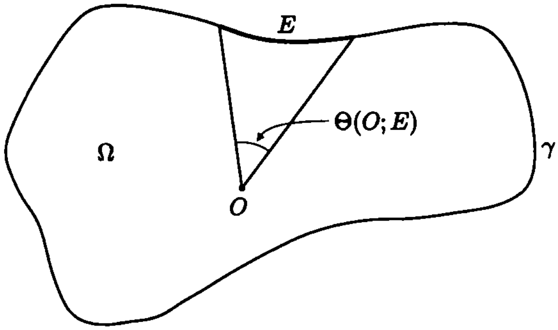

Figure 1

Using the polar representation of the boundary curve γ, a more explicit formulation can be given to this problem. In fact, (4) can be rewritten in the form

$$\max\{\Theta(O;E) : \int_{\Theta(O;E)} \sqrt{\rho^2(O;\theta) + [\rho'(O;\theta)]^2}d\theta = C\}, \tag{5}$$

where the classical expression $ds = \sqrt{\rho^2(O;\theta) + [\rho'(O;\theta)]^2}d\theta$ is used for the differential of arc of a C^1curve γ given in polar coordinates by the equation $\rho = \rho(O;\theta)$. The double meaning of $\Theta(O;E)$ both as a set and a measure should not cause confusion in (5).

In many situations it occurs that an optimal set $E^*(O)$ for problem (4) is but a single arc of length C. In the formulation (5), this case corresponds to optimal angular coordinates of de form $\Theta(O;E^*(O)) = [\alpha, \alpha \oplus C]$, expression in which $0 \le \alpha < 2\pi$ and '$\oplus$' stands for the sum modulus 2π. For instance, if Ω is a circle of radius r and O is its center, then $\Theta(O;E) = r^{-1}|E|$ is constant on the measurable subsets E of $\partial\Omega$ with $|E| = C$; therefore, every one of these sets, in particular an arc of length C, is optimal for problem (4). When this property holds for every $0 < C < |\partial\Omega|$; i.e., when problem (4) is solved by an arc of length C whichever be $0 < C < |\partial\Omega|$, we say that the internal visual angle $\Theta(O;\cdot)$ is *coalescent with respect to the length of arc* (or simply *coalescent*). For a *convex* plane domain Ω, the internal visual angle $\Theta(O;\cdot)$ is naturally defined for every $O \in \Omega$. Then, we can ask whether or not a convex domain exists such that the visual angle $\Theta(O;\cdot)$ turns out to be coalescent for every $O \in \Omega$ and, in the affirmative case, we can look for suitable characterization of such domains.

A general attack of questions related to coalescence of measures in abstract measure spaces was made in [2]. As a matter of fact, the optimization problems (3) and (4) are particular cases of the following more general one:

$$\sup\{v(E) : \mu(E) = C\}, \tag{6}$$

where μ and ν are two σ-finite Borel measures on a topological space X. When ν is absolutely continuous with respect to μ, the existence of the *Radon-Nikodym derivative* $d\nu/d\mu$ is guaranteed and it is in terms of this derivative that the following analytical equivalent of coalescence was derived ([2], Theorem 12).

THEOREM 1. *Suppose that* δ_f, *the distribution function of* $f = d\nu/d\mu$, *is continuous on its support. Then, problem* (6) *has a connected optimal solution (for every* C *in the range of* μ*) provided that* f *is a* μ*-connected function. Conversely, if problem* (6) *has a connected optimal solution for every* C *in the range of* μ *and* $\sup\{v(E) : \mu(E) = C\} < +\infty$ *(for every* $0 < C < \mu(X)$ *in the range of* μ*); then* $f = d\nu/d\mu$ *is a* μ*-connected function.*

The *range* of a measure μ is constituted by the set of the (extended) real values reached by μ or, equivalently, the set $\mu(\mathcal{A}) \subseteq \overline{\mathbb{R}}$, where $\mathcal{A}$ is the underlying σ-algebra. A Borel subset B of X is said to be μ*-connected* when it is "connected in a measure-theoretic sense"; i.e., if there exists a μ-null set N such that $B \cup N$ is connected. A function $f : X \rightarrow \overline{\mathbb{R}}$ is named μ*-connected* when all their level sets are μ-connected. For example, the graph of a continuous L-connected function $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is "unimodal"; i.e., looks like a "bump". The intuition behind Theorem 1 is simple: a solution to problem (6) is basically given by a level set of the Radon-Nikodym derivative $d\nu/d\mu$, so that these solutions will be all connected if (and only if) the level sets of $d\nu/d\mu$ are all connected.

The following results, whose proof is the main concern of this paper, provide a suitable response to the above posed questions on the harmonic measure and visual angles.

THEOREM 2. *The circle is the unique Dini-smooth domain* Ω *such that the harmonic measure* $\omega(x; \Omega, \cdot)$ *is coalescent whichever be the point* $x \in \Omega$.

