DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

Piotr Pawlas, Tomasz Rychlik, Dominik Szynal

ON A CHARACTERIZATION OF DISTRIBUTIONS BY EXPECTATIONS OF TRANSFORMED GENERALIZED ORDER STATISTICS

Abstract. We generalize results of Swanepoel [10] on the characterization of distributions by expectations of transformed order statistics using lower generalized order statistics. In particular we characterize continuous distributions via expectations of kth lower record values.

1. Introduction

Kamps [6], [7] presented the concept of generalized order statistics. In particular, ordinary order statistics, record values, kth record values are contained in this model.

We introduce the model of lower generalized order statistics presented in [9]. Let $\{X_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ and pdf $f(x)$, and $X_{1:n}, \dots, X_{n:n}$ denote the order statistics from the sample $X_1, \dots, X_n$.

DEFINITION 1.1. *Let*

$$n \in \mathbb{N},\ k \geq 1,\ \widetilde{m} = (m_1, \dots, m_{n-1}) \in \mathbb{R}^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j,\ 1 \leq r \leq n-1,$$

be parameters such that

$$\gamma_r = k + (n-r) + M_r \geq 1 \text{ for } r \in \{1, \dots, n-1\}.$$

If random variables $Z(1,n,\widetilde{m},k), \dots, Z(n,n,\widetilde{m},k)$ have the joint density function of the form

$$f^{Z(1,n,\widetilde{m},k),\dots,Z(n,n,\widetilde{m},k)}(x_1, \dots, x_n)$$
$$= k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} (F(x_i))^{m_i} f(x_i)\Big)(F(x_n))^{k-1} f(x_n),$$

Key words and phrases: characterization, kth lower record, lower generalized order statistics, complete sequence of functions.

for $F^{-1}(1) > x_1 \geq \ldots \geq x_n > F^{-1}(0)$, *then they are called lower generalized order statistics based on* F.

For $m = 0$, $k = 1$, the lower generalized order statistics $Z(r, n, \tilde{m}, k)$ reduce to the order statistics $X_{n-r+1:n}$ from the sample $X_1, \ldots, X_n$, while for $m = -1$, $k \in N$, we obtain kth lower record values (cf.[8]).

From the above definition we obtain

COROLLARY 1.1. *The marginal density function of the r-th lower generalized order statistics for* $m_1 = \ldots = m_{n-1} = m$ *based on an absolutely continuous distribution function* F *and density function* f *is given by*

$$f^{Z(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!} (F(x))^{\gamma_r - 1} (g_m(F(x)))^{r-1} f(x), \tag{1}$$

with

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \ r = 1, 2, \ldots, n,$$

$$g_m(x) = \begin{cases} \frac{1}{m+1}\left(1 - x^{m+1}\right), \textit{for } m \neq -1, \\ -\log x, \qquad \textit{for } m = -1. \end{cases}$$

By (1), the expectation of a measurable function G of a single lower generalized order statistics $Z(r, n, m, k)$ based on distribution function F can be written as

$$E_F G(Z(r,n,m,k)) = \int_0^1 G\left((F^{-1}(t)\right) \phi_{r,n}(t) dt, \tag{2}$$

under the assumption that the integral of the right hand side exists, where

$$\phi_{r,n}(t) = \frac{c_{r-1}}{(r-1)!} t^{\gamma_r - 1} g_m^{r-1}(t), \quad t \in [0, 1).$$

We use the following notion.

DEFINITION 1.2. *Let* $L(a, b)$ *denote the space of integrable functions on* (a, b). *A sequence* $(f_n) \subset L(a, b)$, $n \geq 1$, *is called complete on* $L(a, b)$, *if for all functions* $g \in L(a, b)$ *the conditions*

$$\int_a^b g(x) f_n(x) dx = 0, \ n \in \mathbb{N},$$

imply $g(x) = 0$ *a.e in* (a, b).

The following lemmas present some complete sequences of functions.

