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ON THE COHOMOLOGY AND GEOMETRY OF PRINCIPAL SHEAVES

Abstract. We study the cohomological classification of *principal sheaves*, the latter being defined in a slightly different way than in [6], a fact allowing to consider on them geometrical objects like connections. The classification of *vector sheaves* (studied in [10]) is now a corollary of the classification of their principal sheaves of *frames*. In particular, principal sheaves with an abelian structural sheaf, equipped (the former) with a connection, admit a hypercohomological classification generalizing that of *Maxwell fields* given in [10].

Introduction

The present note is placed within the framework of *Abstract Differential Geometry* expounded in [10], in combination with certain ideas from [20].

In this framework, we start with *algebraized spaces*, i.e., spaces not bearing any smooth structure in the ordinary sense, and apply purely sheaf-theoretic methods, without recurrence to any kind of calculus. Such spaces include the smooth manifolds, the differential spaces in the sense of [16], and other generalized structures, such as those of [14], [15], [17] etc. They are also the base space of *principal and vector sheaves*, over which one can extend a great part of the classical geometry of fiber bundles, in particular the *theory of connections* and related topics.

This point of view seems to be advantageous especially for theoretical physics, where the spaces involved are far from being smooth and often admit singularities. Therefore, algebraic methods are most welcome. For relevant comments we refer to [7] (see also [8]), as well as to [12], [13], and [11] for recent applications of Abstract Differential Geometry in this direction.

Here we are mainly concerned with the cohomological classification of principal and vector sheaves. Principal sheaves are meant in a slightly dif-

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ferent way than in the original definition of [6], i.e., locally they look like sheaves of groups. This slight deviation allows one to define connections and related geometrical objects (see [20]), a fact not considered in the fundamental work of A. Grothendieck. The main Theorem 2.9, in conjunction with Theorem 2.6, gives a detailed account of the classification of these principal sheaves.

On the other hand, vector sheaves and their classification have been studied by A. Mallios in [10]. However, using the (principal) *sheaves of frames* associated with vector sheaves, we derive the cohomological classification of vector sheaves from that of principal sheaves. This is the content of the other main Theorem 3.5.

Finally, we examine the particular case of principal sheaves with abelian structural sheaf, the former being equipped also with a connection. As we show in Theorem 4.6, sheaves of this category admit a hypercohomological classification with coefficients in a two-term complex, determined by an appropriate operator of (abstract) logarithmic differential. A by-product of this result is the classification of *Maxwell fields* (i.e., line sheaves equipped with connections), also obtained straightforwardly in [10, Chap. VI, Theorem 18.2]).

1. Preliminaries

For the basic theory of sheaves and their cohomology, we refer to [3], [5] and [10, Vol. I], whose main notations and terminology are followed here.

Given a sheaf $\mathcal{S} \equiv (\mathcal{S}, X, \pi)$ and an open $U \subseteq X$, we denote by $\mathcal{S}(U)$ the set of (continuous) sections of $\mathcal{S}$ over U. A morphism of sheaves $f : \mathcal{S} \to \mathcal{S}'$ induces the corresponding morphism of presheaves (of sections) $\{f_U : \mathcal{S}(U) \to \mathcal{S}'(U)\}$, for all open $U \subseteq X$.

It is often convenient to identify a sheaf with the sheaf of germs of its sections. Similarly, a morphism f can be identified with the morphism generated by the presheaf morphism (f_U). For simplicity, we usually write $f(s)$, instead of $f_U(s)$, for any section $s \in \mathcal{S}(U)$. In this case, the difference between the original morphism f and the induced morphism of sections will be understood by the context or by an explicit mention of the range of the morphism at hand.

As mentioned in the Introduction, we start with an *algebraized space* $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ a sheaf of associative, commutative, and unital $\mathbb{K}$-algebras ($\mathbb{K} = R$, C) over X.

To an $(X, \mathcal{A})$, we associate a *differential triad* $(\mathcal{A}, d, \Omega^1)$, where Ω^1 is an $\mathcal{A}$-*module* over X, and $d : \mathcal{A} \to \Omega^1$ a $\mathbb{K}$-linear morphism satisfying the *Leibniz condition* $d(a \cdot b) = a \cdot d(b) + b \cdot d(a)$, for every $(a, b) \in \mathcal{A} \times_X \mathcal{A}$. Given an algebraized space $(X, \mathcal{A})$, we obtain a differential triad by the sheafification of Kähler's theory of differentials (see [10, Chap. XI, Section 5], [2, Chap. 3]). For many examples and other details we refer to [10, Sections VI.1, VI.2, and Chap. X].

2. Classification of principal sheaves

In this section X is assumed to be just a topological space. It will be completed with a differential triad in Section 4, where we introduce connections.

Fixing a *sheaf of groups* $(\mathcal{G}, X, \pi_G)$, we give the following basic definition, which is a slight variant of A. Grothendieck's original one (see [6, p. 32]).

DEFINITION 2.1. A *principal sheaf of structure type* $\mathcal{G}$ and with *structural sheaf* $\mathcal{G}$ is a sheaf of sets $(\mathcal{P}, X, \pi)$ such that:

i) $\mathcal{G}$ acts on the right of $\mathcal{P}$.

ii) There exists a *coordinatizing open covering* $\mathcal{U} = \{U_\alpha \subseteq X \mid \alpha \in I\}$ of X and corresponding isomorphisms (: *coordinates*) $\phi_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$, satisfying the equivariance property $\phi_\alpha(s \cdot g) = \phi_\alpha(s) \cdot g$, for every $(s, g) \in (\mathcal{P} \times_X \mathcal{G})|_{U_\alpha}$.

For brevity, a principal sheaf as before is called a $\mathcal{G}$-principal sheaf, denoted by $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$.

The local structure of a principal sheaf implies that $\mathcal{G}$ acts freely on $\mathcal{P}$ and freely transitively on its stalks. As a result, we obtain

LEMMA 2.2. *The map* $\boldsymbol{k} : \mathcal{P} \times_X \mathcal{P} \to \mathcal{G}$, *given by* $q = p \cdot \boldsymbol{k}(p, q)$, *is a morphism of sheaves satisfying equalities*

$$\boldsymbol{k}(p \cdot g, q) = g^{-1} \cdot \boldsymbol{k}(p, q)\,; \quad \boldsymbol{k}(p, q \cdot g) = \boldsymbol{k}(p, q) \cdot g.$$

Proof. Clearly, $\boldsymbol{k}$ is well defined by the properties of the action of $\mathcal{G}$ on $\mathcal{P}$ mentioned before the statement. On the other hand, the set

$$(\mathcal{P} \times_X \mathcal{P})|_{U_\alpha} = \pi^{-1}(U_\alpha) \times_{U_\alpha} \pi^{-1}(U_\alpha) \tag{2.1}$$

is open in $\mathcal{P} \times_X \mathcal{P}$, for every $U_\alpha \in \mathcal{U}$. Then, for any $p, q \in (\mathcal{P} \times_X \mathcal{P})|_{U_\alpha}$, we check that $\boldsymbol{k}(p, q) = \phi_\alpha(p)^{-1} \cdot \phi_\alpha(q)$, thus proving the continuity of $\boldsymbol{k}$ on (2.1), from which follows that $\boldsymbol{k}$ is a morphism. The equalities of the statement are routinely checked. ∎

A coordinatizing open covering $\mathcal{U}$ induces the family of (local) *natural sections* of $\mathcal{P}$

$$s_\alpha = \phi_\alpha^{-1} \circ \mathbf{1}|_{U_\alpha} \in \mathcal{P}(U_\alpha), \qquad \alpha \in I,$$

if $\mathbf{1} : X \to \mathcal{G}$ is the unit section of $\mathcal{G}$ (: $\mathbf{1}(x) := e_x$, the neutral element of the stalk $\mathcal{G}_x$). Equivalently, $s_\alpha = \phi_\alpha^{-1}(\mathbf{1}|_{U_\alpha})$ with ϕ_α denoting now the induced morphism of sections.