THEOREM 3. *The circle is the unique convex* C^1 *domain such that the internal visual angle* $\Theta(O; \cdot)$ *is coalescent whichever be the point* $O \in \Omega$.

The proof we will give in Section 2 for Theorem 2 is based on the Theorem 1 and conformal invariance of the Laplace equation. Beyond a moderate geometric appeal, the problem of characterization of domains such that the family of internal visual angles is coalescent can be considered as a slight variation of the case corresponding to the harmonic measure. After all, when $\Omega = \mathbb{R}^2_+$ (the half plane), the harmonic measure coincides, up to a multiplicative constant, with the internal visual angle. Nevertheless, conformal invariance is a powerful tool which is absent for internal visual angles, a fact that makes the study of visual angles to be considerably more involved than that one needed for harmonic measures. Consistently, in Section 3 we develop from scratch a proof for Theorem 3. This proof is organized as follows: without resorting to Theorem 1, a characterization of optimal sets for problem (4) is firstly developed in subsection 3.1 (Theorem 4) and it is expressed in terms of distributions functions of the function

$$\theta \mapsto \sqrt{\rho^2(O;\theta) + [\rho'(O;\theta)]^2}.$$

As a consequence of this characterization, an analytic criterion for coalescence is obtained (Theorem 5) and applied to prove a part of Theorem 3. Filling the gap existing between the analytic condition furnished by Theorem 5 and the geometric content of Theorem 3, in subsection 3.2 we prove a result (Theorem 6) that provides a characterization of the circle using the set-valued map $O \mapsto \arg\min \rho(O;\cdot)$. Finally, the remaining part of Theorem 3 is proved in subsection 3.3 by assembling Theorems 5 and 6. The final Section 4 gathers together some general observations and remarks.

Some special, perhaps infrequent, notations are used along this paper. For instance, $\arg\min f$ will denote the set of minimizers of a given function f and F^+, F^- will stand for certain distributions functions associated with f. The very meaning of every particular notation will always be opportunely declared.

2. Proof of Theorem 2

To prove Theorem 2, it is sufficient to restrict ourselves to consider C-windows E composed by a finite union of arcs, as we will make from now on. In this case, it is well known that the harmonic measure $\omega(z;\Omega,E)$ coincides with the solution $u(z)$ to the Dirichlet problem

$$\text{(7)} \qquad \begin{cases} \Delta u(z) = 0, & z \in \Omega \\ u(z) = 1, & z \in E \\ u(z) = 0, & z \in \partial\Omega \backslash E \end{cases}.$$

When Ω is a circle, the Poisson kernel enables us to write the solution to problem (7) in an explicit way. Namely, if $\Omega = B_1(0)$, we have

$$\text{(8)} \qquad \omega(z;\Omega,E) = \frac{1}{2\pi} \int_E \frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\, d\theta$$

where $z = re^{i\phi}$ and, without any risk of confusion, the set $E \subseteq \partial B_1(0)$ has been identified with the set of its angular coordinates.

Now, we are to complete our argument. First suppose that Ω is a circle; since the Poisson kernel

$$\theta \mapsto P(r,\phi;\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}$$

is connected (when considered as a function on S^1, of course), the Poisson formula (8) and Theorem 1 ensure that

$$\sup\{\omega(z; B_1(0), E) : |E| = C\} = \omega(z; B_1(0), E^*),$$

where E^* is an arc of length C and $z \in B_1(0)$ is arbitrary. This proves a part of Theorem 2. In order to prove the converse, choose a Dini-smooth domain Ω such that, for very $z \in \Omega$ and $0 < C < |\Gamma|$, the identity

$$\sup\{\omega(z; \Omega, E) : |E| = C\} = \omega(z; \Omega, E^*)$$

holds with E^* a single arc of length C. Let $\Phi : B_1(0) \to \Omega$ conformally maps the unit disk $B_1(0)$ on the domain Ω. Since the boundary $\partial\Omega$ was supposed to be a sufficiently regular (Dini-smooth; see, for instance, [3], pg. 48) Jordan curve, the Riemann map Φ and its derivative Φ' both extend continuously up to the boundary $\partial\Omega$. Then, the function $v(z) = \omega(\Phi^{-1}(z); B_1(0), \Phi^{-1}(E))$ is the harmonic measure of the set $\Phi^{-1}(E) \subseteq S^1$ and we can write

$$|E| = \int_E ds = \int_{\Phi^{-1}(E)} |\Phi'(e^{i\theta})| \, d\theta.$$

In consequence, we have

$$\begin{aligned} \sup\{\omega(z; \Omega, E) \ &: \ |E| = C\} \\ &= \sup\{\omega(\Phi^{-1}(z); B_1(0), \Phi^{-1}(E)) : \int_{\Phi^{-1}(E)} |\Phi'(e^{i\theta})| \, d\theta = C\} \\ &= \sup\{\omega(\Phi^{-1}(z); B_1(0), A) : \int_A |\Phi'(e^{i\theta})| \, d\theta = C\}, \end{aligned}$$

