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DISTRIBUTIVE ATOMIC EFFECT ALGEBRAS

Abstract. Motivated by the use of fuzzy or unsharp quantum logics as carriers of probability measures there have been recently introduced effect algebras (D-posets). We extend a result by Greechie, Foulis and Pulmannová of finite distributive effect algebras to all Archimedean atomic distributive effect algebras. We show that every such an effect algebra is join and meet dense in a complete effect algebra being a direct product of finite chains and distributive diamonds. This proves that every such effect algebra has a MacNeille completion being again a distributive effect algebra and both these effect algebras are continuous lattices. Moreover, we show that every faithful or (o)-continuous state (probability) on such an effect algebra is a valuation, hence a subadditive state. Its existence is also proved. Finally, we prove that every complete atomic distributive effect algebra E is a homomorphic image of a complete modular atomic ortholattice regarded as effect algebra and E is an MV-effect algebra (MV-algebra) if and only if it is a homomorphic image of a Boolean algebra regarded as effect algebra.

1. Introduction and basic definitions

Recently, effect algebras as carriers of probability measures in the "quantum probability theory" have been introduced (Foulis and Bennett [4]). In the fuzzy-probability setting an equivalent (in some sense) structure, D-poset was introduced (F. Kôpka [11]). Thus elements of these structures represent quantum events or fuzzy events which have yes-no character that may be unsharp or imprecise. Unfortunately, there are effect algebras (D-posets) admitting no states hence also no probabilities. Moreover, a state or probability ω on a lattice effect algebra E need not be subadditive. It was proved in Riečanová [20] that ω is subadditive iff it is a valuation.

The existence of (o)-continuous states on Archimedean modular atomic effect algebras was proved in Riečanová [16]. But not all of them are also

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subadditive. Further, it is well known that on MV-effect algebras, which are distributive lattices derived from MV-algebras, every state is subadditive (see [8] and [16]). On the other hand distributive effect algebras need not be MV-effect algebras. Nevertheless, as we are going to show, all faithful states (probabilities) on them are subadditive (σ-subadditive). Further, for Archimedean atomic distributive effect algebras we prove: (1) the existence of a MacNeille completion, (2) the existence of a valuation, (3) the existence of a modular ortholattice as a homomorphic pre-image.

A model for an effect algebra is the standard effect algebra of positive self-adjoint operators dominated by the identity on a Hilbert space. In general form, an effect algebra was introduced in [4].

DEFINITION 1.1. A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. Moreover, if we write $a \oplus b = c$ for $a, b, c \in E$, then we mean both that $a \oplus b$ is defined and $a \oplus b = c$. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*).

Recall that a set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E if:

(i) $1 \in Q$,
(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q then $a, b, c \in Q$.

Assume that $(E_1; \oplus_1, 0_1, 1_1)$ and $(E_2; \oplus_2, 0_2, 1_2)$ are effect algebras. An injection $\varphi : E_1 \to E_2$ is called an *embedding* iff $\varphi(1_1) = 1_2$ and for $a, b \in E_1$ we have $a \leq b'$ iff $\varphi(a) \leq (\varphi(b))'$ in which case $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$. We can easily see that then $\varphi(E_1)$ is a sub-effect algebra of E_2 and we say that E_1 and $\varphi(E_1)$ are *isomorphic*, or that E_1 *is up to isomorphism a sub-effect algebra of* E_2. We usually identify E_1 with $\varphi(E_1)$.

We say that a finite system $F = (a_k)_{k=1}^{n}$ of not necessarily different elements of an effect algebra $(E;\oplus,0,1)$ is $\oplus$-*orthogonal* if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus\limits_{k=1}^{n} a_k$ or $\bigoplus F$) exists in E. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus\limits_{k=1}^{n-1} a_k$ exists and $\bigoplus\limits_{k=1}^{n-1} a_k \le a'_n$. An arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily distinct elements of E is called $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_\kappa)_{\kappa\in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G \text{ finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G \text{ finite}\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_\kappa)_{\kappa\in H_1}$).

An effect algebra $(E;\oplus,0,1)$ is called *Archimedean* if for no nonzero element $e \in E$ the elements $ne = e \oplus \oplus e \cdots \oplus e$ (n times) exist for all $n \in N$. An Archimedean effect algebra is called *separable* if every $\oplus$-orthogonal system of elements of E is at most countable. We can show that *every complete effect algebra is Archimedean* [15].

For an element x of an effect algebra E we write $\operatorname{ord}(x) = \infty$ if nx exists for every $n \in N$. We write $\operatorname{ord}(x) = n_x \in N$ if n_x is the greatest positive integer such that $n_x x$ exists in E. Clearly, in an Archimedean effect algebra $n_x < \infty$ for every $x \in E$.

For more details we refer the reader to (Dvurečenskij and Pulmannová [3]) and the references given there. We review only a few properties without proof.