The same local structure induces also the *1-cocycle* $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ of $\mathcal{P}$, given by $g_{\alpha\beta}(x) := (\phi_\alpha \circ \phi_\beta^{-1})(e_x)$, for every $x \in U_{\alpha\beta} := U_\alpha \cap U_\beta$. It is immediate that

$$s_\beta = s_\alpha \cdot g_{\alpha\beta}. \tag{2.2}$$

DEFINITION 2.3. A *morphism* of principal sheaves (over the same base X)

$$(f, \phi, id_X) : (\mathcal{P}, \mathcal{G}, X, \pi) \longrightarrow (\mathcal{P}', \mathcal{G}', X, \pi')$$

is determined by an ordinary morphism of sheaves of sets $f : \mathcal{P} \to \mathcal{P}'$ and a morphism of sheaves of groups $\phi : \mathcal{G} \to \mathcal{G}'$, related together by the *equivariance* property $f(p \cdot g) = f(p) \cdot \phi(g)$, for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$. An *isomorphism* is a morphism where f and ϕ are isomorphisms in their categories.

Restricting ourselves to the category of $\mathcal{G}$-principal sheaves over the same base X, we say that two such sheaves are *equivalent* if they are $(f, id_\mathcal{G}, id_X)$-isomorphic. We obtain an equivalence relation the quotient space of which is denoted by

$$P_\mathcal{G}(X). \tag{2.3}$$

LEMMA 2.4. *Every morphism of the form $(f, id_\mathcal{G}, id_X)$ is an isomorphism.*

Proof. Since f is a local homeomorphism, it suffices to show that f is a bijection. First assume that $f(p) = f(q)$, for any $p, q \in \mathcal{P}$. Since $\pi(p) = \pi(q) := x$, there is a (unique) $g \in \mathcal{G}_x$ such that $q = p \cdot g$. Applying f we see that $f(p) = f(q) = f(p) \cdot g$, which implies that $g = e_x$ and proves the injectivity of f.

To show that f is onto, we take an arbitrary $q \in \mathcal{P}'$ with $\pi'(q) = x$. If $x \in U_\alpha$, we consider the natural section $s_\alpha \in \mathcal{P}(U_\alpha)$ and the element

$$p := s_\alpha(x) \cdot k'\left(f(s_\alpha(x)), q\right) \in \mathcal{P}_x,$$

where k' is the analog of k for $\mathcal{P}'$. Clearly $f(p) = q$. ∎

The following result describes the relationship between isomorphisms of principal sheaves and cocycles, a fact which is crucial for the subsequent (cohomological) classification of principal sheaves.

PROPOSITION 2.5. *Let $\mathcal{P}$, $\mathcal{P}'$ be two $\mathcal{G}$-principal sheaves over the same coordinatizing open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X. Let (s_α), (s'_α) be their respective natural sections and $(g_{\alpha\beta})$, $(g'_{\alpha\beta})$ the corresponding cocycles. Then, for every isomorphism of $\mathcal{P}$ onto $\mathcal{P}'$, there exists a unique 0-cochain $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$ satisfying equalities*

$$f(s_\alpha) = s'_\alpha \cdot h_\alpha, \tag{2.4}$$

$$g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}, \tag{2.5}$$

on U_α and $U_{\alpha\beta}$ respectively ($\alpha, \beta \in I$). Conversely, any 0-cochain satisfying (2.4) determines a unique isomorphism f satisfying also (2.5).

Proof. For any $x \in U_\alpha$, there is a unique $h_\alpha(x) \in \mathcal{G}_x$ such that $f(s_\alpha)(x) = f(s_\alpha(x)) = s'_\alpha(x) \cdot h_\alpha(x)$. This determines a section $h_\alpha \in \mathcal{G}(U_\alpha)$ satisfying (2.4). Its continuity is a consequence of equality $h_\alpha = k' \circ (s'_\alpha, f(s_\alpha))$. Applying f on both sides of (2.2), we have that $f(s_\beta) = f(s_\alpha) \cdot g_{\alpha\beta}$. Substituting $f(s_\alpha)$ and $f(s_\beta)$ with their expressions given by (2.4), and using the analog of (2.2) for $\mathcal{P}'$, we get (2.5).

Conversely, for each $\alpha \in I$, we define the map $f_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{P}'|_{U_\alpha}$ with

$$f_\alpha(p) := s'_\alpha(x) \cdot h_\alpha(x) \cdot g_\alpha(x), \tag{2.6}$$

where $x := \pi(p)$ and

$$g_\alpha(x) = k(s_\alpha(x), p). \tag{2.7}$$

It is clear that $\pi' \circ \phi_\alpha = \pi$ and $f_\alpha = (s'_\alpha \circ \pi) \cdot (h_\alpha \circ \pi) \cdot (k \circ (s_\alpha \circ \pi, id))$, with π and id now restricted on $\mathcal{P}|_{U_\alpha}$. Hence, f_α is a continuous morphism of sheaves over U_α, which is also $\mathcal{G}|_{U_\alpha}$-equivariant by Lemma 2.2. Therefore, Lemma 2.4 implies that f_α is an isomorphism of principal sheaves.

On the other hand, for any $p \in \mathcal{P}$ with $\pi(p) = x \in U_{\alpha\beta}$, we have also the analogs of (2.6) and (2.7)

$$f_\beta(p) = s'_\beta(x) \cdot h_\beta(x) \cdot g_\beta(x), \tag{2.6$'$}$$

$$p = s_\beta(x) \cdot g_\beta(x). \tag{2.7$'$}$$

Then, (2.7) and (2.7′), along with (2.2), yield $g_\alpha(x) = g_{\alpha\beta}(x) \cdot g_\beta(x)$. Therefore, the last equality, the analog of (2.2) for $\mathcal{P}'$, and (2.5) imply that

$$\begin{aligned} & s'_\beta(x) \cdot h_\beta(x) \cdot g_\beta(x) \\ &\quad = s'_\alpha(x) \cdot g'_{\alpha\beta}(x) \cdot h_\beta(x) \cdot g_{\beta\alpha}(x) \cdot g_\alpha(x) \\ &\quad = s'_\alpha(x) \cdot (h_\alpha(x) \cdot g_{\alpha\beta}(x) \cdot h_\beta^{-1}(x)) \cdot h_\beta(x) \cdot g_{\beta\alpha}(x) \cdot g_\alpha(x) \\ &\quad = s'_\alpha(x) \cdot h_\alpha(x) \cdot g_\alpha(x), \end{aligned}$$

which shows that (2.6) and (2.6′) coincide on the overlapping. We obtain an isomorphism f by gluing together all the f_α's.

Equality (2.4) is trivially satisfied. Finally, assume that there is also another isomorphism f' satisfying (2.4). Then, for any p as before,

$$\begin{aligned} f'(p) &= f'(s_\alpha(x) \cdot g_\alpha(x)) = f'(s_\alpha(x)) \cdot g_\alpha(x) \\ &= f(s_\alpha(x)) \cdot g_\alpha(x) = f(s_\alpha(x) \cdot g_\alpha(x)) = f(p); \end{aligned}$$

that is, $f = f'$. This completes the proof. ■

THEOREM 2.6. *Let $\mathcal{U}$ be an open covering of the topological space X, which is a basis of its topology. Then, a 1-cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ determines a* *unique, up to isomorphism, principal sheaf* $(\mathcal{P},\mathcal{G},X,\pi)$ *with corresponding cocycle the given* $(g_{\alpha\beta})$.