and then, setting $\Phi^{-1}(z) = re^{i\phi}$, Theorem 1 (with $d\nu = P(r, \phi; \theta) \, d\theta$ and $d\mu = \left|\Phi'(e^{i\theta})\right| \, d\theta$) enable us to conclude that

$$\theta \mapsto \frac{d\nu}{d\mu} = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2} \frac{1}{|\Phi'(e^{i\theta})|} \tag{9}$$

is a connected function for every $0 \le r < 1$, $0 \le \phi < 2\pi$. In particular, by taking $r = 0$ we see that $\theta \mapsto 1/\left|\Phi'(e^{i\theta})\right|$ must be a connected function (its graph looks like a single bump *when traced on* S^1). Indeed, since the Poisson kernel is an approximate identity, it is easily deduced that $\theta \mapsto 1/\left|\Phi'(e^{i\theta})\right|$ must be a constant (in other case, the graph of the product given by (9) would contain a second bump around $\phi = \theta$); therefore, $|\Phi'|$ is a constant on S^1. Since Φ' does not vanish on $B_1(0)$, an application of the Maximum Modulus Theorem shows that Φ' reduces to a constant a; thus, $\Phi(z) = az + b$ and $\Omega = \Phi(B_1(0))$ is a circle. This finishes the proof of Theorem 2.

3. Proof of Theorem 3

3.1. Optimal sets and an analytic condition for coalescence

With the purpose of obtaining a characterization of optimal sets corresponding to problem (5), some preliminary concepts are needed. For a continuous and positive function $f : S^1 \to \mathbb{R}$ we define

$$F^+(\lambda) = \int_{\{\theta: f(\theta) \le \lambda\}} f(\theta)\, d\theta, \ \lambda \in \mathbb{R}, \tag{10}$$

and

$$F^-(\lambda) = \int_{\{\theta: f(\theta) < \lambda\}} f(\theta)\, d\theta, \ \lambda \in \mathbb{R}. \tag{11}$$

Integrals extended to the *λ-level set* $\{\theta : f(\theta) \le \lambda\}$ and to the *strict λ-level set* $\{\theta : f(\theta) < \lambda\}$ of function f are respectively involved in the expression of F^+ and F^-, so that these functions can be considered as *distribution functions* corresponding to the measure $d\mu(\theta) = f(\theta)\, d\theta$ on S^1. The main properties of these functions are collected in the following lemma.

LEMMA 1. *If f is a positive and continuous function, then functions F^+ and F^-, respectively defined by* (10) *and* (11), *satisfy the following properties:*

i) *F^+ and F^- are strictly increasing functions on* $[\min f, \max f]$.

ii) *F^+ and F^- are constant on* $\mathbb{R} \setminus [\min f, \max f]$. *Moreover,* $F^+(\lambda) \ge F^-(\lambda)$, $\lambda \in \mathbb{R}$, *and*

$$F^+(\max f) = \int_0^{2\pi} f(\theta)\, d\theta, \quad F^-(\min f) = 0.$$

iii) *F^+ is right-continuous, F^- is left-continuous and*

$$F^+(\lambda^-) = \lim_{\mu \uparrow \lambda} F^+(\mu) = F^-(\lambda), \quad F^-(\lambda^+) = \lim_{\mu \downarrow \lambda} F^+(\mu) = F^+(\lambda).$$

Proof. For $\lambda_1, \lambda_2 \in [\min f, \max f]$, $\lambda_1 < \lambda_2$, we have

$$\begin{aligned} F^+(\lambda_2) - F^+(\lambda_1) &= \int_{\{\theta: \lambda_1 < f(\theta) \le \lambda_2\}} f(\theta)\, d\theta \\ &\ge \int_{\{\theta: \lambda_1 < f(\theta) < \lambda_2\}} f(\theta)\, d\theta. \end{aligned} \tag{12}$$

Since $\{\theta : \lambda_1 < f(\theta) < \lambda_2\}$ is a non-void open set by the continuity of f and since f is positive, we see that the last integral in (12) is positive. Then, $F^+(\lambda_2) - F^+(\lambda_1) > 0$ and F^+ is strictly increasing on $[\min f, \max f]$. That so it is F^- can be shown by means of a similar reasoning. This proves i). Properties ii) are immediate. As for properties iii), they are an easy consequence of the following identities

$$\cup_{\mu<\lambda}\{\theta : f(\theta) \le \mu\} = \{\theta : f(\theta) < \lambda\} = \cup_{\mu<\lambda}\{\theta : f(\theta) < \mu\},$$
$$\cap_{\mu>\lambda}\{\theta : f(\theta) \le \mu\} = \{\theta : f(\theta) \le \lambda\} = \cap_{\mu>\lambda}\{\theta : f(\theta) < \mu\},$$

and of basic results in Measure Theory (see, for example, [4], Theor. 10.11, p. 166). ∎

For a regular domain Ω which is starlike with respect to the point $O \in \Omega$, we now consider the functions F^+ and F^- associated with

$$f(\theta) = \sqrt{\rho^2(O;\theta) + [\rho'(O;\theta)]^2},\ 0 \le \theta < 2\pi.$$

In view of Lemma 1-i), ii), given a C with $0 < C < |\partial\Omega| = \int_0^{2\pi} f(\theta)\, d\theta$, one at least from the following three alternatives hold:

A) There exists a λ such that $F^+(\lambda) = C$.
B) There exists a λ such that $F^-(\lambda) = C$.
C) There exists a λ such that $F^-(\lambda) < C < F^+(\lambda)$.

