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ON PSEUDO-EFFECT ALGEBRAS WHICH CAN BE COVERED BY PSEUDO MV-ALGEBRAS

Abstract. Pseudo-effect algebras are partial algebras $(E; +, 0, 1)$ which were recently introduced. They have a partially defined addition $+$ which is only associative and not necessary commutative and with two complements, left and right ones. They are a non-commutative generalization of orthomodular posets and MV-algebras, respectively. We define five kinds of compatibilities, and we introduce a block as a maximal set of mutually compatible elements. The compatibility is a property of the physical system which corresponds to the distributivity, or equivalently, to "classical mechanics"-type phenomena. We show that any lattice pseudo-effect algebra under a natural condition can be covered by blocks, and any block is a pseudo MV-algebra. This result generalizes the analogical result of Riečanová for effect algebras. If the pseudo-effect algebra with the condition is, in addition, a σ-complete lattice, then it is a commutative effect algebra which can be covered by σ-complete MV-algebras.

1. Introduction

Today there exists a whole family of non-commutative generalizations of MV-algebras which were introduced by Chang [Cha] in fifties: pseudo MV-algebras of Georgescu and Iorgulescu [GeIo] and generalized MV-algebras of Rachůnek [Rac] which, in addition, are equivalent. Also a non-commutative version of BL-algebras, pseudo BL-algebras, have been introduced in [DGI]. In addition, pseudo-effect algebras, which are partial non-commutative algebras, were recently introduced by the authors [DvVe I, DvVe II]. Non-commutative algebras are algebraic non-commutative analogs of non-commutative reasoning.

A non-commutative reasoning can be met in the every-day life very often. Many human processes are depending on the order of variables. On the other

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hand; today there exists even a concurrent programming language using a non-commutative logic [Bau].

Quantum mechanical measurements are also in general non-commutative; the result of some experiment may depend on the order of the measurements. Consider, for example, a beam of particles which are prepared in a certain state, and which are sent through a sequence of three polarizing filters F_1, F_2, F_3. It is well-known that the order of the filters makes in general a difference. For example, let the filter be polarizing in planes perpendicular to the particle beam, such that F_1 polarizes vertically, F_2 horizontally and F_3 at a 45° angle. If we place the filters in the order F_1, F_2, F_3, then no particles are detected, but in the order F_1, F_3, F_2, particles are detected; the difference is due to quantum interference.

Such phenomena are in the literature nowadays presented also as sequential conjunctions or sequentially independent effects by Gudder and Nagy [GuNa] or sequential probability models by Foulis [Fou]. Our structure, the pseudo-effect algebra, is different, and it arises typically from not necessarily commutative po-groups, which have been studied in physics for many years.

An important case of (Abelian) po-groups used in physics is $B(H)$, the system of all Hermitian operators of a separable Hilbert space H, and the system of all effect operators $E(H)$, i.e. the system of all Hermitian operators A on H such that $O \leq A \leq I$, where O and I are the zero and identity operators. Then $E(H)$ is the interval in $B(H)$, and it is one of the most important examples of effect algebras. In addition, if M is a maximal system of all mutually commuting operators from $E(H)$, then M can be converted into an MV-algebra, [CGP].

In 1994, effect algebras entered quantum structures which generalize MV-algebras. Quantum structures are algebraic structures which are connected with mathematical foundations of quantum mechanics. The most important examples of them are orthomodular lattices, orthomodular posets, orthoalgebras and effect algebras [FoBe, DvPu]. They are not distributive structures, but a local distributivity is expressed by the compatibility. The blocks, maximal sets of mutually compatible elements, are sometimes Boolean algebras, and in quantum structures it means that blocks reflect a so-called locally classical part of a quantum mechanical system [Var, DvPu].

Recently Riečanová [Rie] showed that every lattice effect algebra can be covered by blocks and every block is an MV-algebra.

In the present paper we generalize this result to pseudo-effect algebras introduced by the authors. Such algebras are sometimes unit intervals in cones of unital po-groups [DvVe I, DvVe II]. We introduce five kinds of compatibilities of elements of pseudo-effect algebras and show that in a case of lattice pseudo-effect algebras four of them coincide. We show that any block in a lattice pseudo-effect algebra is a distributive lattice with special kinds of the Riesz decomposition properties. If, in addition, they satisfy the difference compatibility property, then any block is a pseudo-effect subalgebra of the pseudo-effect algebra E which, in addition is a pseudo MV-algebra, and E is a set-theoretical union of its blocks. Moreover, if such a pseudo-effect algebra E is a σ-complete lattice, then every block is a σ-complete MV-algebra, and E is commutative.

In addition, an open problem is formulated.

We recall a similar problem for effect algebras with the Riesz interpolation property was studied in [Dvu 1], and Jenča [Jen] studied blocks of mutually compatible elements satisfying the Riesz decomposition property. However, such blocks are not necessary MV-algebras.

2. Pseudo-effect algebras

A partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* ([DvVe I, DvVe II]) if, for all $a, b, c \in E$, the following holds

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;

(ii) for any $a \in E$, there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$;

(iii) if $a+b$ exists, there are elements $d, e \in E$ such that $a+b = d+a = b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a+c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. If E is a lattice under $\leq$, we say that E is a *lattice pseudo-effect algebra.* If $+$ is commutative, i.e. if $a+b = b+a$, E is said to be an *effect algebra.*

Let $/$, $\setminus$ be two partial binary operations on E such that, for $a, b \in E$, a / b is defined iff $b \setminus a$ is defined iff $a \leq b$, and such that in this case we have

$$(b \setminus a) + a = a + (a / b) = b. \tag{2.1}$$

Then

$$a = (b \setminus a) / b = b \setminus (a / b). \tag{2.2}$$

If $a \leq b \leq c$, then

$$\begin{aligned} (c \setminus a) \setminus (b \setminus a) &= c \setminus b, \\ (a / b) / (a / c) &= b / c, \\ (c \setminus b) / (c \setminus a) &= b \setminus a, \\ (a / c) \setminus (b / c) &= a / b. \end{aligned}$$

Let $E = (E; +, 0, 1)$ be a pseudo-effect algebra. We define $a^- := 1 \setminus a$ and $a^\sim := a / 1$ for any $a \in E$.

For example if (G, u) is a unital (not necessary Abelian) po-group with a strong unit u (sometimes it is sufficient to assume only $u > 0$), and

$$\Gamma(G, u) := [0, u] = \{g \in G : \ 0 \le g \le u\},$$

then $(\Gamma(G, u); +, 0, u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G, u)$. In [DvVe II], there are conditions showing when a pseudo-effect algebra can be represented in this way.

We recall that a *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

In [Dvu] it was shown that every pseudo MV-algebra is isomorphic to $\Gamma(G, u)$, where (G, u) is a unital ℓ-group with a strong unit u, where $a \oplus b := (a + b) \wedge u$, $a \odot b = (a - u + b) \vee 0$ and $a^- = u - a$ and $a^\sim = -a + u$.

If M is a pseudo MV-algebra, then the partial operation $a + b$ is defined iff $a \le b^-$, and then $a + b := a \oplus b$, and $(M; +, 0, 1)$ is a pseudo-effect algebra.

For two elements $a, b \in E$, we write $a \,\mathbf{com}\, b$ if, for any $x, y \in E$ with $x \le a$, $y \le b$, we have $x + y, y + x$ are defined in E, and $x + y = y + x$.

PROPOSITION 2.1. *Let E be a pseudo-effect algebra. For $a, b, c \in E$, let $a \,\mathbf{com}\, b$, and $c \le a$, $c \le b$. Then*

$$(a \setminus c) + (b \setminus c) = ((a + b) \setminus c) \setminus c,$$
$$(c / b) + (c / a) = c / (c / (a + b)).$$

Proof. Put $u = (a + b) \setminus c$. Then $a + b = u + c$ and $a + (b \setminus c) + c = u + c$, $a + (b \setminus c) = u$. Hence $((a + b) \setminus c) \setminus c = (a + (b \setminus c)) \setminus c = ((b \setminus c) + a) \setminus c = (b \setminus c) + (a \setminus c) = (a \setminus c) + (b \setminus c)$. ∎

PROPOSITION 2.2. *Let E be a pseudo-effect algebra, $a, b, c \in E$, $a, b \le c$. If $a \vee b \in E$, then $(c \setminus a) \wedge (c \setminus b) \in E$, $(a / c) \wedge (b / c) \in E$, and*

$$c \setminus (a \vee b) = (c \setminus a) \wedge (c \setminus b),$$

$$(a \vee b) / c = (a / c) \wedge (b / c).$$

In particular, if $a + b = b + a$, then

$$(a \vee b) / (a + b) = a \wedge b = (a + b) \setminus (a \vee b).$$

In addition, if $c \geq \bigvee_i a_i \in E$, then $\bigwedge_i (c \setminus a_i)$, $\bigwedge_i (a_i / c) \in E$, and

$$c \setminus \Big(\bigvee_i a_i\Big) = \bigwedge_i (c \setminus a_i), \qquad \Big(\bigvee_i a_i\Big) / c = \bigwedge_i (a_i / c).$$

Proof. We have $a \leq a \vee b \leq c$, $b \leq a \vee b \leq c$, so that $c \setminus (a \vee b) \leq c \setminus a, c \setminus b$. If $w \leq c \setminus a, c \setminus b$, then by (2.2), $a = (c \setminus a) / c \leq w / c$, $b \leq (c \setminus b) / c \leq w / c$, so that $a \vee b \leq w / c$ and $w = c \setminus (w / c) \leq c \setminus (a \vee b)$. Thus $c \setminus (a \vee b) = (c \setminus a) \wedge (c \setminus b)$.