LEMMA 1.2. *Elements of an effect algebra $(E;\oplus,0,1)$ satisfy the properties:*

(i) *$a \oplus b$ is defined iff $a \le b'$,*
(ii) *$a \le a \oplus b$,*
(iii) *if $a \oplus b$ and $a \vee b$ exist then $a \wedge b$ exists and $a \oplus b = (a \wedge b) \oplus (a \vee b)$,*
(iv) *$a \oplus b \le a \oplus c$ iff $b \le c$ and $a \oplus c$ is defined,*
(v) *$a \ominus b = 0$ iff $a = b$,*
(vi) *if $u \le a$, $v \le b$ and $a \oplus b$ is defined then $u \oplus v$ is defined,*
(vii) *if E is a lattice and $a, b \le c'$ then $(a \vee b) \oplus c = (a \oplus c) \vee (b \oplus c)$.*

2. Algebraic properties of atomic distributive effect algebras

It is well known that lattice effect algebras are a common generalization of orthomodular lattices and MV-algebras (see [10], [1], [2], [7], [8]). As posets, MV-algebras are distributive lattices.

DEFINITION 2.1. A lattice effect algebra is called distributive iff, as a poset, it forms a distributive lattice.

Recall that elements x and y of a lattice effect algebra are called *compatible* (written $x \leftrightarrow y$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$. If every two elements of E are compatible then E is called an *MV-effect algebra*. Every *MV-effect algebra* M can be organized into an MV-algebra by extending partial operation $\oplus$ onto the total binary operation $\widehat{\oplus}$ by putting $x \widehat{\oplus} y = x \oplus (x' \wedge y)$ for all $x, y \in M$ (Kôpka and Chovanec [12]). In a lattice effect algebra E every maximal subset $M \subseteq E$ of mutually compatible elements is a sub-lattice and a sub-effect algebra of E. In fact M is an MV-effect algebra called block of E. Moreover, E is a union of its blocks (Riečanová [17]). Every MV-effect algebra (hence every block of a lattice effect algebra) is distributive. On the other hand *there are distributive effect algebras which are not MV-effect algebras*. The smallest one is a *distributive diamond* $E = \{0, a, b, 1\}$ in which $1 = 2a = 2b$. Evidently elements a, b are not compatible because $a \wedge b = 0$ and $a \oplus (b \ominus (a \wedge b))$ is not defined in E. Hence E is not an MV-effect algebra. The five-element effect algebra $E = \{0, a, b, c, 1\}$ in which $1 = a \oplus c = 2b$ is not distributive because $a \vee b = 2b = 1$ but $a \wedge c = b \wedge c = 0$ and hence $(a \vee b) \wedge c \neq (a \wedge c) \vee (b \wedge c)$. We will call it a *non-distributive diamond*.

Every *finite chain* $0 < a < 2a < \ldots < 1 = n_a a$ is a distributive effect algebra in which every pair of elements is compatible, hence it is an MV-effect algebra.

An element a of an effect algebra E is called an *atom* if $0 \leq b < a$ implies $b = 0$ and E is called *atomic* if for every $x \in E$, $x \neq 0$ there is an atom $a \in E$ with $a \leq x$. Clearly every finite effect algebra is atomic. Greechie, Foulis and Pulmannová [6] have proved that *every finite distributive effect algebra E is a cartesian product of finite chains and distributive diamonds*. Here we extend this result onto Archimedean atomic distributive effect algebras.

The notion of a *central element* of an effect algebra E has been introduced in [6]. In [13] it was proved that an element $z \in E$ is central iff for every $x \in E$ there exist elements $x \wedge z$ and $x \wedge z'$ for which $x = (x \wedge z) \vee (x \wedge z')$. It follows that $1 = z \vee z'$, which is equivalent with the condition $z \wedge z' = 0$. Thus for a distributive effect algebra E the set of all central elements of E (called a *center* of E) is $C(E) = \{z \in E \mid z \wedge z' = 0\}$. In every effect algebra E the center $C(E)$ is a Boolean algebra [6]. Moreover, $C(E)$ is a sub-lattice and a sub-effect algebra of E and for every $Q \subseteq C(E)$ such that $q = \bigvee Q$ exists in E we have $q \in C(E)$, [18].

THEOREM 2.2. *Let a be an atom of an Archimedean atomic distributive effect algebra E with* $\mathrm{ord}(a) = n_a$. *Then*

(i) *$n_a a \in C(E)$ and $ka \notin C(E)$ for all positive integers $k < n_a$.*

(ii) *For every $x \in E$ with $a \leq x \leq n_a a$ there is a positive integer k such that $x = ka$.*

(iii) $[0, n_a a] = \{0, a, \ldots, n_a a\}$ *iff there is no atom* $b \neq a$ *with* $b < n_a a$; *otherwise* $n_a = 2$ *and there is a unique atom* $b \neq a$ *with* $b < n_a a$ *in which case* $2b = 2a$, *hence* $[0, n_a a] = \{0, a, b, 2a = 2b\}$ *is a distributive diamond.*

(iv) $n_a a$ *is an atom of* $C(E)$.

(v) $C(E)$ *is atomic and for every atom* $p \in C(E)$ *there is an atom* $b \in E$ *with* $p = n_b b$.

Proof. (i) If $k < n_a$ then $a \oplus ka$ is defined which implies that $a \leq (ka) \wedge (ka)'$. Hence $ka \notin C(E)$. Further, $a \wedge (n_a a)' = 0$ as otherwise $a \wedge (n_a a)' = a$ which gives $a \oplus (n_a a)$ is defined, a contradiction. Thus $a \oplus (n_a a)' = a \vee (n_a a)'$. If $n_a \geq 2$ then $(2a) \oplus (n_a a)' = a \oplus (a \vee (n_a a)') = (a \oplus a) \vee (a \oplus (n_a a)') = (2a) \vee (n_a a)'$ which gives that $(2a) \wedge (n_a a)' = 0$. By induction $(n_a a) \wedge (n_a a)' = 0$ which gives $n_a a \in C(E)$.