Since F^+ and F^- are strictly increasing functions, one and only one value of λ exists such that A), B) or C) holds. Furthermore, $\lambda > 0$ by the positivity of f. Note that A) and B) simultaneously hold when λ is a point of continuity of F^+. The nature of optimal solutions to problem (4) depends on what alternative A), B) or C) do occur. Concretely, we can state the following:

THEOREM 4. *The λ-level set*

$$E^* = \{\theta : f(\theta) \le \lambda\} \tag{13}$$

or the strict λ-level set

$$E^* = \{\theta : f(\theta) < \lambda\} \tag{14}$$

are optimal for problem (4) *depending on whether alternative* A) *or* B) *holds, respectively. In the case in which alternative* C) *holds, any measurable subset E^* of γ is optimal provided that*

$$\{\theta : f(\theta) < \lambda\} \subseteq E^* \subseteq \{\theta : f(\theta) \le \lambda\} \tag{15}$$

and $\int_{E^*} f(\theta)\, d\theta = C$. All these optimal sets are unique in a measure-theoretic sense: if A is another optimal set then $\Theta(O; A \,\triangle\, E^*) = 0$.

Proof. Let us prove that the indicated sets are optimal. First suppose that alternative A) occurs and denote by A any measurable subset of γ satisfying the restriction

$$\int_A f(\theta)\, d\theta = C.$$

By calling $E^* = \{\theta : f(\theta) \le \lambda\}$, λ being the unique solution to $F^+(\lambda) = C$, we have

$$\begin{aligned}\int_{A\setminus E^*} f(\theta)\, d\theta + \int_{A\cap E^*} f(\theta)\, d\theta &= \int_A f(\theta)\, d\theta \\ &= C\end{aligned}$$

$$= F^+(\lambda) = \int_{E^*} f(\theta)\,d\theta = \int_{E^*\setminus A} f(\theta)\,d\theta + \int_{A\cap E^*} f(\theta)\,d\theta,$$

whence we obtain

$$\lambda\Theta(O; A\setminus E^*) \le \int_{A\setminus E^*} f(\theta)\,d\theta = \int_{E^*\setminus A} f(\theta)\,d\theta \le \lambda\Theta(O; E^*\setminus A), \tag{16}$$

or, in view of $\lambda > 0$,

$$\Theta(O; A\setminus E^*) \le \Theta(O; E^*\setminus A). \tag{17}$$

From inequality (17), we easily derive the following ones

$$\begin{aligned}\Theta(O;A) &= \Theta(O; A\setminus E^*) + \Theta(O; A\cap E^*)\\ &\le \Theta(O; E^*\setminus A) + \Theta(O; A\cap E^*)\\ &= \Theta(O; E^*),\end{aligned}$$

which prove the optimality of E^* when alternative A) holds. That (14) and (15) are optimal when, respectively, alternative B) or C) occurs is similarly proved.

Now we show the uniqueness in the sense of Measure Theory of the exhibited optimal sets. To this end, suppose that A is an optimal subset for problem (4) and that alternative A) occurs. If it were $\Theta(O; E^*\setminus A) > 0$ then, reasoning as before we would have

$$\lambda\Theta(O; A\setminus E^*) < \int_{A\setminus E^*} f(\theta)\,d\theta = \int_{E^*\setminus A} f(\theta)\,d\theta \le \lambda\Theta(O; E^*\setminus A) \tag{18}$$

and hence

$$\Theta(O;A) < \Theta(O;E^*).$$

This inequality is in contradiction with the supposed optimality of A; therefore, it must be $\Theta(O; E^*\setminus A) = 0$ and, since f is bounded away from 0, the middle equality in (18) implies that $\Theta(O; A\setminus E^*) = 0$. Thus, $\Theta(O; A\triangle E^*) = \Theta(O; E^*\setminus A) + \Theta(O; A\setminus E^*) = 0$, as it was affirmed.