The second equality can be proved in an analogical way. The third equation follows from the first two ones. ∎

PROPOSITION 2.3. *Let E be a lattice pseudo-effect algebra, $a, b, c \in E$, and $a, b \leq c$. Then*

$$c \setminus (a \wedge b) = (c \setminus a) \vee (c \setminus b),$$
$$(a \wedge b) / c = (a / c) \vee (b / c).$$

Proof. From $a \wedge b \leq a \leq c$ and $a \wedge b \leq b \leq c$, we have $c \setminus a \leq c \setminus (a \wedge b)$, $c \setminus b \leq c \setminus (a \wedge b)$. Suppose $w \geq c \setminus a, c \setminus b$. Then $c \setminus a = (c \setminus a) \wedge c \leq w \wedge c \leq c$, so that $(w \wedge c) / c \leq (c \setminus a) / c = a$.

In a similar way, $(w \wedge c) / c \leq b$. Therefore, $(w \wedge c) / c \leq a \wedge b$. Hence, $c \setminus (a \wedge b) \leq c \setminus ((w \wedge c) / c) = w \wedge c \leq w$, which implies $c \setminus (a \wedge b) = (c \setminus a) \vee (c \setminus b)$.

The proof of the second equation is similar. ∎

PROPOSITION 2.4. *Let E be a lattice pseudo-effect algebra. For any $a, b \in E$,*

$$(a \vee b) \setminus (a \wedge b) = ((a \vee b) \setminus a) + (a \setminus (a \wedge b))$$
$$(a \wedge b) / (a \vee b) = ((a \wedge b) / a) + (a / (a \vee b)).$$

Proof. Calculate $a \vee b = ((a \vee b) \setminus a) + (a \setminus (a \wedge b)) + a \wedge b$. Then $(a \vee b) \setminus (a \wedge b) = (a \vee b) \setminus a + (a \setminus (a \wedge b))$. Similarly $a \vee b = a + a / (a \vee b) = (a \wedge b) + (a \wedge b) / a + a / (a \vee b)$. ∎

PROPOSITION 2.5. *Let E be a pseudo effect-algebra and let $c \leq \bigwedge_i a_i \in E$. Then $\bigwedge_i (a_i \setminus c)$, $\bigwedge_i (c / a_i) \in E$, and*

$$\Big(\bigwedge_i a_i\Big) \setminus c = \bigwedge_i (a_i \setminus c), \qquad c / \Big(\bigwedge_i a_i\Big) = \bigwedge_i (c / a_i).$$

Proof. It is clear that $(\bigwedge_i a_i) \setminus c \leq a_i \setminus c$. Let $w \leq a_i \setminus c$ for any i. Then $w + c \leq a_i$, so that $w + c \leq \bigwedge_i a_i$, i.e., $w \leq (\bigwedge_i a_i) \setminus c$.

In a similar way we prove the second equality. ∎

PROPOSITION 2.6. *Let E be a pseudo-effect algebra, $a = \bigvee_i a_i \in E$. Then $\bigwedge_i(a \setminus a_i)$, $\bigwedge_i(a_i \mathbin{/} a) \in E$, and*

$$\bigwedge_i (a \setminus a_i) = 0 = \bigwedge_i (a_i \mathbin{/} a).$$

Proof. Let $w \leq a \setminus a_i$ for any a_i. Then $a_i = (a \setminus a_i) \mathbin{/} a \leq w \mathbin{/} a$, hence, $a \leq w \mathbin{/} a$, which gives $w = a \setminus (w \mathbin{/} a) \leq a \setminus a = 0$.

Similarly for the second equality. ∎

PROPOSITION 2.7. *Let E be a lattice pseudo effect algebra. If $a = \bigvee_i a_i \in E$ and $c \leq a_i$ for any i, then $\bigvee_i(a_i \setminus c)$, $\bigvee_i(c \mathbin{/} a_i) \in E$, and*

$$a \setminus c = \bigvee_i (a_i \setminus c), \qquad c \mathbin{/} a = \bigvee_i (c \mathbin{/} a_i).$$

Proof. Since $c \leq a_i \leq a$, then $a_i \setminus c \leq a \setminus c$ for any i. Let $a_i \setminus c \leq v$ for any i. Then $a_i \setminus c \leq v \wedge (a \setminus c) \leq a \setminus c$, so that $(a \setminus c) \setminus (v \wedge (a \setminus c)) \leq (a \setminus c) \setminus (a_i \setminus c) = a \setminus a_i$ for any i. By Proposition 2.6, $(a \setminus c) \setminus (v \wedge (a \setminus c)) = 0$ which implies $a \setminus c = v \wedge (a \setminus c)$, i.e., $a \setminus c \leq v$, consequently $a \setminus c = \bigvee_i(a_i \setminus c)$.

In a similar way we prove the second equality. ∎

PROPOSITION 2.8. *Let E be a lattice pseudo-effect algebra. Then, for any $a, b \in E$,*

$$(a \setminus (a \wedge b)) \wedge (b \setminus (a \wedge b)) = 0 = ((a \wedge b) \mathbin{/} a) \wedge ((a \wedge b) \mathbin{/} b)$$
$$((a \vee b) \setminus a) \wedge ((a \vee b) \setminus b) = 0 = (a \mathbin{/} (a \vee b)) \wedge (b \mathbin{/} (a \vee b)).$$

Proof. Put $c = a \wedge b$ in Proposition 2.5.

For the second equality, let $z \leq (a \vee b) \setminus a, (a \vee b) \setminus b$. Then $z + a \leq a \vee b$ and $z + b \leq a \vee b$, so that $a \leq z \mathbin{/} (a \vee b)$ and $b \leq z \mathbin{/} (a \vee b)$ which gives $a \vee b \leq z \mathbin{/} (a \vee b)$, i.e., $z = 0$. ∎

PROPOSITION 2.9. *Let E be a lattice pseudo-effect algebra. If $x+y, x+z \in E$, then*

$$x + (y \wedge z) = (x + y) \wedge (x + z),$$
$$x + (y \vee z) = (x + y) \vee (x \vee z).$$

If $y + x, z + x \in E$, then

$$(y \wedge z) + x = (y + x) \wedge (z + x),$$
$$(y \vee z) + x = (y + x) \vee (z \vee x).$$

Proof. By Proposition 2.5, $x \mathbin{/} ((x + y) \wedge (x + z)) = y \wedge z$, i.e., $x + (y \wedge z) = (x + y) \wedge (x + z)$. In a similar way we obtain the second equality using Proposition 2.7. ∎

PROPOSITION 2.10. *Let E be a lattice pseudo-effect algebra. If $a \wedge b = 0$ and $a + b, b + a \in E$, then*

$$a \vee b = (a + b) \wedge (b + a).$$

Proof. Let $w = (a + b) \wedge (b + a)$. Then $w \geq a \vee b$, and $w \setminus (a \vee b) \leq (a + b) \setminus a = b$ and $w \setminus (a \wedge b) \leq (b + a) \setminus a = b$, so that $w \setminus (a \vee b) = 0$, i.e., $w = a \vee b$. ∎

3. Compatibilities

In the present section, we introduce five kinds of compatibilities of elements of a pseudo-effect algebra. We show that in the case of a lattice pseudo-effect algebra four of them coincide. We prove that a set of mutually compatible elements, an analog of Boolean subalgebra, is always a distributive lattice with two kinds of the Riesz decomposition properties.

We say that a poset E (i) satisfies the *Riesz interpolation property*, (RIP) for short, if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_j$ for $i, j = 1, 2$ implies there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$, and (ii) is an *antilattice*, if only comparable elements of E have an infimum. It is clear that if E is a lattice, then it satisfies (RIP), and any linearly ordered poset is an antilattice.

We introduce five kinds of the compatibilities of elements of a pseudo-effect algebra. We say that two elements a and b of a pseudo-effect algebra E are (i) *compatible* (and we write $a \leftrightarrow b$) if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c = b_1 + a_1 + c \in E$; (ii) *strongly compatible* (and we write $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$) if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, $a_1 + b_1 + c = b_1 + a_1 + c \in E$, and $a_1 \wedge b_1 = 0$, (iii) *weakly compatible*, (and we write $a \stackrel{\mathrm{w}}{\longleftrightarrow} b$) if there exist three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c \in E$ and $b_1 + a_1 + c \in E$, (iv) *ultra weakly compatible* (and we write $a \stackrel{\mathrm{uw}}{\longleftrightarrow} b$) if there exist three elements $a_1, b_1, c \in E$ such that either $a = a_1 + b_1 + c \in E$, or $b_1 + a_1 + c \in E$, and (v) *ultra strongly compatible* (and we write $a \stackrel{\mathrm{us}}{\longleftrightarrow} b$) if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, $a_1 \wedge b_1 = 0$, $a_1 \,\mathbf{com}\, b_1$, and $a_1 + b_1 + c \in E$. It is evident that (v) implies (ii), (ii) implies (i), (i) implies (iii), and (iii) implies (iv). If E is an effect algebra, then (ii) and (v) are equivalent, and so are (i), (iii) and (iv). If E is a lattice effect algebra, then (i) and (ii) are equivalent [Rie].