(ii) Let $a \leq x \leq ka$. Then $\{a, 2a, \ldots, n_a a, x\}$ is a set of pairwise compatible elements and hence there is a block M of E such that $\{a, 2a, \ldots, n_a a, x\} \subseteq M$ (see [17]). Because $x \leq a \oplus a \oplus \ldots \oplus a$ (k-times), we conclude using the Riesz decomposition property for MV-algebras that $x = la$ for some positive integer l.

(iii) By (ii) $[0, n_a a] = \{0, a, \ldots, n_a a\}$ iff there is no atom $b \neq a$ with $b < n_a a$. Assume to the contrary that there is an atom $b \neq a$ with $b < n_a a$. Then $a < a \vee b \leq n_a a$ which by (ii) gives $a \vee b = la$ for some positive integer l. Since E is also a modular lattice, intervals $[a, a \vee b]$ and $[a \wedge b, b]$ are isomorphic [5, p. 212], which gives that $[a, la]$ and $[0, b]$ are isomorphic. It follows that $l = 2$, as b is an atom. Moreover, intervals $[b, a \vee b]$ and $[a \wedge b, a]$ are isomorphic, which gives that $[b, 2a]$ and $[0, a]$ are isomorphic. It follows that there is an atom c such that $b \oplus c = 2a$, as a is an atom. Evidently $c \neq a$ because $a \neq b$. If $c \neq b$ then $\{0, a, b, c, 2a\}$ is a non-distributive diamond, a contradiction. Hence $c = b$ and thus $2a = 2b$. Assume $n_a > 2$. Then $3a = (2b) \oplus a = b \oplus (b \oplus a) = b \oplus (b \vee a) = (b \oplus b) \vee (b \oplus a) = 2b \vee (b \vee a) = 2b \vee a$, which implies that $(2b) \wedge a = 0$, a contradiction. We conclude that $n_a = n_b = 2$ and $[0, n_a a] = \{0, a, b, 2a = 2b\}$ is a distributive diamond.

(iv) This is clear by (iii) and (i).

(v) Assume that $z \in C(E)$, $z \neq 0$. Then there exists an atom $b \in E$ with $b \leq z$. Moreover, $n_b b = ((n_b b) \wedge z) \vee ((n_b b) \wedge z')$. If $(n_b b) \wedge z' \neq 0$ then $(n_b b) \wedge z' = n_b b$ because $(n_b b) \wedge z' \in C(E)$ and $n_b b$ is an atom of $C(E)$ by (iv). But then $b \leq z \wedge z'$, a contradiction. Thus $n_b b = (n_b b) \wedge z$ which gives $n_b b \leq z$. Clearly, if z is an atom of $C(E)$ then $n_b b = z$.

3. MacNeille completions of distributive atomic effect algebras

It is well known that every poset $(P; \leq)$ has the *MacNeille completion* (completion by cuts). By J. Schmidt [22] the MacNeille completion of a poset P is any complete lattice $\widehat{P}$ into which the poset P can be supremum and infimum densely embedded, i.e., for each $x \in \widehat{P}$ there are $Q, M \subseteq P$ such that $x = \bigvee \varphi(M) = \bigwedge \varphi(Q)$, where $\varphi : P \to \widehat{P}$ is the embedding. We usually identify P with $\varphi(P)$.

A complete effect algebra $(\widehat{E}, \widehat{\oplus}, \widehat{0}, \widehat{1})$ is called a *MacNeille completion of an effect algebra* $(E; \oplus, 0, 1)$ if, up to isomorphism, E is a sub-effect algebra of $\widehat{E}$ and, as posets, $\widehat{E}$ is a MacNeille completion of E.

It is known that there are (even finite) effect algebras the MacNeille completion of which are not again effect algebras [15].

For an effect algebra E and $p \in C(E)$ the interval $[0, p]$ is an effect algebra with inherited $\oplus$-operation and the unit element p (see [6] and [13]). An effect algebra E is called *order continuous* ((o)-continuous for brevity) if for any net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E such that $x_{\alpha_1} \leq x_{\alpha_2}$ for all $\alpha_1 \leq \alpha_2$ and $\bigvee_{\alpha \in \mathcal{E}} x_\alpha = x$ (written $x_\alpha \uparrow x$) we have $\bigvee_{\alpha \in \mathcal{E}} (x_\alpha \wedge y) = x \wedge y$ for all $y \in E$ [16]. An (o)-continuous effect algebra E is a continuous lattice in order convergence.

Recall that a direct product $\prod_{\varkappa \in H} E_\varkappa$ of effect algebras $E_\varkappa$, we mean a Cartesian product with "coordinatewise" defined $\oplus$, 0 and 1 (see [6] and [16]).