Proof. Let $(\mathcal{G}(U_\alpha),\zeta_{\alpha\beta})$ be the presheaf of sections of $\mathcal{G}$, with restriction maps the group morphisms $\zeta_{\alpha\beta}:\mathcal{G}(U_\alpha)\longrightarrow\mathcal{G}(U_\beta):\sigma\mapsto\sigma|_{U_\beta}$, if $U_\beta\subseteq U_\alpha$. We consider the maps

$$\varrho_{\alpha\beta}:=g_{\beta\alpha}\cdot\zeta_{\alpha\beta}:\mathcal{G}(U_\alpha)\longrightarrow\mathcal{G}(U_\beta),\qquad U_\beta\subseteq U_\alpha.$$

Then, for any $U_\gamma\subseteq U_\beta\subseteq U_\alpha$ and $\sigma\in\mathcal{G}(U_\alpha)$,

$$\begin{aligned}(\varrho_{\beta\gamma}\circ\varrho_{\alpha\beta})(\sigma)&=\varrho_{\beta\gamma}\left(g_{\beta\alpha}\cdot\zeta_{\alpha\beta}(\sigma)\right)=g_{\gamma\beta}\cdot\zeta_{\beta\gamma}\left(g_{\beta\alpha}\cdot\zeta_{\alpha\beta}(\sigma)\right)\\&=g_{\gamma\beta}\cdot g_{\beta\alpha}\cdot\zeta_{\beta\gamma}\left(\zeta_{\alpha\beta}(\sigma)\right)=g_{\gamma\alpha}\cdot\zeta_{\alpha\gamma}(\sigma)=\varrho_{\alpha\gamma}(\sigma),\end{aligned}$$

from which follows that $\varrho_{\alpha\gamma}=\varrho_{\beta\gamma}\circ\varrho_{\alpha\gamma}$. Therefore, in virtue of the hypothesis about the covering, the association $U_\alpha\mapsto\mathcal{G}(U_\alpha)$ and the maps $(\varrho_{\alpha\beta})$ determine a presheaf $(\mathcal{G}(U_\alpha),\varrho_{\alpha\beta})$ which, in turn, generates a sheaf of sets denoted by $(\mathcal{P},X,\pi)$. We show that this is the sought principal sheaf.

i) There is a *right action* $\delta:\mathcal{P}\times_X\mathcal{G}\to\mathcal{P}$ obtained as follows: for each $\alpha\in I$, we define the map $\delta_\alpha:\mathcal{G}(U_\alpha)\times\mathcal{G}(U_\alpha)\longrightarrow\mathcal{G}(U_\alpha)$, with $\delta_\alpha(\sigma,g):=\sigma\cdot g$. Each δ_α is an action such that $\varrho_{\alpha\beta}\circ\delta_\alpha=\delta_\beta\circ(\varrho_{\alpha\beta}\times\zeta_{\alpha\beta})$, for every $U_\beta\subseteq U_\alpha$. Then, δ is generated by the presheaf morphism (δ_α).

ii) To find the local structure of $\mathcal{P}$, we fix an open set $U_\alpha\in\mathcal{U}$. Then, all the U_β's, with $U_\beta\subseteq U_\alpha$, form a basis of the topology of U_α. For any such U_β, we define the map

$$\psi_{\alpha,U_\beta}:\mathcal{G}(U_\beta)\longrightarrow\mathcal{G}(U_\beta):\sigma\mapsto g_{\alpha\beta}\cdot\sigma, \tag{2.8}$$

whose domain is the group of sections of the presheaf $(\mathcal{G}(U_\alpha),\varrho_{\alpha\beta})$, generating $\mathcal{P}$, while its range is the group of sections of $(\mathcal{G}(U_\alpha),\zeta_{\alpha\beta})$, generating the group $\mathcal{G}$.

It is straightforward that (2.8) is a $\mathcal{G}(U_\beta)$-equivariant bijection, with inverse given by $\psi^{-1}_{\alpha,U_\beta}(\tau)=g_{\beta\alpha}\cdot\tau$, for every $\tau\in\mathcal{G}(U_\beta)$. Moreover, for every U_γ, with $U_\gamma\subseteq U_\beta\subseteq U_\alpha$, and any $\sigma\in\mathcal{G}(U_\beta)$,

$$\begin{aligned}(\zeta_{\beta\gamma}\circ\psi_{\alpha,U_\beta})(\sigma)&=(g_{\alpha\beta}\cdot\sigma)|_{U_\gamma}=g_{\alpha\gamma}\cdot(g_{\gamma\beta}\cdot\sigma|_{U_\gamma})\\&=\psi_{\alpha,U_\gamma}(g_{\gamma\beta}\cdot\sigma|_{U_\gamma})=(\psi_{\alpha,U_\gamma}\circ\varrho_{\beta\gamma})(\sigma).\end{aligned}$$

This shows that the family $(\psi_{\alpha,U_\beta}):(\mathcal{G}(U_\beta),\varrho_{\beta\gamma})\to(\mathcal{G}(U_\beta),\zeta_{\beta\gamma})$, for all U_β's running in U_α, is a $(\mathcal{G}(U_\beta))$-equivariant presheaf isomorphism, generating thus a $\mathcal{G}|_{U_\alpha}$-equivariant sheaf isomorphism $\psi_\alpha:\mathcal{P}|_{U_\alpha}\to\mathcal{G}|_{U_\alpha}$. Therefore, $(\psi_\alpha)_{\alpha\in I}$ is a family of coordinates of $\mathcal{P}$ with respect to $\mathcal{U}$.

iii) Let us denote by $(\bar{g}_{\alpha\beta})$ the cocycle of $\mathcal{P}$, with respect to $\mathcal{U}$ and the local structure just defined. For an $x\in U_{\alpha\beta}$, by definition, we have that $\bar{g}_{\alpha\beta}(x)=(\psi_\alpha\circ\psi_\beta^{-1})(e_x)$. Since $\mathcal{G}$ can be identified with the sheaf of germs of its (continuous) sections, we may write

$$e_x = [\mathbf{1}|_{U_\gamma}] \equiv \mathbf{1}|_{U_\gamma}(x)$$

for some $U_\gamma \subseteq U_{\alpha\beta}$ with $x \in U_\gamma$ (which, of course, always exists). Thus, (2.8) implies that

$$\begin{aligned}\bar{g}_{\alpha\beta}(x) &= (\psi_\alpha \circ \psi_\beta^{-1})(e_x) = \psi_\alpha(\psi_\beta^{-1}(\mathbf{1}|_{U_\gamma}(x))) \\ &= \psi_\alpha(\psi_{\beta,U_\gamma}^{-1}(\mathbf{1}|_{U_\gamma}(x))) = \psi_\alpha(g_{\gamma\beta}(x)) = \psi_\alpha(e_x)\cdot g_{\gamma\beta}(x) \\ &= \psi_\alpha(\mathbf{1}|_{U_\gamma}(x))\cdot g_{\gamma\beta}(x) = g_{\alpha\gamma}(x)\cdot g_{\gamma\beta}(x) = g_{\alpha\beta}(x)\end{aligned}$$

Hence, $(\bar{g}_{\alpha\beta}) = (g_{\alpha\beta})$.