LEMMA 1.1. *Let*

$I_1 = \{(r_n, n)$: *for fixed* $\mu \in \mathbb{N}$, $n \geq \mu$, $1 \leq r_\mu \leq \mu$, *and* $r_\mu \leq r_n \leq r_\mu + n - \mu\}$,

$I_2 = \{(r_n, n) : n \geq 2,\ 1 \leq r_n \leq n\}$,

$I_3 = \{(r_{ij}, n_j) : n_j \to \infty$ *as* $j \to \infty$, *and* $r_{ij} = i,\ i = 1, \ldots, n_j\}$,

$I_4 = \{(r_{ij}, n_j) : \mu_j, n_j \in \mathbb{N},\ j = 1, 2, \ldots,\ \mu_{j+1} > n_j \geq \mu_j > 1$,
$j = 1, 2, \ldots, \sum_{j=1}^{\infty} \sum_{\nu=\mu_j}^{n_j} \frac{1}{\nu - 1} = \infty$, *and* $r_{ij} = i,\ i = \mu_j, \ldots n_j\}$.

For any set I_j, $j = 1, 2, 3, 4$, *of pairs of indices* $(r, n), 1 \leq r \leq n$, *the sequence of polynomials*

$$\{x^{r-1} (1-x)^{n-r}\}_{(r,n) \in I_j}$$

is complete on $L(0, 1)$.

The statements of Lemma 1.1 for $j = 1, 2, 3, 4$, were proven in [1], [2], [4], [5], respectively.

LEMMA 1.2 (cf.[3]). *Let* n_0 *be any fixed non-negative integer and a nonnegative absolutely continuous function* $h(x)$ *have a nonzero derivative a.e. on* (a, b). *Then the sequence of functions*

$$\{(h(x))^n e^{-h(x)} : n \geq n_0\}$$

is complete in $L(a, b)$ *iff* $h(x)$ *is strictly monotone on* (a, b).

2. A characterization

From now on we assume that F is an absolutely continuous cdf and F_0 is a continuous function. Theorem 2.1 was proven in [10].

THEOREM 2.1. *We have* $F = F_0$ *if and only if for all* $n \geq 1$ *and* $1 \leq r \leq n$,

$$E_F\left(F_0(X_{r:n})\right) = \frac{r}{n+1},$$

where $E_F\left(X\right)) = \int_{-\infty}^{+\infty} x dF_X(x)$.

The following result extends Theorem 2.1 to the lower generalized order statistics with parameter $m > -1$.

THEOREM 2.2. *Let* $Z(r, n, m, k)$ *denote the generalized order statistic based on* F *with* $m > -1$. *Then* $F = F_0$ *if and only if for some* $j = 1, 2, 3, 4$, *and all* $(r, n) \in I_j$, *we have*

$$E_F\left(F_0(Z(r, n, m, k))\right) = \frac{c_{r-1}}{(r-1)!} \left(\frac{1}{m+1}\right)^r B\left(\frac{\gamma_r + 1}{m+1}, r\right), \tag{3}$$

where

$$B\left(\alpha, \beta\right) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx,$$

and I_j, $j = 1, 2, 3, 4$, *are defined in Lemma* 1.1.

Proof. Suppose that $F = F_0$. By (1), we have

$$E_F F_0 (Z(r,n,m,k)) = E_F F (Z(r,n,m,k))$$
$$= \frac{c_{r-1}}{(r-1)!}(\frac{1}{m+1})^{r-1} \int_{-\infty}^{\infty} (F(x))^{\gamma_r} \left(1-(F(x))^{m+1}\right)^{r-1} f(x)dx.$$

By (2),

$$E_F F (Z(r,n,m,k)) = \frac{c_{r-1}}{(r-1)!}(\frac{1}{m+1})^{r-1} \int_0^1 t^{\gamma_r}(1-t^{m+1})^{r-1}dt.$$

Since

$$\lambda \int_0^1 x^{\mu-1}(1-x^{\lambda})^{\nu-1}dx = B\left(\frac{\mu}{\lambda},\nu\right),$$

we get

$$E_F F_0 (Z(r,n,m,k)) = E_F F (Z(r,n,m,k))$$
$$= \frac{c_{r-1}}{(r-1)!}\left(\frac{1}{m+1}\right)^{r} B\left(\frac{\gamma_r+1}{m+1},r\right),$$

where $\gamma_r = k + (n-r)(m+1)$.