We note that if E is an effect algebra with (RIP), then $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$ iff $a \leftrightarrow b$ and $a \wedge b \in E$, [Dvu 1]. Therefore it can happen for such effect algebras that $a \leftrightarrow b$ but $a \stackrel{\mathrm{c}}{\not\longleftrightarrow} b$. In addition, it can happen that $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$ but $a \stackrel{\mathrm{c}}{\not\longleftrightarrow} 1 - b$, or $a \stackrel{\mathrm{c}}{\not\longleftrightarrow} 1 - a$ as the following example shows.

EXAMPLE 3.1. Let G be the additive group $\mathbb{R}^2$ with the positive cone of all (x,y) such that either $x=y=0$ or $x>0$ and $y>0$. Then $u=(1,1)$ is a strong unit for G. The effect algebra $E=\Gamma(G,u)$ is an antilattice having (RIP), but E is not a lattice, and $a=(0.8;0.2)\leftrightarrow 1-a=(0.2;0.8)$ and $a \not\stackrel{\mathrm{c}}{\longleftrightarrow} 1-a$.

REMARK 3.2. (0) $a\leftrightarrow b$ if and only if $b\leftrightarrow a$; $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$ iff $b \stackrel{\mathrm{c}}{\longleftrightarrow} a$.

(i) strong compatibility implies compatibility.

(ii) $a \stackrel{\mathrm{c}}{\longleftrightarrow} a$, $(a=0+a, a=0+a)$.

(iii) If $a\le b$, $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$ $(a=0+a, b=(b\setminus a)+a)$.

(iv) $0 \stackrel{\mathrm{c}}{\longleftrightarrow} a \stackrel{\mathrm{c}}{\longleftrightarrow} 1$.

(v) If E is a pseudo MV-algebra, then any two elements are strongly compatible.

PROPOSITION 3.3. *Let E be a pseudo-effect algebra. Then $a\leftrightarrow b$ if and only if there exist three elements $a_1',b_1',c\in E$ such that $a=c+a_1'$, $b=c+b_1'$, and $c+a_1'+b_1'=c+b_1'+a_1'\in E$.*

In addition, $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$ if and only if there three elements $a_1',b_1',c\in E$ such that $a=c+a_1'$, $b=c+b_1'$, $c+a_1'+b_1'=c+b_1'+a_1'\in E$, and $a_1'\wedge b_1'=0$.

Proof. Let $a\leftrightarrow b$. Then $a=a_1+c, b=b_1+c$, where $a_1+b_1+c=b_1+a_1+c\in E$, so that $a=c+a_1', b=c+b_1'$. Therefore, $u:=a_1+b_1+c=a_1+c+b_1'=c+a_1'+b_1'\in E$ and $u=b_1+a_1+c=b_1+c+a_1'=c+b_1'+a_1$ which proves $a_1'+b_1'=b_1'+a_1'$.

If, in addition, $a \stackrel{\mathrm{c}}{\longleftrightarrow} b$ and $w\le a_1',b_1'$, then $a=c+w+w \mathbin{/} a_1'$, $b=c+w+w \mathbin{/} b_1'$, so that $a=w'+(w \mathbin{/} a_1')'+c$ and $b=w'+(w \mathbin{/} b_1')'+c$. Hence $w'\le a_1,b_1$ i.e., $w'=0$ and $w=0$. ∎

PROPOSITION 3.4. *Let a and b be two elements of a lattice pseudo-effect algebra E such there are three elements $a_1,b_1,c\in E$ such that $a=a_1+c$, $b=b_1+c$, $a_1+b_1+c\in E$, $a_1+b_1=b_1+a_1$, and $a_1\wedge b_1=0$. Then $a\vee b=a_1+b_1+c$, $a\wedge b=c$, $a_1=a\setminus(a\wedge b)$, and $b_1=b\setminus(a\wedge b)$.*

In addition, if $a=a_1+c$, $b=b_1+c$, $a_1\wedge b_1=0$, and $a_1+b_1+c\in E$ or $b_1+a_1+c\in E$, then $a\wedge b=c$.

Proof. We have $c\le a,b$. Let $d\le a,b$. Then $c\le c\vee d\le a,b$ implies $(c\vee d)\setminus c\le a\setminus c=a_1$, $(c\vee d)\setminus c\le b\setminus c=b_1$, which gives $(c\vee d)\setminus c\le a_1\wedge b_1=0$, i.e., $c=c\vee d$ and $d\le c$.

Similarly, if $u:=a_1+b_1+c$, then $a,b\le u$. Assume $a,b\le w$. Then $a,b\le w\wedge u\le u$, i.e., $u\setminus(w\wedge u)\le u\setminus a=b_1$, $u\setminus(w\wedge u)\le u\setminus b=a_1$ which gives $u\setminus(w\wedge u)\le a_1\wedge b_1=0$, i.e., $w\wedge u=u$ and $u\le w$.

The last assertion is now clear. ∎

It is worthy to recall that Proposition 3.4 is valid also for the case $a = c + a_1$, $b = c + b_1$, where $c + a_1 + b_1 = c + b_1 + a_1$, and $a_1 \wedge b_1 = 0$. Then $a \vee b = c + a_1 + b_1$, $a \wedge b = c$, $a_1 = (a \wedge b) / a$, and $b_1 = (a \wedge b) / b$.

On the other hand, it is possible to show that if $a \leftrightarrow b$, then a_1, b_1, c are not necessarily determined uniquely even in a commutative case of E.

PROPOSITION 3.5. *Let a and b be elements of a pseudo-effect algebra E.*

(i) *$a \leftrightarrow b$ if and only if there are two elements $u, v \in E$ such that $v \leq a, b \leq u$, $u \setminus a = b \setminus v$, $u \setminus b = a \setminus v$, and $(u \setminus a) + (u \setminus b) = (u \setminus b) + (u \setminus a)$.*

(ii) *$a \overset{c}{\longleftrightarrow} b$ if and only if there are two elements $u, v \in E$ such that $v \leq a, b \leq u$, $u \setminus a = b \setminus v$, $u \setminus b = a \setminus v$, $(u \setminus a) + (u \setminus b) = (u \setminus b) + (u \setminus a)$, $(u \setminus a) \wedge (u \setminus b) = 0$.*

Proof. (i) Let $a = a_1 + c$, $b = b_1 + c$, $a_1 + b_1 = b_1 + a_1$, $a_1 + b_1 + c \in E$. Put $v = c$ and $u = a_1 + b_1 + c$. Then $v \leq a, b \leq u$, $u \setminus a = (b_1 + a) \setminus a = b_1$, $b \setminus v = (b_1 + c) \setminus c = b_1$, $u \setminus b = (a_1 + b) \setminus b = a_1$, and $a \setminus v = (a_1 + c) \setminus c = a_1$.

Conversely, suppose the conditions are fulfilled. Define $c = v$, $b_1 = b \setminus v$, $a_1 = a \setminus v$. Then $a = (a \setminus v) + v = a_1 + c$; $b = (b \setminus v) + v = b_1 + c$, and $u = (u \setminus b) + b = (a \setminus v) + (b_1 + c) = a_1 + b_1 + c$.

On the other hand, $u = (u \setminus a) + a = (b \setminus v) + (a_1 + c) = b_1 + a_1 + c$. By cancellation, we have $a_1 + b_1 = b_1 + a_1$.

(ii) It is now easy. ∎

PROPOSITION 3.6. *Let E be a lattice pseudo-effect algebra and $a, b \in E$. The following statements are equivalent*

(i) $a \overset{c}{\longleftrightarrow} b$.

(ii) $(a \vee b) \setminus a = b \setminus (a \wedge b)$ *and* $(a \vee b) \setminus b = a \setminus (a \wedge b)$.

(iii) $(a \wedge b) / a = b / (a \vee b)$ *and* $(a \wedge b) / b = a / (a \vee b)$.

Proof. (i) $\Rightarrow$(ii). Let $a \leftrightarrow b$. Then by Proposition 3.7

$$a = a \setminus (a \wedge b) + (a \wedge b),$$
$$b = b \setminus (a \wedge b) + (a \wedge b),$$

and $a \vee b = a \setminus (a \wedge b) + b \setminus (a \wedge b) + (a \wedge b)$. On the other hand, $a \vee b = (a \vee b) \setminus b + b \setminus (a \wedge b) + (a \wedge b)$, which implies $(a \vee b) \setminus b = a \setminus (a \wedge b)$.