Finally, assume that there is also another principal sheaf $(\mathcal{P}', \mathcal{G}, X, \pi')$ with the same cocycle $(g_{\alpha\beta})$. Since (2.5) is trivially satisfied, Proposition 2.5 implies that $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic. The proof is now complete. ∎

COROLLARY 2.7. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf with coordinatizing covering $\mathcal{U}$ and cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$. Assume that $\mathcal{V} = \{V_i | i \in J\}$ is an open refinement of $\mathcal{U}$, which is also a basis of the topology of X. Then $\mathcal{P}$ is isomorphic to a principal sheaf $\bar{\mathcal{P}} \equiv (\bar{\mathcal{P}}, \mathcal{G}, X, \bar{\pi})$ with coordinatizing covering $\mathcal{V}$ and corresponding cocycle $(\bar{g}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$, obtained by an appropriate restriction of $(g_{\alpha\beta})$.*

Proof. For a refining map $\tau : J \to I$ $(: V_i \subseteq U_{\tau(i)})$, we set $\bar{g}_{ij} := g_{\tau(i)\tau(j)}|_{V_{ij}}$, for all $i, j \in J$. We obtain a cocycle $(\bar{g}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$ inducing, by Theorem 2.6, a principal sheaf $\bar{\mathcal{P}}$ as in the statement.

Let us denote by $(\bar{\phi}_i)$ the coordinates and by $(\bar{s}_i)$ the natural sections of $\bar{\mathcal{P}}$, with respect to $\mathcal{V}$. For each $i \in J$, we define the isomorphism

$$f_i := \phi_{\tau(i)}^{-1} \circ \bar{\phi}_i : \bar{\mathcal{P}}|_{V_i} \longrightarrow \mathcal{P}|_{V_i}$$

where $\phi_{\tau(i)}^{-1}$ is, in fact, restricted on the subsheaf $\mathcal{G}|_{V_i} \subseteq \mathcal{G}|_{U_{\tau(i)}}$.

We shall show that $f_i = f_j$ on $\bar{\mathcal{P}}|_{V_{ij}}$. Indeed, for any p in the previous overlapping, with $\bar{\pi}(p) = x$, there are unique $a_i, a_j \in \mathcal{G}_x$ such that $\bar{s}_i(x)\cdot a_i = p = \bar{s}_j(x)\cdot a_j$. Since $a_j = \bar{g}_{ji}(x)\cdot a_i$, we check that

$$\begin{aligned}f_j(p) &= (\phi_{\tau(j)}^{-1} \circ \bar{\phi}_j)(\bar{s}_j(x)\cdot a_j) = \phi_{\tau(j)}^{-1}(\bar{g}_{ji}(x)\cdot a_i) \\ &= \phi_{\tau(j)}^{-1}(\bar{g}_{ji}(x))\cdot a_i = \phi_{\tau(j)}^{-1}((\phi_{\tau(j)} \circ \phi_{\tau(i)}^{-1})(e_x))\cdot a_i \\ &= \phi_{\tau(i)}^{-1}(a_i) = (\phi_{\tau(i)}^{-1} \circ \bar{\phi}_i)(p) = f_i(p),\end{aligned}$$

as claimed. Gluing together the isomorphisms (f_i) we obtain the isomorphism of the statement. ∎

A direct combination of Proposition 2.5 and Corollary 2.7 proves also the following isomorphism criterion for principal sheaves with different coordinatizing coverings.

COROLLARY 2.8. *Let $(\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf with cocycle $(g_{\alpha\beta})$ over a coordinatizing open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$, which is also a basis of the topology of X. Let $(\mathcal{Q}, \mathcal{G}, X, \pi')$ be another principal sheaf with cocycle $(\gamma_{\alpha'\beta'})$ over $\mathcal{U}' = (U_{\alpha'})_{\alpha'\in I'}$, also a basis of the topology of X. If $\mathcal{V} = (V_i)_{i\in J}$ is a common refinement of $\mathcal{U}$ and $\mathcal{U}'$, we denote by $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ the principal sheaves obtained from $\mathcal{P}$ and $\mathcal{Q}$, respectively, by restricting their cocycles on $\mathcal{V}$. Then the following conditions are equivalent:*

i) *$\mathcal{P}$ and $\mathcal{Q}$ are isomorphic.*

ii) *$\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ are isomorphic.*

iii) *If $\tau : J \longrightarrow I$ and $\tau' : J \longrightarrow I'$ are refining maps for the previous coverings, and $(\bar{g}_{ij}) := (g_{\tau(i)\tau(j)})$, $(\bar{\gamma}_{ij}) := (\gamma_{\tau'(i)\tau'(j)})$ are the cocycles of $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ respectively, then there exists a 0-cochain $(h_i) \in C^0(\mathcal{V}, \mathcal{G})$ such that $\bar{\gamma}_{ij} = h_i \cdot \bar{g}_{ij} \cdot h_j^{-1}$, for every $i, j \in J$.*

Recalling the notation (2.4), we are now in a position to prove the main

THEOREM 2.9 (*Classification of principal sheaves*).

$$\boldsymbol{P}_{\mathcal{G}}(X) \cong H^1(X, \mathcal{G}).$$

For details concerning the *1st cohomology set* we refer to [6, Chap. V], [9, Chap. 1], [10, Chap. III]. However, for the reader's convenience, we recall the following facts needed in the proof.

Let X be a topological space, $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ an open covering of it, and $\mathcal{G}$ a sheaf of (not necessarily abelian) groups. Two cocycles $(f_{\alpha\beta}), (f'_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ are said to be *cohomologous* if there is a 0-cochain $h \equiv (h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$ such that $f'_{\alpha\beta} = h_\alpha \cdot f_{\alpha\beta} \cdot h_\beta^{-1}$ holds over $U_{\alpha\beta}$, for all $\alpha, \beta \in I$. The equivalence class of $(f_{\alpha\beta})$ is denoted by $[(f_{\alpha\beta})]_{\mathcal{U}}$ and the corresponding quotient space by $H^1(\mathcal{U}, \mathcal{G})$.

If $\mathcal{V} = (V_i)_{i\in J}$ is an open refinement of $\mathcal{U}$, any refining map $\tau : J \longrightarrow I$ induces the map

$$t^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(\mathcal{V}, \mathcal{G}) : [(f_{\alpha\beta})]_{\mathcal{U}} \longmapsto [(f_{\tau(i)\tau(j)})|_{V_{ij}}]_{\mathcal{V}},$$

which is independent of the choice of τ. As is known,

$$H^1(X, \mathcal{G}) := \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{G}), \tag{2.9}$$

with $\mathcal{U}$ is running the set of all *proper* open coverings of X. For every $\mathcal{U}$, there is a *canonical injection* $t_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{G}) \to H^1(X, \mathcal{G})$. Then, we set

$$[(f_{\alpha\beta})] := t_{\mathcal{U}}([(f_{\alpha\beta})]_{\mathcal{U}}). \tag{2.10}$$

Proof of Theorem 2.9. We define the map $\Phi : \boldsymbol{P}_{\mathcal{G}}(X) \to H^1(X, \mathcal{G})$ as follows: for a class $[\mathcal{P}] \in \boldsymbol{P}_{\mathcal{G}}(X)$, we set $\Phi([\mathcal{P}]) := [(g_{\alpha\beta})]$, if $\mathcal{U} = (U_\alpha)_{\alpha\in I}$

is an arbitrary coordinatizing open covering with corresponding cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ for the representative sheaf $\mathcal{P}$.

