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ON (α, β)-DERIVATIONS OF SEMIPRIME RINGS

Abstract. We show that if α and β are centralizing automorphisms and d a centralizing (α, β)-derivation of a semiprime ring R, then d is commuting. Some results on α-derivations and centralizing derivations of semiprime rings follow as applications of this result.

1. Introduction

Throughout, R denotes a ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$. Then $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ hold in R. R is *prime* if $aRb = 0$ implies either $a = 0$ or $b = 0$; it is *semiprime* if $aRa = 0$ implies $a = 0$. A prime ring is obviously semiprime. An additive mapping d from R into itself is called a *derivation* if $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. A mapping f from R into itself is *commuting* if $[f(x), x] = 0$; and *centralizing* if $[f(x), x] \in Z(R)$ for all $x \in R$. We call a mapping $f : R \to R$ *central* if $f(x) \in Z(R)$ for all $x \in R$. Recall that if f is an additive commuting mapping from R into itself, then a linearization of $[f(x), x] = 0$ yields $[f(x), y] = [x, f(y)]$ for all $x, y \in R$. The study of centralizing and commuting mappings was initiated by Posner [12]. Considerable work has been done on centralizing and commuting mappings as well as centralizing and commuting derivations of prime and semiprime rings during the last couple of decades (see, e.g., [1–4, 6, 10, 11, 12] and references therein). Derivations are generalized as α- or skew-derivations and (α, β)-derivations and have been applied in various situations; in particular, in the solution of some functional equations (see, e.g., Brešar [5]). Let α, β be automorphisms of R. An additive mapping d of R into itself is called an (α, β)-*derivation* if $d(xy) = \alpha(x)d(y) + d(x)\beta(y)$ for all $x, y \in R$. If $\beta = 1$, where 1 is the

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identity mapping of R, then d is called an α-derivation or a skew-derivation. For instance, $d = \alpha - \beta$ is an (α, β)-derivation and $d = \alpha - 1$ is an α-derivation. Of course, a $(1,1)$-derivation or a 1-derivation is a derivation. For more information on α-derivations and (α, β)-derivations, we refer to [5, 7, 8, 9, 13].

Bell and Martindale [2, Lemma 4] have proved that if d is a centralizing derivation on a nonzero left ideal U of a semiprime ring R, then d is commuting on U. The result has played a fundamental role in the development of the theory of commuting and centralizing mappings. The main purpose of this paper is to extend this result to (α, β)-derivations. We consider our results on the whole ring R rather than left ideals for simplicity. We essentially show (Theorem 2.4) that if α, β are centralizing automorphisms of a semiprime ring R and d is a centralizing (α, β)-derivation of R, then d is commuting. The result of Bell and Martindale [2, Lemma 4] follows as a corollary of our result.

2. Results

We shall need the following results.

LEMMA 2.1 [6, Proposition 3.1]. *Let U be a Jordan subring of a semiprime ring R. If an additive mapping f of R into itself is centralizing on U, then $2[f(x), x] = 0$ for all $x \in U$.*

LEMMA 2.2 [13, Proposition 2.3]. *Let d be a commuting α-derivation of a semiprime ring R, then d is central.*

LEMMA 2.3. *Let α and β be centralizing automorphisms and d a centralizing (α, β)-derivation of a semiprime ring R, then the identity $[d(x), x]^2 + (x[\alpha(x), d(x)] + [\alpha(x), d(x)]x)d(x) = 0$ holds for all $x \in R$.*

Proof. By assumption $[d(x), x] \in Z(R)$. Linearizing this, we get

$$[d(x), y] + [d(y), x] \in Z(R) \text{ for all } x, y \in R. \tag{1}$$

Since d is additive, therefore by Lemma 2.1, we have

$$2[d(x), x] = 0 \text{ for all } x \in R. \tag{2}$$

Linearizing (2), we get

$$2([d(x), y] + [d(y), x]) = 0 \text{ for all } x, y \in R. \tag{3}$$

Using (1)–(3) and the fact that $[d(x), x] \in Z(R)$, the following identity follows easily

$$[d(x), xy + yx] + [d(y), x^2] = 0 \text{ for all } x, y \in R. \tag{4}$$

Replacing y by yx in (4), we get $[d(x), xyx + yx^2] + [d(yx), x^2] = 0$, which gives $[d(x), xy + yx]x + (xy + yx)[d(x), x] + [\alpha(y)d(x) + d(y)\beta(x), x^2] = 0$ for all $x, y \in R$. This further gives

$$(5)\qquad [d(x), xy+yx]x+([x,y]+2yx)[d(x),x]+\alpha(y)[d(x),x^2]+[\alpha(y),x^2]d(x) +d(y)[\beta(x),x^2] + [d(y),x^2]\beta(x) \text{ for all } x,y \in R.$$