Similarly, $a \vee b = b \setminus (a \wedge b) + a \setminus (a \wedge b) + (a \wedge b)$ and $a \vee b = (a \vee b) \setminus a + a \setminus (a \wedge b) + (a \wedge b)$, i.e., $(a \vee b) \setminus a = b \setminus (a \wedge b)$.

(ii)$\Rightarrow$(i). We have $a = a \setminus (a \wedge b) + a \wedge b$ and $b = b \setminus (a \wedge b) + a \wedge b$. Put $a_1 = a \setminus (a \wedge b)$, $b_1 = b \setminus (a \wedge b)$, and $c = a \wedge b$. Then $a \vee b = (a \vee b) \setminus a + a = (a \vee b) \setminus a + a \setminus (a \wedge b) + a \wedge b = (a \vee b) \setminus b + b \setminus (a \wedge b) + a \wedge b$. Then $a_1 + b_1 + c = b_1 + a_1 + c \in E$, and by Proposition 2.8, $a_1 \wedge b_1 = 0$.

The equivalence (i) and (iii) follows from Proposition 3.3 and from similar reasoning as those in the equivalence (i) and (ii). ∎

PROPOSITION 3.7. *Let E be a lattice pseudo-effect algebra. If $a \stackrel{c}{\longleftrightarrow} b$, then $(1 \setminus a) \stackrel{c}{\longleftrightarrow} (1 \setminus b)$, and $(a / 1) \stackrel{c}{\longleftrightarrow} (b / 1)$.*

Proof. We have by Proposition 3.4 $a \vee b = a \setminus (a \wedge b) + b \setminus (a \wedge b) + a \wedge b$, so that, $1 = 1 \setminus (a \vee b) + a \setminus (a \wedge b) + b$. Hence

$$1 \setminus b = 1 \setminus (a \vee b) + a \setminus (a \wedge b).$$

Similarly, $1 = 1 \setminus (a \vee b) + b \setminus (a \wedge b) + a$, i.e.,

$$1 \setminus a = 1 \setminus (a \vee b) + b \setminus (a \wedge b).$$

Since $a \setminus (a \wedge b) + b \setminus (a \wedge b) = b \setminus (a \wedge b) + a \setminus (a \wedge b)$, by Proposition 3.3, we conclude that $(1 \setminus a) \stackrel{c}{\longleftrightarrow} (1 \setminus b)$.

If now we express $a \vee b = a \wedge b + (a \wedge b) / b + (a \wedge b) / a$ and $1 = a \wedge b + (a \wedge b) / b + (a \wedge b) / a + (a \vee b) / 1$, we have $b / 1 = (a \wedge b) / a + (a \vee b) / 1$ and $a / 1 = (a \wedge b) / b + (a \vee b) / 1$, i.e., $(a / 1) \stackrel{c}{\longleftrightarrow} (b / 1)$. ∎

THEOREM 3.8. *Let E be a lattice pseudo-effect algebra and $a, b \in E$. Then the following assertions are equivalent.*

(i) $a \leftrightarrow b$.
(ii) $a \stackrel{c}{\longleftrightarrow} b$.
(iii) $a \stackrel{w}{\longleftrightarrow} b$.

Proof. It is clear that (ii) implies (i), and (i) implies (iii).

(i)⇒(ii). Let $a \leftrightarrow b$, i.e., $a = a_1 + c$ and $b = b_1 + c$. Therefore $a = c + a_1'$ and $b = c + b_1'$ for some $a_1', b_1' \in E$. Then $u := a_1 + b_1 + c \in E$ and $a, b \leq a \vee b \leq u$. Hence $(a \vee b) \setminus a \leq u \setminus a = b_1 \leq b$ and $(a \vee b) \setminus b \leq u \setminus b = a_1 \leq a$. In a similar way we have $a / (a \vee b) \leq a / u = a / (a_1 + b_1 + c) = a / (a_1 + c + b_1') = a / (c + a_1' + b_1') = b_1' \leq b$, as well as $b / (a \vee b) \leq a_1' \leq a$.

Put $w_1 := ((a \vee b) \setminus b) / a = ((a \vee b) \setminus b) / ((a \vee b) \setminus (a / (a \vee b)) = b \setminus (a / (a \vee b))$, when we have used (2.2) and equations below (2.2).

In a similar way, $w_2 := ((a \vee b) \setminus a) / b = a \setminus (b / (a \vee b))$. Define $w = w_1 \vee w_2$. We assert $w = a \wedge b$. We have

$$a \setminus w \leq a \setminus (((a \vee b) \setminus b) / a) = (a \vee b) \setminus b,$$
$$b \setminus w \leq b \setminus (((a \vee b) \setminus a) / b) = (a \vee b) \setminus a.$$

Hence by Proposition 2.5,

$$(a \wedge b) \setminus w = (a \setminus w) \wedge (b \setminus w) \leq ((a \vee b) \setminus b) \wedge ((a \vee b) \setminus a) = 0,$$

i.e., $a \wedge b = w$.

Define $a' := a \setminus w \leq (a \vee b) \setminus b \leq a_1$ and $b' := b \setminus w \leq (a \vee b) \setminus a \leq b_1$. Then $u_1 := a' + b' + w \in E$ and $u_2 := b' + a' + w \in E$. We show that $u_1 \wedge u_2 = a \vee b$. It is clear that $u_1 \wedge u_2 \geq a \vee b$. Assume $d \geq a, b$. Then

$a, b \leq a \vee b \leq d \wedge u_1 \wedge u_2$. Hence

$$(u_1 \wedge u_2) \setminus (d \wedge u_1 \wedge u_2) \leq (u_1 \wedge u_2) \setminus a \leq u_2 \setminus a = b',$$
$$(u_1 \wedge u_2) \setminus (d \wedge u_1 \wedge u_2) \leq (u_1 \wedge u_2) \setminus b \leq u_1 \setminus a = a'.$$

Since $a' \wedge b' = 0$, we have $u_1 \wedge u_2 = d \wedge u_1 \wedge u_2$, i.e., $d \geq u_1 \wedge u_2$, which proves $a \vee b = u_1 \wedge u_2$.

Calculate

$$a \vee b = (a \vee b) \setminus a + a \setminus (a \wedge b) + a \wedge b \geq b' + a' + w \geq a \vee b,$$
$$a \vee b = (a \vee b) \setminus a + b \setminus (a \wedge b) + a \wedge b \geq a' + b' + w \geq a \vee b,$$

which proves $u_1 = u_2 = a \vee b$.

Since $a' \wedge b' = 0$, by Proposition 3.4, we have $a \vee b = a' + b' + w = b' + a' + w$, $a' = a \setminus (a \wedge b)$, $b' = b \setminus (a \wedge b)$, $w = a \wedge b$, i.e.,

$$(a \vee b) \setminus a = b' = b \setminus (a \wedge b),$$
$$(a \vee b) \setminus b = a' = a \setminus (a \wedge b),$$

and

$$w_1 = ((a \vee b) \setminus b) / a = (a \setminus (a \wedge b)) / a = a \wedge b,$$
$$w_2 = ((a \vee b) \setminus a) / b = (b \setminus (a \wedge a)) / b = a \wedge b,$$

i.e. $w_1 = w_2 = w = a \wedge b$.

(iii)$\Rightarrow$(i). It follows the main ideas of the proof of the the previous implication.

Thus, let $a = a_1 + c = c + a_1'$ and $b = b_1 + c = c + b_1'$ for some $a_1', b_1' \in E$, and let $u_1 := a_1 + b_1 + c \in E$ and $u_2 := b_1 + a_1 + c \in E$. Set $u = u_1 \wedge u_2$. Hence $(a \vee b) \setminus a \leq u \setminus a \leq u_2 \setminus a = b_1 \leq b$ and $(a \vee b) \setminus b \leq u \setminus b \leq u_1 \setminus b = a_1 \leq a$. In addition, $a / (a \vee b) \leq a / u_1 = a / (a_1 + b_1 + c) = a / (c + a_1' + b_1') = b_1' \leq b$ and $b / (a \vee b) \leq b / u_2 = b / (c + b_1' + a_1') = a_1' \leq a$.

Put $w_1 := ((a \vee b) \setminus b) / a = ((a \vee b) \setminus b) / ((a \vee b) \setminus (a / (a \vee b)) = b \setminus (a / (a \vee b))$. Similarly we have for $w_2 := ((a \vee b) \setminus a) / b = a \setminus (b / (a \vee b))$. Define $w = w_1 \vee w_2$. We have as above $w = a \wedge b$.

Define $a' := a \setminus w \leq (a \vee b) \setminus b \leq a_1$ and $b' := b \setminus w \leq (a \vee b) \setminus a \leq b_1$. Then $u_1' := a' + b' + w \in E$ and $u_2' := b' + a' + w \in E$. We show that $u_1' \wedge u_2' = a \vee b$. It is clear that $u_1' \wedge u_2' \geq a \vee b$. Assume $d \geq a, b$. Then $a, b \leq a \vee b \leq d \wedge u_1' \wedge u_2'$. Hence

$$(u_1' \wedge u_2') \setminus (d \wedge u_1' \wedge u_2') \leq (u_1' \wedge u_2') \setminus a \leq u_2' \setminus a = b',$$
$$(u_1' \wedge u_2') \setminus (d \wedge u_1' \wedge u_2') \leq (u_1' \wedge u_2') \setminus b \leq u_1' \setminus a = a'.$$

Since $a' \wedge b' = 0$, we have $u_1' \wedge u_2' = d \wedge u_1' \wedge u_2'$, i.e., $d \geq u_1' \wedge u_2'$, which proves $a \vee b = u_1' \wedge u_2'$.