The proof of uniqueness is analogous for alternative B) and we will omit its details, but further discussion is needed to prove alternative C). If alternative C) holds and inclusions (15) do not hold in the measure-theoretic sense by a measurable set A satisfying $\int_A f(\theta)\,d\theta = C$, then we obtain as before that $\lambda\Theta(O; A\setminus E^*) \le \lambda\Theta(O; E^*\setminus A)$ and, since $\lambda > 0$,

$$\Theta(O;A) \le \Theta(O;E^*). \tag{19}$$

By setting $E_+^* = \{\theta : f(\theta) \leq \lambda\}$ and $E_-^* = \{\theta : f(\theta) < \lambda\}$, we can see that one of the following inequalities

$$\Theta(O; E_-^* \setminus A) > 0 \tag{20}$$

or

$$\Theta(O; A \setminus E_+^*) > 0 \tag{21}$$

holds. First suppose that inequality (20) holds. Then we obtain strict inequality in the last inequality (16) and, taking into account that $E_-^* \subset E^*$, we deduce

$$\Theta(O; A \setminus E^*) < \Theta(O; E^* \setminus A);$$

hence strict inequality in (19), a contradiction. Likewise, if (21) occurs, then the first inequality (16) will be strict and consequently $\Theta(O; A \setminus E^*) < \Theta(O; E^* \setminus A)$, leading again to strict inequality in (19), a contradiction to the optimality of E^*. This completes the proof. ∎

In order to state in a concise way our analytic condition for coalescence of visual angles, the concept of a connected function defined on S^1 becomes useful: we say that a function $f : S^1 \to \mathbb{R}$ is *connected on* S^1 when, for every $\lambda \in \mathbb{R}$, its λ-level set is a connected subset of S^1. Since the complement $S^1 \setminus E$ is connected whenever E it is, a function $f : S^1 \to \mathbb{R}$ is connected if and only if $\{x \in S_1 : f(x) > \lambda\}$ is connected for every λ. Furthermore, a continuous function f is connected on S^1 if and only if their strict λ-level sets are connected. In fact, assume f is connected and, for any λ, let θ_1, θ_2 be two distinct points of the strict λ-level set $\{\theta : f(\theta) < \lambda\}$. Without loss of generality, we can suppose that $f(\theta_1) = \lambda - \delta_1$, $f(\theta_2) = \lambda - \delta_2$ with $0 < \delta_1 \leq \delta_2$ and therefore, $\theta_1, \theta_2 \in \{\theta : f(\theta) \leq \lambda - \delta_1\}$. By the connectivity of $\{\theta : f(\theta) \leq \lambda - \delta_1\}$, at least one of the supplementary arcs of $S^1 \setminus \{e^{i\theta_1}, e^{i\theta_2}\}$ is included in this set, and hence in $\{\theta : f(\theta) < \lambda\}$. Therefore $\{\theta : f(\theta) < \lambda\}$ is connected. Conversely, if the strict level sets of f are connected and the λ-level set $\{\theta : f(\theta) \leq \lambda\}$ was not connected for some λ, then $\{\theta : f(\theta) > \lambda\}$ would not be connected as well and a contradiction can be reached using a similar argument.

We are now ready to prove the following:

THEOREM 5. *The internal visual angle* $\Theta(O, \cdot)$ *is coalescent if and only if the function* $f = \sqrt{\rho^2(O; \cdot) + [\rho'(O; \cdot)]^2}$ *is connected on* S^1.

Proof. First suppose that f is a connected function and choose $0 < C < |\partial\Omega|$. Since f is continuous, the previous discussion shows that the sets $\{\theta : f(\theta) < \lambda\}$ and $\{\theta : f(\theta) \leq \lambda\}$ are connected for every value of λ, in particular for that value corresponding to any alternative A), B) or C) of Theorem 4. It is clear, by this reason, that there exists a connected optimal solution to problem (4) when alternative A) or B) holds. Since

$\{\theta : f(\theta) < \lambda\} \subseteq \{\theta : f(\theta) \leq \lambda\}$, we can choose a connected measurable set E^* such that $\int_{E^*} f(\theta)\, d\theta = C$ and $\{\theta : f(\theta) < \lambda\} \subseteq E^* \subseteq \{\theta : f(\theta) \leq \lambda\}$; therefore, problem (4) admits a connected optimal solution also in the case C).

Conversely, assume that the internal visual angle $\Theta(O, \cdot)$ is coalescent. Given a positive λ, from Theorem 4 we conclude that every set E^* satisfying

$$\{\theta : f(\theta) < \lambda\} \subseteq E^* \subseteq \{\theta : f(\theta) \leq \lambda\}$$

must differ from a connected set in a null set at most. Now, since f is a continuous function, we see that this can occur if and only if the level sets $\{\theta : f(\theta) < \lambda\}$ and $\{\theta : f(\theta) \leq \lambda\}$ are connected; i.e., f is a connected function. ∎

Take for instance the case $\Omega = B_r(0)$, the circle of radius r centered at the origin. Even if the calculations involved in this case are simple enough, we think that a thorough discussion may be helpful. To begin with, fix an interior point $O = r_0 e^{i\phi} \in B_r$ (see Figure 2); then we have

$$\begin{aligned} \rho^2(O;\theta) &= |r_0 e^{i\phi} - r e^{i\theta}|^2 \\ &= (r_0 e^{i\phi} - r e^{i\theta})(r_0 e^{-i\phi} - r e^{-i\theta}) \\ &= r_0^2 - 2 r_0 r \cos(\theta - \phi) + r^2, \ 0 \leq \theta < 2\pi, \end{aligned} \tag{22}$$

and differentiating with respect to θ, we deduce

$$2\rho(O;\theta)\rho'(O;\theta) = 2 r_0 r \sin(\theta - \phi), \ 0 \leq \theta < 2\pi,$$

or

$$\rho(O;\theta)\rho'(O;\theta) = r_0 r \sin(\theta - \phi), \ 0 \leq \theta < 2\pi. \tag{23}$$