We show that Φ is *well defined*, i.e., independent of the choice of the representative and its cocycle. To this end let $\mathcal{Q}$ be any principal sheaf with $[\mathcal{P}] = [\mathcal{Q}]$, whose cocycle $(\gamma_{\alpha'\beta'})$ is defined over a coordinatizing covering $\mathcal{U}' = (U_{\alpha'})_{\alpha' \in I'}$. We choose an arbitrary common refinement $\mathcal{V} \subseteq \mathcal{U} \cap \mathcal{U}'$, $\mathcal{V} = (V_i)_{i \in J}$, forming also a basis of the topology of X. Considering any refining maps $\tau : J \to I$ and $\tau' : J \to I'$, Corollary 2.7 implies that $\mathcal{P}$ is isomorphic to a principal sheaf $\bar{\mathcal{P}}$ with corresponding cocycle $(\bar{g}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$ given by

$$\bar{g}_{ij} = g_{\tau(i)\tau(j)}|V_{ij}; \quad i, j \in J. \tag{2.11}$$

Similarly, $\mathcal{Q}$ is isomorphic to $\bar{\mathcal{Q}}$ with cocycle $(\bar{\gamma}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$ given by

$$\bar{\gamma}_{ij} = \gamma_{\tau'(i)\tau'(j)}|V_{ij}; \quad i, j \in J. \tag{2.12}$$

Since, by the assumption, $\mathcal{P} \cong \bar{\mathcal{P}} \cong \mathcal{Q} \cong \bar{\mathcal{Q}}$, Corollary 2.8 implies that

$$[(\bar{g}_{ij})]_{\mathcal{V}} = [(\bar{\gamma}_{ij})]_{\mathcal{V}}. \tag{2.13}$$

On the other hand, condition $t_{\mathcal{U}} = t_{\mathcal{V}} \circ t^{\mathcal{U}}_{\mathcal{V}}$ and its analog for $\mathcal{U}'$ (cf., for instance, [1, p. 89]), along with equalities (2.9)–(2.10), imply that

$$\begin{aligned}
[(g_{\alpha\beta})] &= t_{\mathcal{U}}\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) = (t_{\mathcal{V}} \circ t^{\mathcal{U}}_{\mathcal{V}})\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) \\
&= t_{\mathcal{V}}\left([(g_{\tau(i)\tau(j)}|V_{ij})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\bar{g}_{ij})]_{\mathcal{V}}\right) \\
&= t_{\mathcal{V}}\left([(\bar{\gamma}_{ij})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\gamma_{\tau'(i)\tau'(j)}|V_{ij})]_{\mathcal{V}}\right) \\
&= (t_{\mathcal{V}} \circ t^{\mathcal{U}'}_{\mathcal{V}})\left([(\gamma_{\alpha'\beta'})]_{\mathcal{V}}\right) = t_{\mathcal{U}'}\left([(\gamma_{\alpha'\beta'})]_{\mathcal{U}'}\right) \\
&= [(\gamma_{\alpha'\beta'})],
\end{aligned} \tag{2.14}$$

which proves the previous assertion.

Here it is worthy to note that, since all the cocycles used above are taken over open coordinatizing coverings, the direct limit (2.9) should be taken with respect to all proper (open) coordinatizing coverings $\mathcal{U}$ of X. This is possible because the latter form a cofinal subset of the set of all proper (open) coverings of X. For relevant details we refer, e.g., to [9] and [10, Vol. I, p. 127].

To show that Φ is *injective*, assume that $\Phi([\mathcal{P}]) = \Phi([\mathcal{Q}])$, for any $[\mathcal{P}], [\mathcal{Q}] \in P_{\mathcal{G}}(X)$. If $(g_{\alpha\beta})$ and $(\gamma_{\alpha'\beta'})$ are the cocycles over any open coordinatizing coverings of the representatives $\mathcal{P}$ and $\mathcal{Q}$, respectively, then $[(g_{\alpha\beta})] = [(\gamma_{\alpha'\beta'})]$. Hence, as in the preceding part of the proof,

$$t_{\mathcal{V}}\left([(g_{\tau(i)\tau(j)}|V_{ij})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\gamma_{\tau'(i)\tau'(j)}|V_{ij})]_{\mathcal{V}}\right),$$

or, in virtue of (2.11) and (2.9),

$$t_{\mathcal{V}}\left([(\bar{g}_{ij})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\bar{\gamma}_{ij})]_{\mathcal{V}}\right) \in H^1(X, \mathcal{G}),$$

and, by the injectivity of $t_{\mathcal{V}}$, $[(\bar{g}_{ij})]_{\mathcal{V}} = [(\bar{\gamma}_{ij})]_{\mathcal{V}}$. Consequently (see Corollary 2.8), $\mathcal{P} \cong \bar{\mathcal{P}} \cong \bar{\mathcal{Q}} \cong \mathcal{Q}$, thus proving the injectivity of Φ.

Finally, let $[(g_{\alpha\beta})] \in H^1(X, \mathcal{G})$ be an arbitrarily chosen cohomology class with the representative cocycle $(g_{\alpha\beta})$ defined over some open covering $\mathcal{U}$ of X. If $\mathcal{U}$ is a basis of the topology of X, then Proposition 2.5 ensures the existence of a principal sheaf $\mathcal{P}$ with $\Phi([\mathcal{P}]) = [(g_{\alpha\beta})]$. If $\mathcal{U}$ is not necessarily a basis of the topology, then we can always find a refinement $\mathcal{V}$ of $\mathcal{U}$ with this property. Then, taking the restriction $(\bar{g}_{ij})$ of $(g_{\alpha\beta})$ on $\mathcal{V}$ and the corresponding sheaf $\bar{\mathcal{P}}$, as in (2.11), we have that

$$\begin{aligned}\Phi([\mathcal{P}]) &= [(\bar{g}_{ij})] = t_{\mathcal{V}}\left([(g_{\tau(i)\tau(j)}|_{V_{ij}})]_{\mathcal{V}}\right)\\ &= (t_{\mathcal{V}} \circ t^{\mathcal{U}}_{\mathcal{V}})\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) = t_{\mathcal{U}}\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) = [(g_{\alpha\beta})],\end{aligned}$$

which completes the *surjectivity* of Φ and the proof. ∎

3. Vector sheaves

In this section we obtain the classification of vector sheaves by applying Theorem 2.9 to their sheaves of frames.

DEFINITION 3.1. Let $(X, \mathcal{A})$ be an algebraized space. A *vector sheaf* $\mathcal{E} \equiv (\mathcal{E}, X, p)$ of rank n is a locally free $\mathcal{A}$-*module*; that is, there is an open coordinatizing covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X and $\mathcal{A}|_{U_\alpha}$-isomorphisms (: coordinates) $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha} \cong (\mathcal{A}|_{U_\alpha})^n$.

For a coordinatizing covering as before, the transformations of coordinates (actually $\mathcal{A}|_{U_{\alpha\beta}}$-isomorphisms of modules) $g_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1}$ define the *cocycle* $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ of $\mathcal{E}$, where $\mathcal{GL}(n, \mathcal{A})$ is the *general linear group sheaf* generated by the complete presheaf

$$X \supseteq U \longmapsto GL(n, \mathcal{A}(U)) \cong \mathrm{Iso}_{\mathcal{A}(U)}(\mathcal{A}^n|_U, \mathcal{A}^n|_U).$$

Hence, $g_{\alpha\beta} \in \mathrm{Iso}_{\mathcal{A}|_{U_{\alpha\beta}}}(\mathcal{A}^n|_{U_{\alpha\beta}}, \mathcal{A}^n|_{U_{\alpha\beta}}) \cong GL(n, \mathcal{A}(U_{\alpha\beta})) \cong \mathcal{GL}(n, \mathcal{A})(U_{\alpha\beta})$. Working as in the proof of Theorem 2.6, we can show that a cocycle $(g_{\alpha\beta}) \in \mathcal{GL}(n, \mathcal{A})$ determines both a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf and a vector sheaf of rank n (see also [10, Vol. 1, p. 359]). The link between these two sheaves is provided by the sheaves of frames defined right below.

Given a vector sheaf $\mathcal{E}$ (of rank n), with coordinatizing covering $\mathcal{U}$, we consider the presheaf

$$U \longmapsto \mathrm{Iso}_{\mathcal{A}|_U}(\mathcal{A}^n|_U, \mathcal{E}|_U), \tag{3.1}$$

where U is running now the basis of topology $\mathcal{B}$ of X, consisting of the open sets $V \subseteq X$ such that $V \subseteq U_\alpha$, for some $U_\alpha \in \mathcal{U}$.