Calculate

$$a \vee b = (a \vee b) \setminus a + a \setminus (a \wedge b) + a \wedge b \geq b' + a' + w \geq a \vee b,$$
$$a \vee b = (a \vee b) \setminus a + b \setminus (a \wedge b) + a \wedge b \geq a' + b' + w \geq a \vee b,$$

which proves $u_1' = u_2' = a \vee b$, that is $a \leftrightarrow b$. ■

PROPOSITION 3.9. *Let a and b be elements of a lattice pseudo-effect algebra E such that $a \wedge b = 0$.*

(1) *If $a \leftrightarrow b$, then $a + b, b + a \in E$ and $a + b = a \vee b = b + a$.*
(2) *If $a + b, b + a \in E$, then $a \leftrightarrow b$.*

Proof. (1) Since $a = a \setminus (a \wedge b) + a \wedge b$ and $b = b \setminus (a \wedge b) + a \wedge b$, so that

$$a \vee b = a \setminus (a \wedge b) + b \setminus (a \wedge b) + a \wedge b = a + b$$
$$= b \setminus (a \wedge b) + a \setminus (a \wedge b) + a \wedge b = b + a.$$

(2) Use Theorem 3.8. ■

PROPOSITION 3.10. *Let E be a lattice pseudo-effect algebra. Then $a \overset{\mathrm{w}}{\longleftrightarrow} b$ if and only if $a \overset{\mathrm{us}}{\longleftrightarrow} b$, and $a \overset{\mathrm{uw}}{\longleftrightarrow} b$ if and only if either $(a \vee b) \setminus b = a \setminus (a \wedge b)$ or $(a \vee b) \setminus a = b \setminus (a \wedge b)$, or equivalently, either $a \,/\, (a \vee b) = (a \wedge b) \,/\, b$ or $b \,/\, (a \vee b) = (a \wedge b) \,/\, a$, or equivalently either $a \setminus (a \wedge b) \leq b^-$ or $b \setminus (a \wedge b) \leq a^-$, or equivalently, either $(a \wedge b) \,/\, a \leq b^\sim$ or $(a \wedge b) \,/\, b \leq a^\sim$.*

In particular, if $a \wedge b = 0$ and $a + b \in E$, then $a \vee b = a + b$.

Proof. Assume $a \overset{\mathrm{w}}{\longleftrightarrow} b$. By Theorem 3.8, $a \overset{\mathrm{c}}{\longleftrightarrow} b$, that is $a = a_1 + c$, $b = b_1 + c$, $a_1 \wedge b_1 = 0$ and $a_1 + b_1 + c = b_1 + a_1 + c \in E$. Assume $x \leq a_1$ and $y \leq b_1$. Then $x \wedge y = 0$, and $x + y, y + x \in E$. By Proposition 3.9, $x + y = x \vee y = y + x$, which proves that $a \overset{\mathrm{us}}{\longleftrightarrow} b$.

Assume now $a \overset{\mathrm{uw}}{\longleftrightarrow} b$. First assume $a \wedge b = 0$ and $a + b \in E$. We assert $a \vee b = a + b$. Indeed, calculate $(a \vee b) \setminus b \leq (a + b) \setminus b = a$ and $a \,/\, (a \vee b) \leq a \,/\, (a + b) = b$. Then $w := ((a \vee b) \setminus b) \,/\, a = ((a \vee b) \setminus b) \,/\, ((a \vee b) \setminus (a \,/\, (a \vee b))) = b \setminus (a \,/\, (a \vee b)) \leq b$. Hence $w \leq a \wedge b = 0$, i.e., $(a \vee b) \setminus b = a$ and, finally, $a \vee b = a + b$.

Second, let $a = a_1 + c$, $b = b_1 + c$ and, for example, $u := a_1 + b_1 + c \in E$. Therefore for $a' := a \setminus (a \wedge b) \leq a_1$ and $b' := b \setminus (a \wedge b) \leq b_1$. Since $a' \wedge b' = 0$ and $a' + b' \in E$ while $a' + b' \leq a_1 + b_1 \in E$, by Proposition 2.8 and by the first part of the present proof, $a' \vee b' = a' + b'$. Using Proposition 2.9, $a \vee b = (a' \vee b') + a \wedge b = a' + b' + c' = a' + b$, where $c' = a \wedge b$. Therefore, $(a \vee b) \setminus b = a \setminus (a \wedge b)$.

In a similar way we proceed with the second possibility. ■

We recall that the last statement generalizes that from Proposition 2.10.

PROPOSITION 3.11. *Let E be a lattice pseudo-effect algebra. Then $a \leftrightarrow b$ if and only if $a \setminus (a \wedge b) \leftrightarrow b \setminus (a \wedge b)$. In addition, $a \overset{\mathrm{uw}}{\longleftrightarrow} b$ if and only if $a \setminus (a \wedge b) \overset{\mathrm{uw}}{\longleftrightarrow} b \setminus (a \wedge b)$.*

Proof. One direction is clear. Suppose now $a_1 := a \setminus (a \wedge b) \leftrightarrow b \setminus (a \wedge b) =: b_1$. By Proposition 3.9, $a_1 \vee a_2 = a_1 + a_2$ and from Proposition 2.9 we have $(a_1 \vee a_2) + c \in E$, where $c = a \wedge b$. Then $a \vee b = (a_1 \vee b_1) + c = a_1 + b_1 + c = b_1 + a_1 + c$, i.e., $a \leftrightarrow b$.

Assume now $a \setminus (a \wedge b) \overset{\text{uw}}{\longleftrightarrow} b \setminus (a \wedge b)$. By Proposition 3.10, e.g., $(a \vee b) \setminus b = a \setminus (a \wedge b)$. Hence, $a \vee b = b \setminus (a \wedge b) + b \geq b \setminus (a \wedge b) + b \setminus (a \wedge b) \in E$, that is, $a \setminus (a \wedge b) \overset{\text{uw}}{\longleftrightarrow} b \setminus (a \wedge b)$.

Conversely, if $a' := a \setminus (a \wedge b) \overset{\text{uw}}{\longleftrightarrow} b \setminus (a \wedge b) =: b'$, then $a' = a_1 + c$ and $b' = b_1 + c$, where e.g., $a_1 + b_1 + c \in E$. Since $c \leq a' \wedge b' = 0$, we have $a' + b' \in E$. Therefore, $a' \vee b' = a' + b'$ by Proposition 3.10, we have $a \vee b = (a' \vee b') + a \wedge b = a' + b' + c \in E$, and finally, $a \overset{\text{uw}}{\longleftrightarrow} b$. ∎

PROPOSITION 3.12. *Let E be a lattice pseudo-effect algebra, let $a_i \leftrightarrow b$ for any $i \in I$, and $a := \bigvee_{i \in I} a_i \in E$. Then $b \leftrightarrow a$ and*

$$\bigvee_i (a_i \wedge b) = \Big(\bigvee_i a_i\Big) \wedge b.$$

Proof. The proof will follow from the following Claims.

CLAIM 1. $a \leq (b \setminus (a \wedge b)) \,/\, 1$, $b \setminus (a \wedge b) + a \setminus (a \wedge b) + a \wedge b \in E$.
$a \vee b \leq b \setminus (a \wedge b) + a \setminus (a \wedge b) + a \wedge b$.
$(a \vee b) \setminus a \leq b \setminus (a \wedge b)$.

We have $a_i \leq (b \setminus (a_i \wedge b)) \,/\, 1 \leq (b \setminus (a \wedge b)) \,/\, 1$, so that $a \leq (b \setminus (a \wedge b)) \,/\, 1$, so that $b \setminus (a \wedge b) + a \in E$. Therefore, $a \vee b \leq b \setminus (a \wedge b) + a \setminus (a \wedge b) + a \wedge b$ and $(a \vee b) \setminus a \leq b \setminus (a \wedge b)$.

CLAIM 2. $\bigwedge_i (b \setminus (a_i \wedge b)) = b \setminus (a \wedge b)$.

$b \setminus (a \wedge b) \leq b \setminus (a_i \wedge b)$ for any a_i. Let $d \leq b \setminus (a_i \wedge b) = (a_i \vee b) \setminus a_i \leq (a \vee b) \setminus a_i$. Then $a_i = ((a \vee b) \setminus a_i) \,/\, (a \vee b) \leq d \,/\, (a \vee b)$ and $a \leq d \,/\, (a \vee b)$, so that $d = (a \vee b) \setminus (d \,/\, (a \vee b)) \leq (a \vee b) \setminus a \leq b \setminus (a \wedge b)$ using Claim 1. This implies Claim 2.