By differentiating once again, it turns out

$$\rho(O;\theta)\rho''(O;\theta) + (\rho'(O;\theta))^2 = r_0 r \cos(\theta - \phi), \ 0 \leq \theta < 2\pi. \tag{24}$$

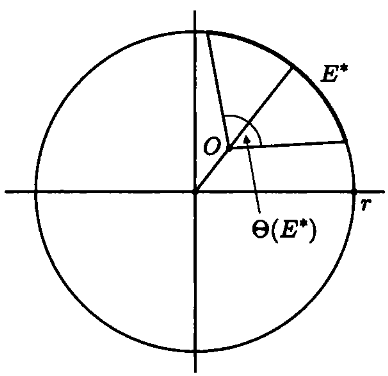

Figure 2

We wish to prove that the function $f(\theta) = \sqrt{\rho^2(O;\theta) + (\rho'(O;\theta))^2}$ is connected on S^1. With this purpose, we note that $f(\theta) \geq \rho(O;\theta) \geq \rho(O;\phi) = f(\phi)$, $0 \leq \theta < 2\pi$, and therefore f attains its global minimum for $\theta = \phi$. Now, we prove that f is strictly increasing on $(\phi, \phi+\pi)$ and strictly decreasing on $(\phi - \pi, \phi)$. In fact, it is easy to see from (23) that $\rho'(O;\theta) > 0$, $\phi < \theta < \phi + \pi$, and $\rho'(O;\theta) < 0$, $\phi - \pi < \theta < \phi$, and taking into account that

$$f'(\theta)f(\theta) = \rho'(O;\theta)(\rho(O;\theta) + \rho''(O;\theta)) \tag{25}$$

it will be enough to show that $\rho(O;\cdot) + \rho''(O;\cdot) > 0$. From (22) and (23), we derive

$$\begin{aligned}
&\rho^3(O;\theta)(\rho(O;\theta) + \rho''(O;\theta)) \\
&\quad = \rho^2(O;\theta)(\rho^2(O;\theta) - (\rho'(O;\theta))^2 + r_0 r\cos(\theta-\phi)) \\
&\quad = \rho^4(O;\theta) - r^2 r_0^2 \sin^2(\theta-\phi) + r_0 r\cos(\theta-\phi)\rho^2(O;\theta) \\
&\quad = (r_0^2 - 2r_0 r\cos(\theta-\phi) + r^2)^2 - r^2 r_0^2\sin^2(\theta-\phi) \\
&\qquad + r_0 r\cos(\theta-\phi)(r_0^2 - 2r_0 r\cos(\theta-\phi) + r^2) \\
&\quad = 3r_0^2 r^2\cos^2(\theta-\phi) - 3r_0 r(r^2 + r_0^2)\cos(\theta-\phi) \\
&\qquad + ((r^2+r_0^2)^2 - r_0^2 r^2) \\
&\quad = P(\cos(\theta-\phi)),
\end{aligned}$$

where P is the quadratic polynomial

$$P(x) = 3r_0^2 r^2\, x^2 - 3r_0 r(r^2 + r_0^2)\, x + ((r^2+r_0^2)^2 - r_0^2 r^2).$$

The discriminant of this polynomial is given by

$$\begin{aligned}
\Delta &= (3r_0 r(r^2+r_0^2))^2 - 4 \times 3r_0^2 r^2 \times ((r^2+r_0^2)^2 - r_0^2 r^2) \\
&= -3r_0^2 r^2 (r^2 - r_0^2)^2 \\
&< 0,
\end{aligned}$$

so that $P(x) > 0$, $x \in \mathbb{R}$, and $\rho(O;\cdot) + \rho''(O;\cdot) > 0$, as we claimed. From the just established property of f, we see that the equation $f(\theta) = \lambda$ has two solutions at most for every $\lambda \in \mathbb{R}$, so proving that f is a connected function. In the light of Theorem 5, this shows that the internal visual angle $\Theta(O;\cdot)$ is coalescent for the circle whatever be the interior point O.

3.2. The map $O \mapsto \arg\min \rho(O;\cdot)$

In the previous subsection we saw that an analytic criterion to decide the coalescence of the internal visual angle $\Theta(O;\cdot)$ is given by the connectedness of the function $f = \sqrt{\rho^2(O;\cdot) + (\rho'(O;\cdot))^2}$. In this subsection a tool is prepared which will serve to link this analytic condition with geometry.