THEOREM 3.1. *For an atomic Archimedean distributive effect algebra E, let $A_{C(E)} = \{p_\varkappa \mid \varkappa \in H\}$ be the set of all atoms of the center $C(E)$ of E. Let $\widehat{E} = \prod_{\varkappa \in H} [0, p_\varkappa]$ be the direct product of effect algebras $[0, p_\varkappa]$, $\varkappa \in H$. Then*

(i) *$\widehat{E}$ is a complete atomic distributive effect algebra which is a MacNeille completion of the effect algebra E.*

(ii) *If E is complete then E is isomorphic with $\prod_{\varkappa \in H} [0, p_\varkappa]$.*

(iii) *E and $\widehat{E}$ are (o)-continuous lattices.*

Proof. Because $C(E)$ is a Boolean algebra, we have $\bigvee_{\varkappa \in H} p_\varkappa = 1$. As $C(E) = B(E) \cap S(E)$, where $B(E) = \{y \in E \mid y \leftrightarrow x \text{ for all } x \in E\}$ and $S(E) = \{x \in E \mid x \wedge x' = 0\}$, we have $p_\varkappa \leftrightarrow x$ for all $\varkappa \in H$ and $x \in E$. By [9] we obtain that $x = x \wedge \left(\bigvee_{\varkappa \in H} p_\varkappa\right) = \bigvee_{\varkappa \in H} (x \wedge p_\varkappa) = \bigoplus_{\varkappa \in H} (x \wedge p_\varkappa)$, because $\bigoplus_{\varkappa \in K} (x \wedge p_\varkappa) = \bigvee_{\varkappa \in H} (x \wedge p_\varkappa)$ for every finite $K \subseteq H$. Conversely, if $\widehat{x} = (x_\varkappa)_{\varkappa \in H} \in \widehat{E}$ then we have for $\varkappa_1 \neq \varkappa_2$, $x_{\varkappa_1} \leq p_{\varkappa_1} \leq p'_{\varkappa_2} \leq x'_{\varkappa_2}$ and hence $x_{\varkappa_1} \oplus x_{\varkappa_2} = x_{\varkappa_1} \vee x_{\varkappa_2}$. By induction $\bigvee_{\varkappa \in K} x_\varkappa = \bigoplus_{\varkappa \in K} x_\varkappa$ for every finite $K \subseteq H$. By definition of $\bigoplus$ we have $\bigvee_{\varkappa \in H} x_\varkappa = \bigoplus_{\varkappa \in H} x_\varkappa$

Assume that $x, y \in E$ with $x \leq y'$. Then $x \wedge p_\varkappa \leq y' \wedge p_\varkappa \leq y' \vee p'_\varkappa = (y \wedge p_\varkappa)'$. As $x = (x \wedge p_\varkappa) \oplus (x \wedge p'_\varkappa)$ and $y = (y \wedge p_\varkappa) \oplus (y \wedge p'_\varkappa)$ we have $x \oplus y = \big((x \wedge p_\varkappa) \oplus (y \wedge p_\varkappa)\big) \vee \big((x \wedge p'_\varkappa) \oplus (y \wedge p'_\varkappa)\big)$, because $p_\varkappa \wedge p'_\varkappa = 0$

and $(x \wedge p'_\varkappa) \oplus (y \wedge p'_\varkappa) \leq p'_\varkappa$. Thus $(x \oplus y) \wedge p_\varkappa = (x \wedge p_\varkappa) \oplus (y \wedge p_\varkappa)$. We conclude that the map $\varphi : E \to \widehat{E}$, defined by $\varphi(x) = (x \wedge p_\varkappa)_{\varkappa \in H}$ for every $x \in E$, is an embedding. We identify E with $\varphi(E)$. Then evidently E is supremum dense (hence also infimum dense) in $\widehat{E}$. This proves that $\widehat{E}$ is the MacNeille completion of E. Further, for every $\varkappa_0 \in H$ and every atom $a_{\varkappa_0} \leq p_{\varkappa_0}$ the element $(x_\varkappa)_{\varkappa \in H} \in \hat{E}$ such that $x_{\varkappa_0} = a_0$ and $x_\varkappa = 0$ for all $\varkappa \neq \varkappa_0$ is an atom of $\widehat{E}$. It follows that $\widehat{E}$ is atomic.

(ii) This follows by part (i)

(iii) Because $[0, p_\varkappa]_{\varkappa \in H}$ are finite lattices, hence (o)-continuous lattices. As $\widehat{E}$ inherits all infima and suprema existing in E, we conclude that E is (o)-continuous.

COROLLARY 3.2. (i) *Every atomic Archimedean distributive effect algebra is sub-directly decomposed into finite chains and distributive diamonds.*

(ii) *An atomic distributive effect algebra has a MacNeille completion that is again a distributive effect algebra iff it is Archimedean.*

Proof. (i) follows from Theorems 2.2 and 3.1. Assertion (ii) is a consequence of Theorem 3.1 and the fact that every complete effect algebra is Archimedean [15].

4. States, probabilities and valuations

Recall that a map $\omega : E \to [0,1]$ is called a (finitely additive) state on an effect algebra E if $\omega(1) = 1$ and $x \leq y' \Longrightarrow \omega(x \oplus y) = \omega(x) + \omega(y)$; ω is called *(o)-continuous* if $x_\alpha \xrightarrow{(o)} x \Longrightarrow \omega(x_\alpha) \to \omega(x)$. Here for a net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E we write $x_\alpha \xrightarrow{(o)} x$ if there exist nets $(u_\alpha)_{\alpha \in \mathcal{E}}$, $(v_\alpha)_{\alpha \in \mathcal{E}}$ such that $u_\alpha \leq x_\alpha \leq v_\alpha$ for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow x$ and $v_\alpha \downarrow x$. A state ω is called *σ-additive* (or a *probability*) if $\omega\left(\bigoplus_{n=1}^\infty x_n\right) = \sum_{n=1}^\infty \omega(x_n)$ for every $\oplus$-orthogonal sequence $(x_n)_{n=1}^\infty$ for which $\bigoplus_{n=1}^\infty x_n$ exists in E. A state ω on E is called faithful if $\omega(x) = 0 \Longrightarrow x = 0$. A state ω on a lattice effect algebra E is called a *valuation* if $x \wedge y = 0 \Longrightarrow \omega(x \vee y) = 0$.