CLAIM 3. $(a \vee b) \setminus a = b \setminus (a \wedge b)$.

By Claim 2, $b \setminus (a \wedge b) = \bigwedge_i (b \setminus (a_i \wedge b))$. Hence, $\bigwedge_i (b \setminus (a_i \wedge b)) = \bigwedge_i ((a_i \vee b) \setminus a_i)$. Let now $x \leq (a_i \vee b) \setminus a_i$ for any $i \in I$. Then $x \leq (a \vee b) \setminus a_i$, so that $a_i = ((a \vee b) \setminus a_i) \,/\, (a \vee b) \leq x \,/\, (a \vee b)$. Consequently, $a \leq x \,/\, (a \vee b)$. Therefore, $x = (a \vee b) \setminus (x \,/\, (a \vee b)) \leq (a \vee b) \setminus a$, which proves Claim 3.

CLAIM 4. $a \wedge b = \bigvee_i (a_i \wedge b)$.

Let $a_i \wedge b \leq e$ for any $i \in I$. Then $a_i \wedge b \leq b \wedge e$ so that $b \setminus (b \wedge e) \leq b \setminus (a_i \wedge b)$ for any $i \in I$. By Claim 2, $b \setminus (b \wedge e) \leq \bigwedge_i (b \setminus (a_i \wedge b)) = b \setminus (a \wedge b)$. Hence $a \wedge b = (b \setminus (a \wedge b)) \,/\, b \leq (b \setminus (b \wedge e)) \,/\, b = b \wedge e \leq e$, so that $a \wedge b = \bigvee_i (a_i \wedge b)$.

CLAIM 5. $a \setminus (a \wedge b) = \bigwedge_i (a \setminus (a_i \wedge b))$.

We have $a \setminus (a \wedge b) \leq a \setminus (a_i \wedge b)$ for any $i \in I$. Assume $w \leq a \setminus (a_i \wedge b)$ for every i. Then $a_i \wedge b = (a \setminus (a_i \wedge b)) / a \leq w / a$, and by Claim 4, $a \wedge b \leq w / a$. Therefore, $w = a \setminus (w / a) \leq a \setminus (a \wedge b)$ which yields Claim 5.

CLAIM 6. $a \setminus (a \wedge b) + b \in E$.

We have $a_i \setminus (a_i \wedge b) \leq 1 \setminus b$ for any i. Then $a_i \leq 1 \setminus b + a_i \wedge b \leq 1 \setminus b + a \wedge b \in E$, so that $a \leq 1 \setminus b + a \wedge b$. Hence $a \setminus (a \wedge b) \leq 1 \setminus b$ and $a \setminus (a \wedge b) + b \leq 1$.

CLAIM 7. $a \setminus (a \wedge b) + b = a \vee b$.

It is clear hat $a \setminus (a \wedge b) + b \geq a, b$, so that $a \setminus (a \wedge b) + b \geq a \vee b$. Let $a \setminus (a \wedge b) + b \geq x \geq a, b$. Then $x \geq a_i, b$ for any i, so that $x \geq a_i \vee b = a_i \setminus (a_i \wedge b) + b$ and

$$\begin{aligned} a_i \setminus (a_i \wedge b) + b &\leq x, \\ a_i \setminus (a_i \wedge b) &\leq x \setminus b, \\ a_i &\leq x \setminus b + a_i \wedge b \in E, \\ a_i &\leq x \setminus b + a \wedge b \in E, \\ a &\leq x \setminus b + a \wedge b, \\ a \setminus (a \wedge b) &\leq x \setminus b, \\ a \setminus (a \wedge b) + b &\leq x, \\ a \setminus (a \wedge b) + b &= a \vee b. \end{aligned}$$

CLAIM 8. $a \leftrightarrow b$.

It follows from Claim 3 and Claim 7 and of (ii) of Proposition 3.6. ∎

COROLLARY 3.13. *Let E be a lattice pseudo-effect algebra. If $a_1 \leftrightarrow b$ and $a_2 \leftrightarrow b$, then $(a_1 \vee a_2) \wedge b = (a_1 \wedge b) \vee (a_2 \wedge b)$ and $(a_1 \wedge a_2) \vee b = (a_1 \vee b) \wedge (a_2 \vee b)$.*

Proof. The first equality follows from Claim 4 of the proof of Proposition 3.12.

For the second one. By Proposition 3.7 and Theorem 3.8, we have $1 \setminus a_1 \leftrightarrow 1 \setminus b \leftrightarrow 1 \setminus a_2$. Then $((1 \setminus a_1) \vee (1 \setminus a_2)) \wedge (1 \setminus b) = 1 \setminus ((a_1 \wedge a_2) \vee b)$. On the other hand, by the first part, it equals to $((1 \setminus a_1) \wedge (1 \setminus b)) \vee ((1 \setminus a_2) \wedge (1 \setminus b)) = (1 \setminus (a_1 \vee b)) \vee (1 \setminus (a_2 \vee b)) = 1 \setminus ((a_1 \vee b) \wedge (a_2 \vee b))$ which entails the desired result. ∎

PROPOSITION 3.14. *Let E be a lattice pseudo-effect algebra, let $a_i \leftrightarrow b$ for any $i \in I$ and let $a = \bigwedge_{i \in I} a_i \in E$. Then $b \leftrightarrow a$.*

Proof. By Proposition 3.7 $1 \setminus b \leftrightarrow 1 \setminus a_i$ for any i. Since $1 \setminus (\bigwedge_i a_i) = \bigvee_i (1 \setminus a_i) \in E$, by Proposition 3.12, $1 \setminus b \leftrightarrow \bigvee_i (1 \setminus a_i)$. Applying again Proposition 3.7, $b \leftrightarrow a$. ∎

We have the following two forms of the Riesz decomposition properties.

PROPOSITION 3.15. *Let a, b, c be elements of a lattice pseudo-algebra such that $a+b \in E$, $c \leq a+b$ and $c \leftrightarrow b$ or $c \leftrightarrow a$. Then there exist two elements $a_1, b_1 \in E$ such that $c = a_1 + b_1$, and $a_1 \leq a$ and $b_1 \leq b$.*

Proof. Let $c \leq a+b$ and define $v := c \setminus (b \wedge c)$, $a_1 := a \wedge v$. Then $a_1 \leq a$ and $a_1 \leq c$. Put $b_1 := a_1 / c$. Then $c = a_1 + b_1$.

To finish the proof, we have to show that $b_1 \leq b$.

$$\begin{aligned} a_1 + b_1 = c &\leq (a+b) \wedge (b \vee c) \\ &= (a+b) \wedge (c \setminus (b \wedge c) + b) \\ &= (a+b) \wedge (v+b) \\ &= (a \wedge v) + b = a_1 + b, \end{aligned}$$

when we have used Proposition 2.8. By the cancellation, $b_1 \leq b$.

In a similar way we proceed when $a \leftrightarrow c$. ■

PROPOSITION 3.16. *Let E be a lattice pseudo-effect algebra. Let $a_1 + a_2 = b_1 + b_2$, where $a_1 \leftrightarrow b_1$ and $a_2 \leftrightarrow b_2$. Then there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that*

$$\begin{aligned} a_1 &= c_{11} + c_{12}, & b_1 &= c_{11} + c_{21}, \\ a_2 &= c_{21} + c_{22}, & b_2 &= c_{12} + c_{22}. \end{aligned}$$

Moreover, we may assume that $c_{12} \wedge c_{21} = 0$, and under this condition the c_{ij}'s are determined uniquely, and $c_{12} + c_{21} = c_{12} \vee c_{21} = c_{21} + c_{12}$.

Proof. We define $c_{11} = a_1 \wedge b_1$, $c_{12} = (a_1 \wedge b_1) / a_1$, $c_{22} = a_2 \wedge b_2$ and $c_{21} = a_2 \setminus (a_2 \wedge b_2)$. Then $a_1 = c_{11} + c_{12}$ and $a_2 = c_{21} + c_{22}$. We show that $c_{21} = a_2 \setminus (a_2 \wedge b_2) = (a_1 \wedge b_1) / b_1$. Put $y = a_1 + a_2 = b_1 + b_2 = b_1 + b_1 / y = y \setminus b_2 + b_2 = a_1 + a_1 / y = y \setminus a_2 + a_2$. By the cancellation, we have $b_2 = b_1 / y$, $b_1 = y \setminus b_2$, $a_1 = y \setminus a_2$ and $a_2 = a_1 / y$.

Calculate and use Proposition 2.2: $c_{21} = a_2 \setminus (a_2 \wedge b_2) = (a_1 / y) \setminus ((a_1 / y) \wedge (b_1 / y)) = (a_1 / y) \setminus ((a_1 \vee b_1) / y) = a_1 / (a_1 \vee b_1) = (a_1 \wedge b_1) / b_1 = c_{11} / b_1$, when we have used Proposition 3.6.