We have already proved (see [18]) the following

PROPOSITION 3.2. *The sheaf $\mathcal{P}(\mathcal{E})$ generated by the presheaf (3.1) is a $\mathcal{GL}(n,\mathcal{A})$-principal sheaf whose cocycle over $\mathcal{U}$ coincides with the cocycle $(g_{\alpha\beta})$ of $\mathcal{E}$. We call $\mathcal{P}(\mathcal{E})$ the sheaf of frames of $\mathcal{E}$.*

COROLLARY 3.3. *For any principal sheaf of the form $(\mathcal{P},\mathcal{GL}(n,\mathcal{A}),X,\pi)$, there is a vector sheaf $\mathcal{E}$ such that $\mathcal{P}\cong\mathcal{P}(\mathcal{E})$.*

Proof. Let $\mathcal{P}$ be a principal sheaf as in the statement, with cocycle $(g_{\alpha\beta})\in\mathcal{GL}(n,\mathcal{A})$. Then, the same cocycle determines a vector sheaf and the sheaf of frames $\mathcal{P}(\mathcal{E})$, both of them having as cocycle the given $(g_{\alpha\beta})$. The result now follows from Proposition 2.5. ∎

A morphism $f\equiv(f,id_X)$ between two vector sheaves $(\mathcal{E},X,p)$ and $(\mathcal{E}',X,p')$ is a morphism of $\mathcal{A}$-modules. The definition of an *isomorphism* between vector sheaves of rank n, over the same base X, is obvious. Analogously to (2.3), we denote by

$$\Phi^n_{\mathcal{A}}(X) \tag{3.2}$$

the set of the resulting isomorphism classes.

For an isomorphism $f:\mathcal{E}\longrightarrow\mathcal{E}'$ we can prove the analog of Proposition 2.5; that is, f is completely known by a 0-cochain $(h_\alpha)\in C^0(\mathcal{U},\mathcal{GL}(n,\mathcal{A}))$ such that $f|\mathcal{E}_{U_\alpha}=\psi'_\alpha\circ h_\alpha\circ\psi_\alpha$ and $g'_{\alpha\beta}=h_\alpha\circ g_{\alpha\beta}\circ h_\beta^{-1}$. As a consequence, we obtain

LEMMA 3.4. *Two vector sheaves $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic if and only if their corresponding sheaves of frames are isomorphic.*

Proof. This is a result of the fact that the cocycles involved in both cases are cohomologous via the same cochain (h_α). ∎

We can prove now the analog of Theorem 2.9, namely the classification of vector sheaves

THEOREM 3.5. *With the notation* (3.2),

$$\Phi^n_{\mathcal{A}}(X)\cong H^1(X,\mathcal{GL}(n,\mathcal{A})).$$

Proof. In virtue of Theorem 2.9, it suffices to show that

$$\Phi^n_{\mathcal{A}}(X)\cong P_{\mathcal{GL}(n,\mathcal{A})}(X).$$

This is a consequence of Corollary 3.3 and Lemma 3.4, along with the vector sheaf analog of Proposition 2.5. ∎

REMARK 3.6. A straightforward proof of Theorem 3.5, without use of principal sheaves, is given in [10, Chap. V, Theorem 2.1]. Our approach shows that the study of vector sheaves and their geometry can be reduced to that of principal sheaves (see also the next section, as well as [18]).

4. Sheaves with connections

As in Section 3, we fix an algebraized space $(X, \mathcal{A})$ together with a differential triad $(\mathcal{A}, d, \Omega^1)$. In order to define connections on a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, we need to enrich the structure of $\mathcal{G}$. Thus, we assume that the following properties are satisfied:

LSG 1. $\mathcal{G}$ admits a *representation* in an $\mathcal{A}$-module of *Lie algebras* $\mathcal{L}$, i.e., there is a morphism of sheaves of groups $\varrho : \mathcal{G} \longrightarrow \mathcal{A}ut(\mathcal{L})$.

LSG 2. There is a *logarithmic differential* $\partial : \mathcal{G} \longrightarrow \Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$ satisfying

$$\partial(g \cdot h) = \varrho(h^{-1}) \cdot \partial(g) + \partial(h), \qquad (g, h) \in \mathcal{G} \times_X \mathcal{G}.$$

The first term in the right side member of the last equality denotes the natural action of $\mathcal{G}$ on (the right of) $\Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$ (see [20] for details). Also, $\mathcal{A}ut(\mathcal{L})$ is the sheaf of groups generated by the complete presheaf

$$U \longmapsto \mathrm{Aut}(\mathcal{L}|_U) := \mathrm{End}(\mathcal{L}|_U)^{\cdot},$$

the upper dot denoting the set of invertible endomorphisms.

DEFINITION 4.1. A sheaf of groups satisfying (LSG 1) and (LSG 2) is called a *Lie sheaf of groups*. It is denoted by $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$

A typical example is provided by the general linear group sheaf

$$\mathcal{GL}(n, \mathcal{A}) \equiv (\mathcal{GL}(n, \mathcal{A}), \mathcal{A}d, \mathcal{M}_n(\mathcal{A}), \widetilde{\partial}),$$

partially defined in Section 3. The *matrix algebra sheaf* $\mathcal{M}_n(\mathcal{A})$ is generated by the (complete) presheaf of $n \times n$ matrices $U \mapsto M_n(\mathcal{A}(U))$, with U running in the topology of X. Thus, for every open $U \subseteq X$

$$\mathcal{GL}(n, \mathcal{A})(U) \cong \mathrm{GL}(n,\ \mathcal{A}(U)) = M_n(\mathcal{A}(U))^{\cdot} \cong \mathcal{M}_n(\mathcal{A})^{\cdot}(U),$$

whence, $\mathcal{GL}(n, \mathcal{A}) = \mathcal{M}_n(\mathcal{A})^{\cdot}$.

The logarithmic differential $\widetilde{\partial} : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \Omega^1 \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$ is defined by $\widetilde{\partial}(a) := a^{-1} \cdot d(a)$, where $d(a) := (da_{ij}) \in M_n(\Omega^1(U))$, for every matrix $a = (a_{ij}) \in M_n(\mathcal{A}(U))$ and $U \subseteq X$ open.

Finally, $\mathcal{A}d : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{A}ut(\mathcal{M}_n(\mathcal{A}))$ is determined by the family of group morphisms $\mathrm{Ad}_U : \mathrm{GL}(n,\ \mathcal{A}(U)) \longrightarrow \mathrm{End}(\mathcal{M}_n(\mathcal{A})|_U)^{\cdot}$ (for all open $U \subseteq X$), each one of which is defined, in turn, as follows: for any $a \in GL(n, \mathcal{A}(U))$, the isomorphism $\mathrm{Ad}_U(a) : \mathcal{M}_n(\mathcal{A})|_U \longrightarrow \mathcal{M}_n(\mathcal{A})|_U$ is given (section-wise) by $\mathrm{Ad}_U(a)(b) := a \cdot b \cdot a^{-1}$, for every $b \in M_n(\mathcal{A}(V))$ and every open $V \subseteq U$.

DEFINITION 4.2. An *abelian Lie sheaf of groups* is a Lie sheaf of groups $(\mathcal{G}, \rho, \mathcal{L}, \partial)$, where $\mathcal{G}$ is a sheaf of abelian groups and ρ the trivial representation.