LEMMA 4.1. (i) *A state ω on a lattice effect algebra E is a valuation iff $\omega(a \vee b) + \omega(a \wedge b) = \omega(a) + \omega(b)$ for all $a, b \in E$.*

(ii) *If there exists a faithful state on an effect algebra E then E is separable.*

The proof can be found in [20].

We say that a state on a lattice effect algebra E is *subadditive* if $\omega(x \vee y) \leq \omega(x) + \omega(y)$ for all $x, y \in E$. If $\omega\left(\bigvee_{n=1}^\infty x_n\right) \leq \sum_{n=1}^\infty \omega(x_n)$ for all $x_n \in E$ with $\bigvee_{n=1}^\infty x_n \in E$ then ω is called *σ-subadditive*.

Note that every valuation on a lattice effect algebra E is subadditive. On the other hand, a state on a lattice effect algebra E need not be subadditive. In Riečanová [20] it has been proved:

LEMMA 4.2. *A state on a lattice effect algebra is subadditive iff it is a valuation.*

The proof of the following Lemma is a routine verification.

LEMMA 4.3. *Let E be a lattice effect algebra and ω be a state on E. If ω is faithful (or E is separable) then the following conditions are equivalent:*

(i) *ω is σ-additive,*
(ii) $x_n \downarrow 0 \Longrightarrow \omega(x_n) \downarrow 0$,
(iii) $x_n \uparrow x \Longrightarrow \omega(x_n) \uparrow \omega(x)$,
(iv) *ω is (o)-continuous,*
(v) $\omega(\bigoplus G) = \bigvee\{\sum_{x \in F} \omega(x) \mid F \subseteq G$ *is finite*$\}$ *for every $\oplus$-orthogonal system G for which $\bigoplus G$ exists in E.*

In this section we show that on every Archimedean atomic distributive effect algebra E there exists a valuation and that every faithful or (o)-continuous state on E is a valuation.

THEOREM 4.4. *If ω is a faithful probability on an Archimedean atomic distributive effect algebra E then*

(i) *ω is a valuation,*
(ii) *ω is σ-subadditive.*

Proof. (i) As ω is faithful, the effect algebra E is separable (see [20]), hence the set $A_{C(E)}$ of all atoms of the center $C(E)$ of E is at most countable. Set $A_{C(E)} = \{p_n \mid n = 1, 2, \dots\}$ By Theorem 2.2, every interval $[0, p_n]$ is either a finite chain or the distributive diamond, hence every state on $[0, p_n]$ is a valuation. It follows that for every restriction $\omega|_{[0,p_n]}$ its multiple $\frac{1}{\omega(p_n)}\omega|_{[0,p_n]}$ is a valuation.

Assume that $x, y \in E$ with $x \wedge y = 0$. Then

$$x \vee y = \bigoplus_{n=1}^{\infty}(x \vee y) \wedge p_n = \bigoplus_{n=1}^{\infty}\big((x \wedge p_n) \vee (y \wedge p_n)\big)$$

(see the proof of Theorem 3.1), under which $p_{n_1} \wedge p_{n_2} = 0$ for all $n_1 \neq n_2$. By σ-additivity of ω we obtain

$$\omega(x \vee y) = \sum_{n=1}^{\infty} \omega\big((x \vee y) \wedge p_n\big) = \sum_{n=1}^{\infty} \omega(x \wedge p_n) + \sum_{n=1}^{\infty}(y \wedge p_n) = \omega(x) + \omega(y),$$

which proves that ω is a valuation.

(ii) By Lemma 4.2, ω is subadditive, which gives

$$\omega\Big(\bigvee_{n=1}^{\infty} x_n\Big) = \omega\Big(\bigvee_{n=1}^{\infty}\Big(\bigvee_{k=1}^{n} x_k\Big)\Big) \leq \lim_{n\to\infty}\sum_{k=1}^{n}\omega(x_k) = \sum_{n=1}^{\infty}\omega(x_k),$$

for all $x_n \in E$, $n = 1, 2, \ldots$, as $\bigvee_{k=1}^{n} x_k \uparrow \bigvee_{n=1}^{\infty} x_n$.

THEOREM 4.5. *Every (o)-continuous state ω on a complete atomic distributive effect algebra E is a valuation. Moreover, E is isomorphic with a direct product $[0, a_0] \times [0, a_0']$ where $a_0 \in C(E)$ is such that $\omega(a_0) = 0$ and the restriction $\omega|_{[0,a_0']}$ is a faithful σ-additive valuation on the effect algebra $[0, a_0']$.*

Proof. Set $A_0 = \{x \in E \mid \omega(x) = 0\}$, $a_0 = \bigvee A_0$, $\mathcal{K} = \{K \subseteq A_0 \mid K$ is finite$\}$ and $x_K = \bigvee K$ for all $K \in \mathcal{K}$. Then $\mathcal{K}$ is directed by set inclusion and for the net $(x_K)_{K\in\mathcal{K}}$ we have $x_K \uparrow a_0$.