By symmetry we have

$$b_2 \setminus (a_2 \wedge b_2) = (a_1 \wedge b_1) / a_1 = c_{12} = c_{11} / a_1.$$

Hence, by Proposition 2.8, $c_{12} \wedge c_{21} = ((a_1 \wedge a_1) / a_1) \wedge ((a_1 \wedge b_1) / b_1) = 0$. In addition, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

Since $a_1 + a_2 = c_{11} + c_{12} + c_{21} + c_{22} = c_{11} + c_{21} + c_{12} + c_{22} = b_1 + b_2$, we conclude $c_{12} + c_{21} = c_{21} + c_{12}$, so that by Proposition 3.10, $c_{12} \leftrightarrow c_{21}$ and $c_{12} + c_{21} = c_{12} \vee c_{21} = c_{21} + c_{12}$.

Uniqueness. Adding the elements c_{11} and c_{21}, respectively, to the equality $c_{12} \wedge c_{21} = 0$, we obtain by Proposition 2.9, $(c_{11} + c_{12}) \wedge (c_{11} + c_{21}) = c_{11}$, so that $c_{11} = a_1 \wedge b_1$, and similarly $c_{22} = a_2 \wedge b_2$. Using the cancellation property, we see that c_{12} and c_{21} are defined consequently in the same way as at the beginning of the present proof. ∎

4. Pseudo-effect algebras and blocks

In the present section, we introduce a block, which is roughly speaking a maximal set of "distributive" or, more precisely, of "Riesz decomposable" elements of a pseudo-effect algebra. We show that if a lattice pseudo-effect algebra E satisfies the difference compatibility property or, equivalently, the weak compatibility property (i.e., the ultra weak and weak compatibilities are equivalent), then every block is a pseudo-effect algebra which is a pseudo MV-algebra, and E can be covered by its blocks. If, in addition, such an algebra is σ-complete, then every block is a σ-complete MV-algebra, and E is a commutative effect algebra.

Let $\{E_t\}_{t\in T}$ be a system of pseudo-effect algebras such that $E_t \cap E_s = \{0,1\}$ for $t \neq s$. The set $E := \bigcup_{t\in T} E_t$ can be organized into a pseudo-effect algebra such that $x+y$ is defined in E iff $x, y \in E_t$ for some $t \in T$ and if $x+y$ is defined in E_t, and in such a case, $x+y$ takes the value from E_t. Then E is a pseudo-effect algebra which is said to be a *horizontal sum* of the system of pseudo-effect algebras $\{E_t\}_{t\in T}$.

A maximal set of mutually compatible elements of a pseudo-effect algebra E is said to be a *block*.

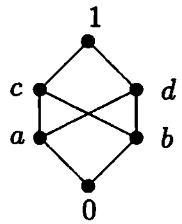
For example, if E is a pseudo MV-algebra, then E is a unique block in E. In addition, if E is a horizontal sum of a system of pseudo MV-algebras $\{E_t\}_{t\in T}$, then E is not necessarily a pseudo MV-algebra, and $\{E_t\}_{t\in T}$ is the system of all blocks in E.

The following example is from [Rie 1].

EXAMPLE 4.1. Let $E = \{0, a, b, c, d, 1\}$, where the addition $+$ is defined in the table.

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	1	×	×
b	b	c	d	×	1	×
c	c	1	×	×	×	×
d	d	×	1	×	×	×
1	1	×	×	×	×	×

$'$	0	a	b	c	d	1
	1	c	d	a	b	0



Then E is an effect algebra which is not a lattice, but all elements of E are strongly compatible and e.g. $c \overset{c}{\longleftrightarrow} d$ and $c \vee d \in E$ but $c \wedge d \notin E$, as well as $a \overset{c}{\longleftrightarrow} b$, $a \wedge b \in E$ but $a \vee b \notin E$. Moreover, E is a unique block, but it is not an MV-algebra.

PROPOSITION 4.2. *If E is a pseudo-effect algebra. If $a \overset{uw}{\longleftrightarrow} b$, then either $a \overset{uw}{\longleftrightarrow} 1 \setminus b$ or $b \overset{uw}{\longleftrightarrow} 1 \setminus a$. If $a \leftrightarrow b$, then $a \overset{uw}{\longleftrightarrow} 1 \setminus b$ and $b \overset{uw}{\longleftrightarrow} a / 1$.*

Proof. Assume that $a = a_1 + c = c' + a_1$, $b = b_1 + c$ and $u = a_1 + b_1 + c \in E$ for some $c' \in E$. Then $1 = 1 \setminus u + a_1 + b$ which gives $1 \setminus b = 1 \setminus u + a_1$. Hence $1 \setminus u + a_1 + c = 1 \setminus u + c' + a_1 \in E$ which proves $a \overset{uw}{\longleftrightarrow} 1 \setminus b$.

In a similar way we proceed with the second possibility. The case $a \leftrightarrow b$ is now evident. ∎

We can ask whether is the ultra weak compatibility equivalent with the weak compatibility in lattice pseudo-effect algebras ?

The partial answer gives the following notion.

We say that a pseudo-effect algebra E has the *weak compatibility property*, (WCP) for short, if, for $a, b \in E$, $a \overset{uw}{\longleftrightarrow} b$ implies $a \overset{w}{\longleftrightarrow} b$.

For example, (i) every pseudo MV-algebra, or (ii) every horizontal sum of pseudo MV-algebras, or (iii) every effect algebra, or (iv) every horizontal sum of the previous algebras has (WCP).

PROPOSITION 4.3. *Let E be a lattice pseudo-effect algebra such that $1 \setminus a = a / 1$ for every $a \in E$. Then E has* (WCP).

Proof. By Proposition 3.11, it is sufficient to verify that if $a \overset{uw}{\longleftrightarrow} b$ for $a, b \in E$ with $a \wedge b = 0$, we have $a \overset{w}{\longleftrightarrow} b$. Assume, e.g., $u = a + b \in E$. Then $1 = a + b + u / 1 = a + a / 1 = a + 1 \setminus a = 1 \setminus a + a$ and therefore, $b \leq 1 \setminus a$ and finally, $b + a \in E$ which proves $a \overset{w}{\longleftrightarrow} b$.

In a similar way we prove that if $b + a \in E$, then $a + b \in E$. ∎

REMARK 4.4. There exists a non-commutative lattice-ordered pseudo-effect algebra E such that $1 \setminus a = a / 1$ for every $a \in E$, [Rac 1]. Such algebras are sometimes connected with cyclically ordered (non-commutative) unital groups in the sense of Rieger [Rig], [Fuc].

PROPOSITION 4.5. *Let a lattice pseudo-effect algebra E satisfy* (WCP).
(i) *If $a \leftrightarrow b$, then $1 \setminus b \leftrightarrow a \leftrightarrow b / 1$.*
(ii) *If $a \leftrightarrow b$ and $c \geq a, b$, then $c \setminus b \leftrightarrow a \leftrightarrow b / c$.*

Proof. (i) By Proposition 4.2, if $a \leftrightarrow b$, then $a \overset{uw}{\longleftrightarrow} 1 \setminus b$ and $b \overset{uw}{\longleftrightarrow} 1 \setminus a$. (WCP) and Theorem 3.8 implies $a \leftrightarrow 1 \setminus b$ and $b \leftrightarrow 1 \setminus a$. Using Proposition 3.7, we have $a / 1 \leftrightarrow b$ and $1 / b \leftrightarrow a$.

(ii) It follows similar reasonings as those in (i). ∎

We say that a pseudo-effect algebra E satisfies the *difference compatibility property*, (DCP) for short, if $a \leftrightarrow b$, $a \leftrightarrow c$ and $b \le c$ imply $c \leftrightarrow c \setminus b$. Every pseudo MV-algebra, or every horizontal sum of pseudo MV-algebras, or every effect algebra, or any horizontal sum of the previous algebras has (DCP). On the other hand, Example 3.1 has (WCP), but not (DCP).

PROPOSITION 4.6. *Let E be a pseudo-effect algebra with* (DCP). (i) *If $a \leftrightarrow b$, then $1 \setminus b \leftrightarrow a \leftrightarrow b / 1$.*

(ii) *If $a \leftrightarrow b$, $a \leftrightarrow c$, and $b + c \in E$, then $a \leftrightarrow b + c$.*

Proof. (i) Since $a \leftrightarrow b$, $a \leftrightarrow 1$, $b \le 1$, we have $a \leftrightarrow 1 \setminus b$. By symmetry we have $1 \setminus a \leftrightarrow b$ and by Propositions 3.3 and 3.7, we have $a = (1 \setminus a) / 1 \leftrightarrow b / 1$.

(ii) Assume $a \leftrightarrow b, c$ and $b + c \in E$. Then $b \le 1 \setminus c$ and by (i) $a \leftrightarrow 1 \setminus c$. Therefore, $a \leftrightarrow (1 \setminus c) \setminus b = 1 \setminus (b + c)$, so that by (i), $a \leftrightarrow b + c$. ∎

In what follows, we prove that in lattice pseudo-effect algebras (WCP) implies (DCP).