Suppose now that condition (3) holds for all $(r,n) \in I_j$ and some $j = 1,2,3,4$. Then

$$\int_{-\infty}^{\infty} [F_0(x) - F(x)] (F(x))^{\gamma_r - 1} \left(1-(F(x))^{m+1}\right)^{r-1} f(x)dx = 0.$$

Setting $F(x) = t$, we obtain

$$\int_0^1 \left[F_0(F^{-1}(t)) - t\right] t^{\gamma_r-1}(1-t^{m+1})^{r-1}dt = 0,$$

and, consequently,

$$\int_0^1 \left[F_0(F^{-1}(u^{\frac{1}{m+1}})) - u^{\frac{1}{m+1}}\right] u^{\alpha} u^{n-r}(1-u)^{r-1}du = 0,$$

where $\alpha = u^{\frac{k-1-m}{m+1}}$. Lemma 1.1 implies that the sequence $\{u^{n-r}(1-u)^{r-1}\}$, $(r,n) \in I_j$, is a complete sequence of functions in $L(0,1)$. Hence, we have $F_0(F^{-1}(u^{\frac{1}{m+1}})) - u^{\frac{1}{m+1}} = 0$ a.e. for $u \in (0,1)$. Since $\frac{1}{m+1} > 0$, $f(u) = u^{\frac{1}{m+1}}$ is continuous and strictly increasing on $[0,1]$, and $f(0) = 0$, $f(1) = 1$, we conclude that $F^{-1}(t) = F_0^{-1}(t)$ for $t \in (0,1)$, and $F(x) = F_0(x)$ a.e. for $x \in \mathbb{R}$.

From the above theorem we obtain the characterizations of distributions by moments of functions of order statistics, Stigler statistics and some sequential order statistics, respectively.

COROLLARY 2.1. *Relation $F = F_0$ holds if and only if for some $j = 1, 2, 3, 4$, and all $(r, n) \in I_j$,*

$$E_F\left(F_0(X_{n-r+1:n})\right) = \frac{n-r+1}{n+1}.$$

COROLLARY 2.2. *We have $F = F_0$ if and only if for some $j = 1, 2, 3, 4$, and all $(r, n) \in I_j$,*

$$E_F\left(F_0(Z(r, n, 0, \alpha - n + 1))\right) = \frac{\alpha - r + 1}{\alpha + 1},$$

where $n - 1 < \alpha \in R$.

COROLLARY 2.3. *The equality $F = F_0$ holds if and only if for some $j = 1, 2, 3, 4$, and all $(r, n) \in I_j$,*

$$E_F\left(F_0(Z(r, n, m, k))\right) = r\binom{n}{r}\Big(\prod_{j=1}^{r} \alpha_j\Big)\left(\frac{1}{m+1}\right)^r B\Big(\frac{\alpha_r(n-r+1)+1}{m+1}, r\Big),$$

where $\alpha_n = k$ and $\alpha_i = [k + (n-i)(m+1)]/(n+1-i)$, $1 \le i \le n-1$.

The results presented above are a stronger than Theorem 2.1, since $I_j \subset \{(r, n) : 1 \le r \le n,\ n \in \mathbb{N}\}$, $j = 1, 2, 3, 4$.

For the upper generalized order statistics we conclude the following result.