An illustrating example is provided by the *group sheaf of units* $\mathcal{A}^{\cdot} \equiv (\mathcal{A}^{\cdot}, \rho^{\cdot}, \mathcal{A}, \widetilde{\partial})$. In this case, $\mathcal{A}^{\cdot} = \mathcal{GL}(1, \mathcal{A})$, $\rho^{\cdot}$ is the trivial representation, and the logarithmic differential reduces to $\widetilde{\partial} : \mathcal{A}^{\cdot} \to \Omega^1 \otimes_{\mathcal{A}} \mathcal{A} \cong \Omega^1$ with $\widetilde{\partial}(s) := s^{-1} \cdot d(s)$, for every $s \in \mathcal{A}^{\cdot}(U)$ and every open $U \subseteq X$.

• From now on we consider $\mathcal{G}$-principal sheaves with $\mathcal{G}$ a Lie sheaf of groups. Also, in order to facilitate our notations, we set $\Omega^1(\mathcal{L}) := \Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$.

DEFINITION 4.3. A *connection* on $(\mathcal{P}, \mathcal{G}, X, \pi)$ is a morphism (of sheaves of sets) $D : \mathcal{P} \to \Omega^1(\mathcal{L})$, such that $D(p \cdot g) = \varrho(g^{-1}) \cdot D(p) + \partial(g)$, for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$.

Equivalently, a connection D is determined by the family of local sections, called (after the classical terminology) *local connection forms*, given by $\omega_\alpha := D(s_\alpha) \in \Omega^1(\mathcal{L})(U_\alpha)$, $\alpha \in I$, and satisfying the compatibility condition (viz. *local gauge transform*) $\omega_\beta = \rho(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial(g_{\alpha\beta})$, over each $U_{\alpha\beta} \neq \emptyset$. In particular, for an abelian Lie sheaf of groups $\mathcal{G}$, the previous condition reduces to

$$\omega_\beta = \omega_\alpha + \partial(g_{\alpha\beta}). \tag{4.1}$$

For the existence of connections on principal sheaves, various examples and other details, we refer to [20].

On the other hand, according to [10, Vol. II], an $\mathcal{A}$-*connection* on a vector sheaf $\mathcal{E}$ is a $\mathbb{K}$-morphism $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ satisfying the *Leibniz-Koszul condition* $\nabla(\alpha \cdot s) = \alpha \cdot \nabla s + s \otimes d\alpha$, for every $(\alpha, s) \in \mathcal{A} \times_X \mathcal{E}$.

The relationship between connections on principal and vector (or, more general, associated) sheaves has been studied in [18], [21]. In particular, we have shown that there exists a bijection

$$\{\mathcal{A}\text{-connections } \nabla \text{ on } \mathcal{E}\} \xrightarrow{\cong} \{\text{connections } D \text{ on } \mathcal{P}(\mathcal{E})\} \tag{$\star$}$$

DEFINITION 4.4. Let $\mathcal{P}$ and $\mathcal{P}'$ be two principal sheaves with the same structural sheaf $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ and base X, equipped with the connections D and D' respectively. We say that $(\mathcal{P}, D)$ and $(\mathcal{P}', D')$ are *(gauge) equivalent* if there is an isomorphism of principal sheaves $f : \mathcal{P} \to \mathcal{P}'$ such that $D = D' \circ f$.

Over a common coordinatizing covering $\mathcal{U}$ for both $\mathcal{P}$ and $\mathcal{P}'$, we have already proved (see [19, Theorem 3.9]) the following criterion of equivalence. In case of different coordinatizing coverings, we may take a common refinement and consider the equivalent principal sheaves of Corollaries 2.7 and 2.8.

LEMMA 4.5. *$(\mathcal{P}, D)$ and $(\mathcal{P}', D')$ are equivalent if and only if there exists a 0-cochain $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$ such that equalities*

(4.2) $$g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}$$

(4.3) $$\omega_\alpha = \rho(h_\alpha^{-1}).\omega'_\alpha + \partial(h_\alpha)$$

hold on $U_{\alpha\beta}$ and U_α respectively, for every $\alpha, \beta \in I$.

Similarly to (2.3), we denote by $P_{\mathcal{G}}(X)^D$ the set of equivalence classes derived from Definition 4.4. On the other hand,

$$\check{H}^1(X, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L})$$

stands for the (Čech) 1-dimensional *hypercohomology* group with coefficients in the 2-term complex $\partial : \mathcal{G} \to \Omega^1(\mathcal{L})$ (see [4, p. 21], [10, Vol. I, p. 224]). Hence, based on the mechanism of [10, Chap. VI, Theorem 18.2], we are in a position to prove the following

THEOREM 4.6. *If $\mathcal{G}$ is an abelian Lie sheaf of groups, then*

$$P_{\mathcal{G}}(X)^D \cong \check{H}^1(X, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L})).$$

Proof. Since we consider only the 1-dimensional hypercohomology with coefficients in the complex $\partial : \mathcal{G} \longrightarrow \Omega^1(\mathcal{L})$, we may consider the following diagram, where the rectangle (I) is commutative

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \\
C^0\big(\mathcal{U}, \Omega^1(\mathcal{L})\big) & \xrightarrow{\delta^{0,1}} & C^1\big(\mathcal{U}, \Omega^1(\mathcal{L})\big) & \xrightarrow{\delta^{1,1}} & \cdots \\
\uparrow{\scriptstyle d^{0,0} = \partial} & (\mathrm{I}) & \uparrow{\scriptstyle d^{1,0} = \partial} & & \\
C^0(\mathcal{U}, \mathcal{G}) & \xrightarrow{\delta^{0,0}} & C^1(\mathcal{U}, \mathcal{G}) & \xrightarrow{\delta^{1,0}} & C^2(\mathcal{U}, \mathcal{G}) \longrightarrow \cdots
\end{array}$$

The horizontal morphisms are the usual coboundary operators and the vertical ones are those induced by ∂. As a result, we obtain the (total) complex

$$\mathcal{S}^0 \xrightarrow{D^0} \mathcal{S}^1 \xrightarrow{D^1} \mathcal{S}^2 \xrightarrow{D^2} \cdots$$

with $\mathcal{S}^0 = C^0(\mathcal{U}, \mathcal{G})$, $\mathcal{S}^1 = C^1(\mathcal{U}, \mathcal{G}) \oplus C^0\left(\mathcal{U}, \Omega^1(\mathcal{L})\right)$, $\mathcal{S}^2 = C^2(\mathcal{U}, \mathcal{G}) \oplus C^1\left(\mathcal{U}, \Omega^1(\mathcal{L})\right)$, $D^0 = \delta^{0,0} + \partial$ and $D^1 = (\delta^{1,0} - \partial) + \delta^{0,1}$. By an easy computation we verify that

(4.4) $$\mathrm{Ker}(D^1) = \mathrm{Ker}(\delta^{1,0} - \partial) \oplus \mathrm{Ker}(\delta^{0,1}),$$

(4.5) $$\operatorname{Im}(D^0) = \operatorname{Im}(\delta^{0,0}) \oplus \operatorname{Im}(\partial).$$

Therefore,

$$\check{H}^1(\mathcal{U}, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L})) := \operatorname{Ker}(D^1)/\operatorname{Im}(D^0) = \frac{\operatorname{Ker}(\delta^{1,0} - \partial) \oplus \operatorname{Ker}(\delta^{0,1})}{\operatorname{Im}(\delta^{0,0}) \oplus \operatorname{Im}(\partial)}.$$

We choose now a pair $(\mathcal{P}, D)$. The principal sheaf $\mathcal{P}$ determines a cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G}) \subseteq C^1(\mathcal{U}, \mathcal{G})$, while D defines the local connection forms $(\omega_\alpha) \in C^0(\mathcal{U}, \Omega^1(\mathcal{L}))$, satisfying (4.1). The same equality implies that

(4.6) $$\partial\left((g_{\alpha\beta})\right) = \delta^{0,1}\left((\omega_\alpha)\right).$$