We write $d(P;Q)$ to denote the Euclidean distance between the points P and Q. The notation $d(O;\partial\Omega)$ indicates the distance from a point O to the boundary of Ω and it is defined to be $d(O;\partial\Omega) = \min_{P\in\partial\Omega} d(O;P)$. For a given $O \in \overline{\Omega}$, we denote by $\arg\min \rho(O;\cdot)$ the set of minimizers $\{P^* \in \partial\Omega : d(O;\partial\Omega) = d(O;P^*)\}$; i.e., the set of points in $\partial\Omega$ nearest to O. Note that $\arg\min \rho(O;\cdot) = \{O\}$ for every $O \in \partial\Omega$. In the proof of our next result, it will be useful to consider the set-valued map $\overline{\Omega} \ni O \mapsto \arg\min \rho(O;\cdot) \subseteq \partial\Omega$. As a matter of fact, this map is *upper semicontinuous* (see, for instance, [1], Theor. 6, pg. 53), which means that for every $O \in \overline{\Omega}$ and for every open set U containing $\arg\min \rho(O;\cdot)$, there exists a neighborhood $\mathcal{U}_O$ of O such that $\arg\min \rho(\mathcal{U}_O;\cdot) \subseteq U$.

THEOREM 6. *Let $\Omega \subset \mathbb{R}^2$ a convex domain bounded by a C^1 curve. Then Ω is a circle if and only if* $\arg\min \rho(O;\cdot)$ *is a connected subset of $\partial\Omega$ for every $O \in \Omega$.*

Proof. The "only if" part of the theorem is immediate: $\arg\min \rho(O;\cdot)$ is the whole boundary when O is the center of the circle and it reduces to a point in other case. In order to prove the converse, assume that $\arg\min \rho(O;\cdot)$ is a connected subset of $\partial\Omega$ for every $O \in \Omega$. Since $\arg\min \rho(O;\cdot) = \{O\}$ for points O belonging to $\partial\Omega$, we see that the values of the upper semicontinuous map $\overline{\Omega} \ni O \mapsto \arg\min \rho(O;\cdot) \subseteq \partial\Omega$ are connected subset of $\partial\Omega$. If $\arg\min \rho(O;\cdot)$ would reduce to a point for every $O \in \overline{\Omega}$, then the function

$$\begin{aligned} \phi : \overline{\Omega} &\to \overline{\Omega} \\ O &\mapsto \phi(O) \end{aligned}$$

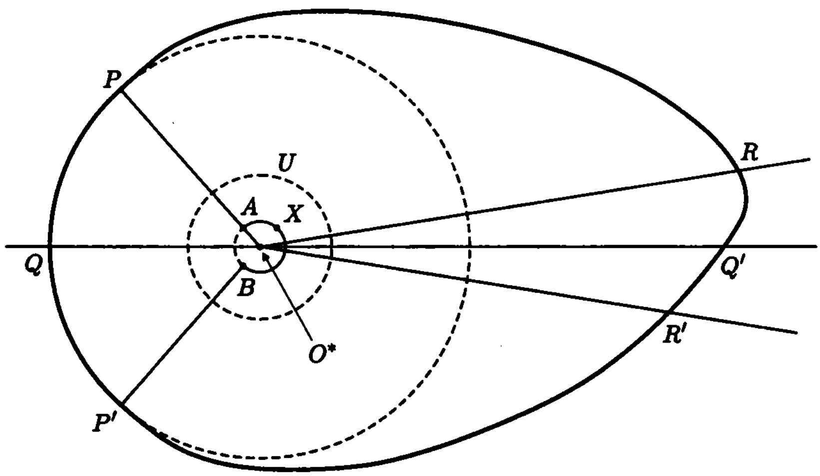

Figure 3

such that $\arg\min \rho(O;\cdot) = \{\phi(O)\}$, would be a continuous function satisfying $\phi|_{\partial\Omega} \equiv \mathrm{id}_{\partial\Omega}$, the identity function on $\partial\Omega$. But such a function can not exist by the Brouwer's fixed point theorem. In fact, choose a point $P \in \Omega$ and define a new function $\widetilde{\phi} : \overline{\Omega} \to \overline{\Omega}$ by making $\widetilde{\phi}(O)$ to be the point where the line segment from $\phi(O)$ to P hits $\partial\Omega$. It is easy to see that $\widetilde{\phi}$ is continuous and that it possesses no fixed point, so violating the Brouwer's Theorem. Then, we conclude that there exists a point $O^* \in \Omega$ such that $\arg\min \rho(O^*;\cdot)$ is an arc of circle. We will see that a new contradiction is reached if we suppose that $\arg\min \rho(O^*;\cdot)$ is not a whole circle. The situation is illustrated in Figure 3, in which $\arg\min \rho(O^*;\cdot)$ is the arc PQP'. By the upper semicontinuity of the map $O \mapsto \arg\min \rho(O;\cdot)$, if we exclude a small arc $RQ'R'$ from the boundary $\partial\Omega$ as indicated in the figure, then the sets $\arg\min \rho(O;\cdot)$ will be contained in the arc $RPQP'R'$ provided that O varies in a small enough ball U centered at O^*. Let us pay attention to the arc AXB of the boundary of a ball contained in U as represented in Figure 3. The continuous function $\delta(X) = d(X, PR) - d(X, P'R')$ satisfies $\delta(A) < 0$ and $\delta(B) > 0$; thus, there exists a point X^* in the arc AXB such that $\delta(X^*) = 0$; that is, $d(X^*, PR) = d(X^*, P'R')$. Since $d(X^*, PQP') > \min\{d(X^*, P); d(X^*, P')\} > d(X^*, PR)$, we realize that $\arg\min \rho(X^*;\cdot)$ can not contain points belonging to the arc PQP' and we then conclude that $\arg\min \rho(X^*;\cdot)$ has two components at least: one of them on the arc PR and the other on $P'R'$. This is in contradiction with the hypotheses of connectedness of $\arg\min \rho(X^*;\cdot)$ so finishing the proof. ∎