Assume that $p \in E$ is an atom of E with $p \leq a_0$. By Theorem 3.1, E is (o)-continuous, which gives $x_K \wedge p \uparrow a_0 \wedge p = p$ and hence there exists $K_0 \in \mathcal{K}$ such that $p = x_{K_0} \wedge p$. Let $K_0 = \{x_1, \ldots, x_n\}$. Then $p = p \wedge \bigvee_{i=1}^{n} x_i = \bigvee_{i=1}^{n} p \wedge x_i$ and hence there exists $i_0 \in \{1, \ldots, n\}$ such that $p \leq x_{i_0}$. It follows that $\omega(p) \leq \omega(x_{i_0}) = 0$. Let $n_p = \mathrm{ord}(p)$. As E is complete, we have $n_p < \infty$ and $\omega(n_p p) = n_p\omega(p) = 0$. This implies that $n_p p \leq a_0$. By [21] there is a set $\{a_\alpha \mid \alpha \in \mathcal{E}\}$ of distinct atoms and positive integers k_α such that $a_0 = \bigoplus_{\alpha\in\mathcal{E}} k_\alpha a_\alpha = \bigvee_{\alpha\in\mathcal{E}} k_\alpha a_\alpha \leq \bigvee_{\alpha\in\mathcal{E}} n_{a_\alpha} a_\alpha \leq a_0$, where $n_{a_\alpha} = \mathrm{ord}(a_\alpha)$. As $n_{a_\alpha} a_\alpha \in C(E)$ for every $\alpha \in \mathcal{E}$, we conclude that $a_0 = \bigvee_{\alpha\in\mathcal{E}} n_{a_\alpha} a_\alpha \in C(E)$. Further, by Lemma 4.3 the (o)-continuity of ω implies $\omega(a_0) = \omega\left(\bigoplus_{\alpha\in\mathcal{E}} k_\alpha a_\alpha\right) = \omega\big(\bigvee\{\bigoplus_{\alpha\in K} k_\alpha a_\alpha \mid K \subseteq \mathcal{E}$ is finite$\}\big) = \bigvee\{\sum_{\alpha\in\mathcal{E}} \omega(k_\alpha a_\alpha) \mid K \subseteq \mathcal{E}$ is finite $\} = 0$, because $\omega(a_\alpha) = 0$ for every $\alpha \in \mathcal{E}$. Thus $\omega(a_0') = 1$.

Let $x \in E$, $x \neq 0$. Then $x = (x\wedge a_0)\vee(x\wedge a_0') = (x\wedge a_0)\oplus(x\wedge a_0')$ because $a_0 \wedge a_0' = 0$. This implies that $\omega(x) = \omega(x \wedge a_0) + \omega(x \wedge a_0') = \omega(x \wedge a_0')$. By assumptions, ω is (o)-continuous on E and hence also the restriction $\omega|_{[0,a_0']}$ is an (o)-continuous map. Further, because $a_0' \in C(E)$ for every $x, y \in E$ with $x \leq y'$ we have $(x \oplus y) \wedge a_0' = (x \wedge a_0') \oplus (y \wedge a_0')$ which gives $\omega\big((x \wedge a_0') \oplus (y \wedge a_0')\big) = \omega\big((x \oplus y) \wedge a_0'\big) = \omega(x \oplus y) = \omega(x) + \omega(y) = \omega(x \wedge a_0') + \omega(y \wedge a_0')$ and hence $\omega|_{[0,a_0']}$ is an (o)-continuous faithful state on $[0, a_0']$. By Lemma 4.3 and Theorem 4.4 we conclude that $\omega|_{[0,a_0']}$ is a σ-additive valuation on $[0, a_0']$. This implies that for $x, y \in E$ with $x\wedge y = 0$ we have $\omega(x\vee y) = \omega\big((x\vee y)\wedge a_0'\big) = \omega\big((x\wedge a_0')\vee(y\wedge a_0')\big) = \omega(x\wedge a_0')+\omega(y\wedge a_0') = \omega(x) + \omega(y)$ which proves that ω is a valuation on E.

Recall the following result from Riečanová [21].

THE SMEARING THEOREM. *For every complete* (o)*-continuous atomic effect algebra* $(E;\oplus,0,1)$ *the following conditions are equivalent:*

(1) *There is a state on the orthomodular lattice*

$$S(E)=\{x\in E\mid x\wedge x'=0\}.$$

(2) *There is a state on* E.

(3) *There is an* (o)*-continuous state on* E.

Note that a proof that $S(E)$ is an orthomodular lattice can be found in [9].

Combining the preceeding theorems about atomic distributive effect algebras and the Smearing Theorem we obtain the following statement:

THEOREM 4.6. *On every Archimedean atomic distributive effect algebra* E *there exists an* (o)*-continuous valuation.*

Proof. Denote by $\widehat{E}$ a MacNeille completion of E and identify E with $\varphi(E)$, where $\varphi: E\to\widehat{E}$ is the embedding. By Theorem 3.1, $\widehat{E}$ is a complete atomic and (o)-continuous effect algebra in which $S(\widehat{E})=C(\widehat{E})$, because $\widehat{E}$ is distributive. As $C(\widehat{E}$ is a complete atomic Boolean algebra, by the Smearing Theorem there exists an (o)-continuous state $\widehat{\omega}$ on $\widehat{E}$. By Theorem 4.5, $\widehat{\omega}$ is an (o)-continuous valuation on $\widehat{E}$. Thus also the restriction $\widehat{\omega}|_E$ has these properties, because $\widehat{E}$ inherits all suprema and infima existing in E.