Since β is additive and centralizing, therefore by Lemma 2.1, we have $2[\beta(x), x] = 0$. Thus $[\beta(x), x^2] = x[\beta(x), x] + [\beta(x), x]x = 2x[\beta(x), x] = 0$. Also $[d(x), x] \in Z(R)$ and $2[d(x), x] = 0$ imply that $[d(x), x^2] = x[d(x), x] + [d(x), x]x = 2x[x, d(x)] = 0$ for all $x \in R$. When we combine these with (4), then from (5), we get $-[d(y), x^2]x + [x, y][d(x), x] + [\alpha(y), x^2]d(x) + [d(y), x^2]\beta(x) = 0$ for all $x, y \in R$, which implies

$$(6)\quad [x,y][d(x),x]+[\alpha(y),x^2]d(x)+[d(y),x^2](\beta(x)-x) = 0 \text{ for all } x,y \in R.$$

Now

$$(7)\qquad \begin{aligned} [d(y),x^2] &= x[d(y),x] + [d(y),x]x = x[d(y),x] + [d(y),x]x \\ &\quad + x[d(x),y] + [d(x),y]x - x[d(x),y] - [d(x),y]x \\ &= x([d(y),x] + [d(x),y]) + ([d(y),x] + [d(x),y])x \\ &\quad - x[d(x),y] - [d(x),y]x \text{ for all } x,y \in R. \end{aligned}$$

Using (1) and (3), from (7), we get

$$(8)\qquad \begin{aligned} [d(y),x^2] &= 2x([d(y),x] + [d(x),y]) - x[d(x),y] - [d(x),y]x \\ &= -x[d(x),y] - [d(x),y]x \text{ for all } x \in R. \end{aligned}$$

Since α is centralizing, therefore it is commuting by [2, Lemma 2]. Thus

$$(9)\qquad \begin{aligned} [\alpha(y),x^2] &= x[\alpha(y),x] + [\alpha(y),x]x = x[y,\alpha(x)] + [y,\alpha(x)]x \\ &= -x[\alpha(x),y] - [\alpha(x),y]x \text{ for all } x,y \in R. \end{aligned}$$

Using (8) and (9), from (6), we get

$$(10)\qquad \begin{aligned} &[x,y][d(x),x] - (x[\alpha(x),y] + [\alpha(x),y]x)d(x) \\ &\quad -(x[d(x),y] + [d(x),y]x)(\beta(x) - x) = 0 \text{ for all } x,y \in R. \end{aligned}$$

Replacing y by $d(x)$ in (10), we get

$$(11)\quad [d(x),x]^2 + (x[\alpha(x),d(x)] + [\alpha(x),d(x)]x)d(x) = 0 \text{ for all } x \in R. \ \blacksquare$$

We now prove our main result.

THEOREM 2.4. *Let α and β be centralizing automorphisms and d a centralizing (α, β)-derivation of a semiprime ring R, then d is commuting.*

Proof. Since d is centralizing, therefore $[d(x), x] \in Z(R)$ for all $x \in R$. Further, Lemma 2.1 gives $2[d(x), x] = 0$ for all $x \in R$. Since α is a centralizing automorphism, therefore α is commuting which implies $(\alpha-1)$ is a commuting α-derivation of R. By Lemma 2.2, we conclude that $(\alpha-1)$ is central. Thus $[(\alpha-1)(x), d(x)] = 0$ for all $x \in R$, which implies $[\alpha(x), d(x)] = [x, d(x)]$ for all $x \in R$. Using this, from Lemma 2.3, we get $0 = [d(x), x]^2 + (x[x, d(x)] + [x, d(x)]x)d(x) = [d(x), x]^2 + (2x[x, d(x)])d(x) = [d(x), x]^2$ for all $x \in R$. Since a semiprime ring has no central nilpotents, therefore $[d(x), x] = 0$ for all $x \in R$. Thus d is commuting. ∎

Taking $\alpha = 1 = \beta$ in Theorem 2.4, we get the following well known result of Bell and Martindale [2, Lemma 4], in the special case when derivation is considered on the whole ring instead of a left ideal.

COROLLARY 2.5. *If d is a centralizing derivation of a semiprime ring R, then d is commuting.*

Taking $\beta = 1$ in Theorem 2.4, the following corollary is immediate.

COROLLARY 2.6. *If α is a centralizing automorphism and d a centralizing α-derivation of a semiprime ring R, then d is commuting.*

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