PROPOSITION 4.7. *Let E be a lattice pseudo-effect algebra satisfying* (WCP). *If $a \leftrightarrow b$, $a \leftrightarrow c$ and $b \le c$, then $a \leftrightarrow c \setminus b$ and $a \leftrightarrow b / c$.*

Proof. By Proposition 3.10, from $a \leftrightarrow c$ we have $c \le a^- + a \wedge c$. On the other hand,

$$\begin{aligned} a \wedge (c \setminus b) + b &\ge (a \wedge b^- \wedge (c \setminus b)) + b \\ &= (a \wedge b^- + b) \wedge ((c \setminus b) + b) \quad \text{(Proposition 2.9)} \\ &= (a \wedge b^- + b) \wedge c \\ &\ge a \wedge c, \end{aligned}$$

while $a \leftrightarrow b^-$ implies $(a \wedge b^-) + (a \wedge b^-) / b^- + (a \wedge b^-) / a = b^- + (a \wedge b^-) / a \in E$, so that $(a \wedge b^-) / a \le b$ and $a \le a \wedge b^- + b$.

Therefore, $b \ge (a \wedge (c \setminus b)) / (a \wedge c)$. Calculate $c = c \setminus b + b \le a^- + a \wedge c = a^- + a \wedge (c \setminus b) + (a \wedge (c \setminus b)) / (a \wedge c)$, so that

$$\begin{aligned} c \setminus b + (a \wedge (c \setminus b)) / (a \wedge c) &\le c \setminus b + b = c \\ &\le a^- + a \wedge (c \setminus b) + (a \wedge (c \setminus b)) / (a \wedge c) \end{aligned}$$

which gives

$$c \setminus b \le a^- + a \wedge (c \setminus b).$$

By Proposition 3.10, this implies $c \setminus b \overset{\text{uw}}{\longleftrightarrow} a$ and by (WCP), $c \setminus b \leftrightarrow a$.

By duality we prove $a \leftrightarrow b / c$. ∎

Finally, we say that a pseudo-effect algebra E satisfies the *compatibility complement property*, (CCP) for short, if $a \leftrightarrow b$ implies $a \leftrightarrow 1 \setminus b$; then also $a \leftrightarrow b / 1$.

We prove that in lattice pseudo-effect algebras three properties (WCP), (DCP) and (CCP) are equivalent.

PROPOSITION 4.8. *Let E be a lattice pseudo-effect algebra. The following three properties are equivalent.*

(i) (WCP).
(ii) (DCP).
(iii) (CCP).

Proof. By Proposition 4.7, (WCP) implies (DCP), and by Proposition 4.6, (DCP) implies (CCP).

We claim (CCP) entails (WCP). Let $a \overset{\mathrm{uw}}{\longleftrightarrow} b$. By Proposition 3.11 it is sufficient to assume that $a \wedge b = 0$, and e.g. $a + b \in E$. Then $a \le b^- = 1 \setminus b$, so that $a \leftrightarrow b^-$. Therefore, $a \leftrightarrow b^- \,/\, 1 = b$. ∎

Now we present the main results of the paper.

THEOREM 4.9. *Let E be a lattice pseudo-effect algebra with* (DCP). *Then every block of E is a pseudo-effect subalgebra of E which is a pseudo MV-algebra. Moreover, any such pseudo-effect algebra E is a set-theoretical union of its blocks.*

Proof. Let M be a block of E. Therefore, $0, 1 \in M$. If $a \in M$, then by (DCP), $1 \setminus a, a \,/\, 1 \in M$, and if $b, c \in M$ and $b + c \in E$, then by Proposition 4.6, $b + c \in M$ which proves that M is a pseudo-effect subalgebra of E. By Proposition 3.12 and Proposition 3.14, M is a lattice in which by Proposition 3.6 $a \setminus (a \wedge b) = (a \vee b) \setminus b$ for all $a, b \in M$, which by [DvVe II, Prop. 8.8] is a necessary and sufficient condition for $(M; \oplus, ^-, ^\sim, 0, 1)$ to be a pseudo MV-algebra, where

$$a \oplus b := ((a^\sim \wedge b) \,/\, a^\sim)^-, \quad a, b \in M.$$

Let now A be any subset of mutually compatible elements of E. Due to Zorn's lemma, there exists a block of E containing A. Since any element of E belongs to some block of E, E can be covered by its blocks. ∎

As a corollary of Theorem 4.9 we have the following important result of Riečanová [Rie]:

COROLLARY 4.10. *Every lattice effect algebra E can be covered by blocks which are MV-algebras, and every block of E is an MV-algebra.*

Proof. Since every effect-algebra satisfies (WCP), the blocks of every lattice effect-algebra are by Theorem 4.9 MV-algebras. ∎

THEOREM 4.11. *Let a pseudo-effect algebra with* (DCP) *be a σ-lattice. Then every block of E is an MV-algebra, and E can be covered by commutative blocks, and in addition, E is a (commutative) effect-algebra.*

Proof. Let $\{a_n\}$ be a sequence of elements of a block M of E. By Proposition 3.12 and Proposition 3.14, $\bigvee_n a_n, \bigwedge_n a_n \in M$, which by Theorem 4.9 means M is a pseudo MV-algebra which is a σ-complete lattice. In view of [Dvu, Thm 4.2], every σ-complete MV-algebra is a (commutative) MV-algebra.

Assume now $a + b \in E$. Then $a \leq 1 \setminus b$ and $a \leftrightarrow 1 \setminus b$, consequently, $a \leftrightarrow b$ by Proposition 4.6. By Theorem 4.9 there exists a block M of E such that $a, b \in M$. Since by above, M is a (commutative) MV-algebra, we have $a + b = b + a$. ∎

The last theorem can be extended as follows. We say that a pseudo-effect algebra E is *Archimedean* if, for an element $a \in E$ such $na := a + \cdots + a \in E$ for any $n \geq 1$, we have $a = 0$.

THEOREM 4.12. *Let E be a lattice pseudo-effect algebra such that every block is a pseudo-effect subalgebra of E. If E is Archimedean, then E is a (commutative) effect algebra.*

Proof. Let M be a block of E. Since E is a lattice such that $a \setminus (a \wedge b) = (a \vee b) \setminus b$, for all $a, b \in M$, we have that M is a pseudo MV-algebra. Now if $a \in M$ and $na \in E$ for any integer $n \geq 1$, then $na \in M$ for any $n \geq 1$, which by the assumptions implies $a = 0$, i.e., M is an Archimedean pseudo MV-algebra. By [Dvu, Thm 4.2], this implies M is an MV-algebra. Hence, if $a + b \in E$, then $a \leq 1 \setminus b$ which means that a and b belong to the same block, therefore, $a + b = b + a$. ∎

THEOREM 4.13. *Every σ-complete effect algebra is Archimedean.*

Proof. It follows from Theorem 4.12 and Corollary 4.10, or it is possible to use directly the definition of the Archimedeanicity and Proposition 2.2 for the elements $a_n = na$. ∎

Finally, we show that properties (WCP), or equivalently (DCP) or (CCP) are necessary for the validity of Theorem 4.8.

PROPOSITION 4.14. *Let E be a lattice pseudo-effect algebra. Then every block of E is a pseudo-effect subalgebra of E if and only if E satisfies* (WCP), *or equivalently E satisfies* (DCP), *or equivalently E satisfies* (CCP).

Proof. If E satisfies e.g. (DCP), then by Theorem 4.9, every block of E is a pseudo-effect subalgebra of E. Conversely, let any block of E be a pseudo-effect subalgebra of E. Assume $a \leftrightarrow b$, then a and b belongs to the same block and hence $a \leftrightarrow 1 \setminus b$, so that E has (CCP). ∎

Finally, we show that the equivalent properties (WCP), (DCP) and (CCP) are not satisfied in every lattice pseudo-effect algebra. We recall that according to Proposition 3.10 and Proposition 4.8, the above properties are equivalent with the following condition: for any $a, b \in E$

(4.1) $$a \wedge b = 0, \quad a + b \quad \text{exists iff} \quad b + a \quad \text{exists.}$$

EXAMPLE 4.15. Let G be the additive free group generated by the two elements g, h; let $v : (G; +) \to (\mathbb{Z}; +)$, where $\mathbb{Z}$ is the additive group of the integers, be the homomorphism determined by the conditions $v(g) = v(h) = 1$; and define a partial order in G by setting $G^+ := \{x \in G : x = 0 \text{ or } v(x) > 0\}$. Then we have for $a, b \in G$

$$a \leq b \text{ iff } a = b \text{ or } v(a) < v(b).$$

Then G is a po-group, but G is not lattice-ordered; $g \vee h$ is not defined in G.

Consider now the interval pseudo-effect algebra $(\Gamma(G, g+h); +, 0, g+h)$. We have $E := \Gamma(G, g+h) = \{a \in G : a = 0 \text{ or } v(a) = 1 \text{ or } a = g+h\}$. It is lattice-ordered; for if $a, b \in E$, then either a and b are comparable, or else $v(a) = v(b) = 1$, in which latter case the only lower bound is 0 and the only upper bound is $g + h$.

E does not fulfil (4.1), since for instance $g \wedge h = 0$, $g + h$ is defined, but $h + g$ is not.

PROBLEM 1. Characterize pseudo-effect algebras which can be covered by pseudo MV-algebras.

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