COROLLARY 4.7. *On every Archimedean atomic MV-effect algebra (MV-algebra) there exists an* (o)*-continuous valuation.*

5. Homomorphisms of effect algebras

The following definitions of a homomorphism and a homomorphic image for effect algebras are particular cases of corresponding definitions for partial algebras introduced in [5].

DEFINITION 5.1. A mapping ω from an effect algebra $(E;\oplus_E,0_E,1_E)$ into an effect algebra $(F;\oplus_F,0_F,1_F)$ is called a *homomorphism* (more precisely effect algebra-homomorphism) if $\omega(x\oplus_E y)=\omega(x)\oplus_F\omega(y)$ for all $x,y\in E$ with $x\le y'$; and $\omega(1_E)=1_F$. A homomorphism ω is called *full* if $\omega(x)\oplus_F\omega(y)=\omega(z)$, $x,y,z\in E$ implies that there exist $a,b,c\in E$ with $\omega(a)=\omega(x)$, $\omega(b)=\omega(y)$, $\omega(c)=\omega(z)$ and $a\oplus_E b=c$. F is called a *homomorphic image* of E if there exists a full homomorphism of E onto F.

Evidently, a homomorphism ω from an effect algebra E into the unit interval $[0,1]$ of reals (in which we define $a\oplus b=a+b$ iff $a+b\le 1$) is a state on E, and conversely. Moreover, a *σ-homomorphism* (i.e., $x_n\uparrow x$ in E implies $\omega(x_n)\uparrow\omega(x)$ in F, $n\in N$) in the case $F=[0,1]$ is a probability on E. If E is a Boolean σ-algebra of subsets of a nonempty set Ω, the σ-homomorphism $\omega : E \to F$ is called an *observable on* F. Recall here that every orthomodular lattice L (including Boolean algebras) becomes an effect algebra if we put $a \oplus b = a \vee b$ iff $a \leq b'$.

The aim of this section is to show that every complete atomic distributive effect algebra E is a homomorphic image of a complete atomic modular ortholattice regarded as effect algebra.

LEMMA 5.2. (i) *Every finite chain is a homomorphic image of a finite Boolean algebra regarded as effect algebra.*

(ii) *The distributive diamond is a homomorphic image of a finite modular ortholattice* MO_2 *(Chinese Lantern) regarded as effect algebra.*

Proof. (i) Assume that the effect algebra E is a finite chain. Then $E = \{0, a, 2a, \ldots, n_a a\}$ where $n_a a = 1$. Assume that B is a Boolean algebra with n_a atoms, $A = \{a_1, a_2, \ldots, a_{n_a}\}$. Then B is isomorphic with the family $P(n_a)$ of all subsets of the set A. We define a mapping $\omega : B \to E$ by the formula $\omega(F) = |F|a$, for every $F \subseteq P(n_a)$ with cardinality $|F|$. Clearly ω is a full homomorphism of B onto E.

(ii) Assume that E is the distributive diamond $\{0, a, b, 1\}$ where $1 = 2a = 2b$. Let $L = \{0, x, x', y, y', 1\}$ be a Chinese Lantern [10]. Then L is a modular ortholattice. Define $\omega : L \to E$ putting $\omega(x) = \omega(x') = a$, $\omega(y) = \omega(y') = b$, $\omega(0) = 0$ and $\omega(1) = 1$. Then, evidently, ω is a full homomorphism of L onto E.

THEOREM 5.3. *Every complete atomic distributive effect algebra E is a homomorphic image of a complete atomic modular ortholattice L regarded as effect algebra.*

Proof. Let $A_{C(E)} = \{p_\varkappa \mid \varkappa \in H\}$ be the set of all atoms of the center $C(E)$ of E. By Theorems 2.2 and 3.1, E is isomorphic with the direct product $\prod_{\varkappa \in H}[0, p_\varkappa]$ under which,for every $\varkappa \in H$, $[0, p_\varkappa]$ is either a finite chain or the distributive diamond. Let for every $\varkappa \in H$, $D_\varkappa$ be the finite Boolean algebra or the Chinese Lantern and $\omega_\varkappa : D_\varkappa \to [0, p_\varkappa]$ be the homomorphism of $D_\varkappa$ onto $[0, p_\varkappa]$ defined in the proof of Lemma 5.2. Clearly $L = \prod_{\varkappa \in H} D_\varkappa$ is a complete atomic effect algebra, which is a modular ortholattice. Define a mapping $\omega : \prod_{\varkappa \in H} D_\varkappa \to \prod_{\varkappa \in H}[0, p_\varkappa]$ by the formula $\omega\left((x_\varkappa)_{\varkappa \in H}\right) = \left(\omega_\varkappa(x_\varkappa)\right)_{\varkappa \in H}$ for every $(x_\varkappa)_{\varkappa \in H} \in \prod_{\varkappa \in H} D_\varkappa$. Evidently, for $x, y \in \prod_{\varkappa \in H} D_\varkappa$ with $x \leq y'$ we have $x = (x_\varkappa)_{\varkappa \in H}$, $y = (y_\varkappa)_{\varkappa \in H}$ where $x_\varkappa, y_\varkappa \in D_\varkappa$ with $x_\varkappa \leq y'_\varkappa$ for all $\varkappa \in H$. Hence $\omega_\varkappa(x_\varkappa \oplus_\varkappa y_\varkappa) = \omega_\varkappa(x_\varkappa) \oplus \omega_\varkappa(y_\varkappa) \leq p_\varkappa$ because $p_\varkappa \in C(E)$. It follows that $\omega(x \oplus_L y) = \omega\left((x_\varkappa \oplus_\varkappa y_\varkappa)_{\varkappa \in H}\right) = \left(\omega_\varkappa(x_\varkappa \oplus_\varkappa y_\varkappa)_{\varkappa \in H}\right) = \left(\omega_\varkappa(x_\varkappa) \oplus \omega_\varkappa(y_\varkappa)\right)_{\varkappa \in H} = \omega(x) \oplus \omega(y)$, which proves that ω is a homomorphism. Further, if $v \in$