3.3. Completion of the proof of Theorem 3

The property of the circle of being the unique convex and C^1 domain Ω such that its internal visual angle $\Theta(O;\cdot)$ is coalescent for every $O \in \Omega$, quickly follows from Theorems 5 and 6. In fact, after Theorem 5, the function $f(O;\cdot) = \sqrt{\rho^2(O;\cdot) + (\rho'(O;\cdot))^2}$ is connected for every $O \in \Omega$. We will show this implies that $\rho(O;\cdot)$ is connected too. In fact, if $\rho(O;\cdot)$ were not connected, then there would exist a λ such that the strict level set $\{\theta : \rho(O;\theta) < \lambda\}$ is not connected. Let (α_1, β_1) and (α_2, β_2) two components of $\{\theta : \rho(O;\theta) < \lambda\}$. In view of the continuity of $\rho(O;\cdot)$, the equalities $\rho(O;\alpha_1) = \lambda = \rho(O;\beta_1)$ and $\rho(O;\alpha_2) = \lambda = \rho(O;\beta_2)$ hold and then, by the Rolle's Theorem there exist $\theta_1 \in (\alpha_1, \beta_1)$ and $\theta_2 \in (\alpha_2, \beta_2)$ such that $\rho'(\theta_i) = 0$, $i = 1, 2$. Therefore $f(O;\theta_i) = \rho(O;\theta_i)$, $i = 1, 2$. Taking into account that $\{\theta : f(O;\theta) < \lambda\} \subseteq \{\theta : \rho(O;\theta) < \lambda\}$, which is immediate from the inequality $\rho(O;\cdot) \leq f(O;\cdot)$, we see that the level set $\{\theta : f(O;\theta) < \lambda\}$ would not be connected. This contradiction proves our claim. Now, if $\rho(O;\cdot)$ is a connected function for every $O \in \Omega$, then

$\arg\min \rho(O;\cdot) = \{(\theta, \rho(\theta)) : \rho(O;\theta) \leq \min \rho(O;\cdot)\}$ is connected (on S^1) so that Theorem 5 applies to conclude that Ω must be a circle. The proof of Theorem 3 is completed.

4. Final remarks

The attentive reader has surely observed that what is actually needed in the proof of Theorem 6 is not Brouwer's theorem, but the following statement (which can be considered to precede Brouwer's theorem in a logical sense): S^1 is not a continuous retract of the disc B_1. Indeed, the entire argument involving Brouwer's theorem can be replaced by the following elementary one: assume, as in the proof, that $\arg\min \rho(O;\cdot)$ is a connected subset of $\partial\Omega$ for every $O \in \Omega$ and consider a point $O^* \in \Omega$ at a maximum distance from $\partial\Omega$; then, the circle centered at O^* with radius $d(O^*, \partial\Omega)$ must intersect $\partial\Omega$ in two points at least and therefore, $\arg\min \rho(O^*;\cdot)$ must be a closed subarc of the circle. On the other side, the author have preferred to present Theorem 3 in the restricted setting of C^1 domains which may, even thinking in the resulting simplifications, be considered irrelevant in many respects. However, other extensions of Theorem 3 (as well as Theorem 2) seem to be more appealing than the easy ones related to the regularity of $\partial\Omega$. We can ask, for example, for the "size" that a subset $\Omega_0 \subseteq \Omega$ should have in order that the circle continues to be the unique convex domain such that the internal visual angle $\Theta(O;\cdot)$ is coalescent whichever be the point $O \in \Omega_0$. An immediate observation in this direction: since visual angles $\Theta(O;\cdot)$ continuously depends on $O \in \Omega$, the coalescence of the whole family $\{\Theta(O;\cdot) : O \in \Omega\}$ is implied by that one of the subfamily $\{\Theta(O;\cdot) : O \in \Omega_0\}$ when Ω_0 is a *dense* subset of Ω. The proof of higher-dimensional versions of Theorems 2 and 3 seems to be considerably more difficult. Furthermore, it is suspected that coalescence of the harmonic measure of three-dimensional domains ceases to characterize only spheres.

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