Hence, applying D^1 on the pair $((g_{\alpha\beta}), (\omega_\alpha))$, and taking into account (4.4) and (4.6) along with the cocycle condition of $(g_{\alpha\beta})$, we see that

$$\begin{aligned} D^1\left((g_{\alpha\beta}), (\omega_\alpha)\right) &= \left(\delta^{1,0}\left((g_{\alpha\beta})\right) - \partial\left((g_{\alpha\beta})\right)\right) + \delta^{0,1}\left((\omega_\alpha)\right) \\ &= \delta^{1,0}\left((g_{\alpha\beta})\right) = g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 0, \end{aligned}$$

which shows that $((g_{\alpha\beta}), (\omega_\alpha)) \in \operatorname{Ker}(D^1)$, thus determining the class

$$[((g_{\alpha\beta}), (\omega_\alpha))]_{\mathcal{U}} \in \check{H}^1(\mathcal{U}, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L})$$

and the corresponding class $[((g_{\alpha\beta}), (\omega_\alpha))] \in \check{H}^1(X, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L}))$. This allows one to define the map

$$\Phi : P_{\mathcal{G}}(X)^D \ni [(\mathcal{P}, D)] \longmapsto [((g_{\alpha\beta}), (\omega_\alpha))] \in \check{H}^1(X, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L})).$$

i) *Φ is well defined.* Assume that $(\mathcal{P}, D)$ and $(\mathcal{P}', D')$ are equivalent. Taking a common coordinatizing covering for both principal sheaves, Proposition 2.5 implies that $g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1} = (h_\alpha \cdot h_\beta^{-1}) \cdot g_{\alpha\beta}$, thus

(4.7) $$(g'_{\alpha\beta}) \cdot (g_{\alpha\beta}^{-1}) = \delta^{0,0}\left((h_\alpha^{-1})\right).$$

The same Proposition, in conjunction with the definition of ∂, yields

(4.8) $$(\omega'_\alpha - \omega_\alpha) = (-\partial(h_\alpha)) = \partial\left((h_\alpha^{-1})\right).$$

Hence, to prove our claim, it suffices to show that

$$[((g_{\alpha\beta}), (\omega_\alpha))]_{\mathcal{U}} = \left[\left((g'_{\alpha\beta}), (\omega'_\alpha)\right)\right]_{\mathcal{U}} \in \operatorname{Ker}(D^1)/\operatorname{Im}(D^0),$$

or, equivalently, $((g'_{\alpha\beta}), (\omega'_\alpha)) - ((g_{\alpha\beta}), (\omega_\alpha)) = ((g'_{\alpha\beta} - g_{\alpha\beta}), (\omega'_\alpha - \omega_\alpha)) \in \operatorname{Im}(D^0)$. This is indeed the case, since (4.7), (4.8) and (4.5) (or the definition of D^0), along with the commutativity of $\mathcal{G}$ (whence the equivalent use of multiplicative and additive notations), lead to

$$\begin{aligned} ((g'_{\alpha\beta} - g_{\alpha\beta}), (\omega'_\alpha - \omega_\alpha)) &= (\delta^{0,0}(h_\alpha^{-1})), \partial(h_\alpha^{-1})) \\ &= (\delta^{0,0}, \partial)((h_\alpha^{-1})) = D^0((h_\alpha^{-1})). \end{aligned}$$

Note that if we use different coordinatizing covers, then we obtain equal classes in the direct limit, working as in the proof of Theorem 2.9.

ii) Φ *is injective*. This is proved by the same arguments, as before, in a reverse way (see also the proof of Theorem 2.9).

iii) Φ *is surjective*. To this end let us take an arbitrary element of $\check{H}^1(X, \mathcal{G} \xrightarrow{\partial} \Omega^1(\mathcal{L}))$, represented by the class $[((g_{\alpha\beta}), (\omega_\alpha))]_{\mathcal{U}}$ of a pair $((g_{\alpha\beta}), (\omega_\alpha)) \in C^1(\mathcal{U}, \mathcal{G}) \oplus C^0(\mathcal{U}, \Omega^1(\mathcal{L}))$. Therefore, equalities

$$\begin{aligned} 0 = D^1 ((g_{\alpha\beta}), (\omega_\alpha)) &= (\delta^{1,0}, \partial) ((g_{\alpha\beta})) + \delta^{0,1} ((\omega_\alpha)) \\ &= \delta^{1,0} ((g_{\alpha\beta})) + \left(-\partial ((g_{\alpha\beta})) + \delta^{0,1} ((\omega_\alpha))\right), \end{aligned}$$

together with (4.2), imply that

$$\delta^{1,0} ((g_{\alpha\beta})) = 0, \tag{4.9}$$

$$\partial ((g_{\alpha\beta})) = \delta^{0,1} ((\omega_\alpha)). \tag{4.10}$$

From (4.9), it follows that $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$, i.e., $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$, which determines a $\mathcal{G}$-principal sheaf $\mathcal{P}$ with cocycle $(g_{\alpha\beta})$ (see Theorem 2.6). On the other hand, (4.10) yields $\partial ((g_{\alpha\beta})) = (\partial(g_{\alpha\beta})) = \delta^{0,1} ((\omega_\alpha)) = (\omega_\beta - \omega_\alpha)$, that is, $\omega_\beta = \omega_\alpha + \partial(g_{\alpha\beta})$, for every $\alpha, \beta \in I$. This is precisely (4.1), which is equivalent to the existence of a connection D on $\mathcal{P}$. Therefore, $\Phi ([(\mathcal{P}, D)]) = [((g_{\alpha\beta}), (\omega_\alpha))]$, by which we complete the proof. ∎

In particular, taking as $\mathcal{G}$ the abelian sheaf of groups $\mathcal{A}^{\cdot}$, we obtain

COROLLARY 4.7. *The following isomorphism holds true:*

$$\boldsymbol{P}_{\mathcal{A}^{\cdot}}(X)^D \cong \check{H}^1(X, \mathcal{A}^{\cdot} \xrightarrow{\tilde{\partial}} \Omega^1).$$

For our final result we need

DEFINITION 4.8. A *line sheaf* is a vector sheaf of rank 1. Furthermore, in the terminology of [10, Vol. II, p. 94], a *Maxwell field* is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a line sheaf and ∇ an $\mathcal{A}$-connection on it.

Line sheaves are classified by $\Phi^1_{\mathcal{A}}(X) \cong H^1(X, \mathcal{A}^{\cdot})$ (cf. Theorem 3.2). Moreover, in analogy with Definition 4.4, two Maxwell fields $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$ (over X) are said to be *equivalent* if there is an isomorphism of line sheaves $f : \mathcal{E} \longrightarrow \mathcal{E}'$ such that

$$\nabla' \circ f = (f \otimes 1_{\Omega^1}) \circ \nabla.$$

The set of resulting classes is denoted by $\Phi^1_{\mathcal{A}}(X)^{\nabla}$. Therefore, we obtain

COROLLARY 4.9. *The following classification of Maxwell fields holds true:*

$$\Phi^1_{\mathcal{A}}(X)^{\nabla} \cong \check{H}^1(X, \mathcal{A}^{\cdot} \xrightarrow{\tilde{\partial}} \Omega^1).$$

Proof. The conclusion is a consequence of Corollaries 3.3 and 4.7, taking also into account Lemma 3.4 and the bijection (⋆). ∎

REMARK 4.10. A direct proof of the previous Corollary (without recurrence to principal sheaves) is given in [10, Vol. II, p. 175]. In the latter, line sheaves are denoted by $\mathcal{L}$, a notation reserved here for the sheaves of Lie algebras $\mathcal{L}$ introduced in the beginning of this section.

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