$\prod_{\varkappa \in H}[0, p_\varkappa]$ then $v = (v_\varkappa)_{\varkappa \in H}$ where $v_\varkappa \in [0, p_\varkappa]$ for all $\varkappa \in H$ and hence there is $x_\varkappa \in D_\varkappa$ with $\omega(x_\varkappa) = v_\varkappa$. This implies that $v = \omega((x_\varkappa)_{\varkappa \in H})$ which proves that ω is a surjection. Since $\omega_\varkappa$ is a full homomorphism for every $\varkappa \in H$, we conclude that ω is full.

Finally note that the distributive diamond cannot be a homomorphic image of a Boolean algebra, because every homomorphism $\omega : E \to F$ of effect algebras E and F maps a compatible pair of elements onto a compatible pair. This is because $x, y \in E$ are compatible iff there are $u, v, z \in E$ such that $x = u \oplus_E z$, $y = v \oplus_E z$ and $u \oplus_E v \oplus_E z$ is defined, which gives $\omega(x) = \omega(u) \oplus_F \omega(z)$, $\omega(y) = \omega(v) \oplus_F \omega(z)$ and $\omega(u \oplus_E v \oplus_E z) = \omega(u) \oplus_F \omega(v) \oplus_F \omega(z)$.

COROLLARY 5.4. *A complete atomic distributive effect algebra E is an MV-effect algebra iff E is a homomorphic image of a Boolean algebra regarded as effect algebra.*

References

[1] C. C. Chang, *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc. 88 (1958), 467–490.

[2] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.

[3] A. Dvurečenskij, S. Pulmannová, *New Trends in Quantum Structures*, Kluwer Academic Publishers, Dordrecht, Boston, London and Ister Science, Bratislava, 2000.

[4] D. Foulis, M. K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys. 24 (1994), 1331–1352.

[5] G. Grätzer, *Universal Algebra*, second edition, Springer Verlag, New York, Heidelberg, Berlin, 1968, 1979.

[6] R. J. Greechie, D. Foulis, S. Pulmannová, *The center of an effect algebra* Order 12 (1995), 91–106.

[7] P. Hájek, *Mathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[8] U. Höhle and E. P. Klement (eds.), *Non-classical Logics and their Applications to Fuzzy Subsets*, Kluwer Academic Publishers, 1995.

[9] G. Jenča, Z. Riečanová, *On sharp elements in lattice ordered effect algebras*, BUSEFAL 80 (1999), 24–29.

[10] G. Kalmbach *Orthomodular lattices*, Academic Press, London, 1983.

[11] F. Kôpka, *D-posets of fuzzy sets*, Tatra Mt. Math. Publ. 1 (1992), 83–87.

[12] F. Kôpka, F. Chovanec, *Boolean D-posets*, Internat. J. Theor. Phys. 34 (1995), 1297–1302.

[13] Z. Riečanová, *Subalgebras, intervals and central elements of generalized effect algebras* Internat. J. Theor. Phys. 38 (1999), 3209–3220.

[14] Z. Riečanová, *Compatibility and central elements in effect algebras*, Tatra Mt. Math. Publ. 16 (1999), 151–158.

[15] Z. Riečanová, *MacNeille completions of D-posets and effect algebras*, Internat. J. Theor. Phys. 39 (2000), 859–869.

[16] Z. Riečanová, *Continuous lattice effect algebras admitting order-continuous states*, Fuzzy Sets and Systems, to appear, preprint, www.elf.stuba.sk/~jenca/preprint/index.htm.

[17] Z. Riečanová, *Generalization of blocks for D-lattices and lattice ordered effect algebras*, Internat. J. Theor. Phys. 39 (2000), 231–237.

[18] Z. Riečanová, *Sharp elements in effect algebras*, Internat. J. Theor. Phys. 40 (2001), 913–920.

[19] Z. Riečanová, *Proper effect algebras admitting no states*, Internat. J. Theor. Phys. 40 (2001), 1683–1691.

[20] Z. Riečanová, *Lattice effect algebras with (o)-continuous faithful valuations*, Fuzzy Sets and Systems 124 (2001), 321–3271.

[21] Z. Riečanová, *Smearings of states defined on sharp elements onto effect algebras*, Internat. J. Theor. Phys. 41 (2002), 1511–1524..

[22] J. Schmidt, *Zur Kennzeichnung der Dedekind-Mac Neilleschen Hulle einer Geordneten Menge*, Archiv d. Math. 7 (1956), 241–249.

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