THEOREM 2.2′. *Under the assumptions of Theorem 2.2, $F = F_0$ if and only if for some $j = 1, 2, 3, 4$, and all $(r, n) \in I_j$,*

$$E_F\left(F_0(X(r, n, m, k))\right) = 1 - \frac{c_{r-1}}{(r-1)!}\left(\frac{1}{m+1}\right)^r B\Big(\frac{\gamma_r + 1}{m+1}, r\Big).$$

CORROLARY 2.1′. *We have $F = F_0$ if and only if for some $j = 1, 2, 3, 4$, and all $(r, n) \in I_j$,*

$$E_F\left(F_0(X_{r:n})\right) = \frac{r}{n+1}.$$

The following theorem gives a characterization of distributions by the expected values of transformed of kth lower record values.

THEOREM 2.3. *We have $F = F_0$ if and only if for fixed positive integers k and n_0, and all $n \ge n_0$,*

$$E_F(F_0(Z_n^{(k)})) = \left(\frac{k}{k+1}\right)^n.$$

Proof. Suppose that $F = F_0$. From (1) with $m = -1$, we easily obtain $E_F(F(Z_n^{(k)})) = \left(\frac{k}{k+1}\right)^n$.

Now we prove the sufficiency of the moment conditions. We have

$$\int_{-\infty}^{\infty} [F_0(x) - F(x)] \, (-\ln F(x))^{n-1} \, (F(x))^{k-1} f(x) dx = 0.$$

Setting $F(x) = t$, we get

$$\int_0^1 \left[F_0(F^{-1}(t) - t\right] (-\ln t)^{n-1} t^{k-1} dt = 0. \tag{4}$$

If $k = 1$, then $F_0(F^{-1}(t)) = t$ and $F_0(t) = F(t)$, by completeness of the sequence $\{t^n, \, n = 0, 1, \ldots\}$ in $L(0,1)$.

For $k \geq 2$, set $h(t) = -(k-1)\ln t > 0$ with $h'(t) = -\frac{k-1}{t} < 0$ for $0 < t < 1$. Then (4) is equivalent to

$$\int_0^1 \left[F_0(F^{-1}(t) - t\right] \left[\frac{h(t)}{k-1}\right]^{n-1} e^{-h(t)} dt = 0,$$

which, by Lemma 2.1, implies that

$$\left[F_0(F^{-1}(t)) - t\right] / h(t) = 0 \text{ a.e. in } (0,1).$$

This enables us to conclude that $F_0(t) = F(t)$.

References

[1] J. S. Huang, *Characterization of distributions by the expected values of the order statistics*, Ann. Inst. Statist. Math. 27 (1975), 87–93.

[2] J. S. Huang, J. S. Hwang, (1975), *L_1-completeness of a class of beta densities*, Statistical Distributions in Scientific Work 3, 137–141, Reidel, Dordrecht.

[3] G. D. Lin, *On a moment problem*, Tôhoku Math. J. 38 (1986), 595–598.

[4] J. S. Hwang, *A note on Bernstein and Müntz-Szasz theorems with applications to the order statistics*, Ann. Inst. Statist. Math. 30 (1978), 167–176.

[5] J. S. Hwang, G. D. Lin, *On a generalized moment problem II*, Proc. Amer. Soc. 91 (1984), 577–580.

[6] U. Kamps, (1995), *A Concept of Generalized Order Statistics*, B. G. Teubner, Stuttgart.

[7] U. Kamps, *A concept of generalized order statistics*, J. Statist. Plann. Inference 48 (1995), 1–23.

[8] P. Pawlas and D. Szynal, *Relations for single and product moments of kth record values from exponential and Gumbel distributions*, J. Appl. Statist. Science 7 (1999), 53–62.

[9] P. Pawlas and D. Szynal, *Recurrence relations for single and product moments of lower generalized order statistics from inverse Weibull distribution*, Demonstratio Math. 24 (2001), 353–358.

[10] J. W. H. Swanepoel, (2001), *A note on the characterization of distributions by expectations of transformed order statistics*, submitted.

Piotr Pawlas
MATHEMATICAL INSTITUTE
THE POLISH ACADEMY OF SCIENCES
ul. Śniadeckich 8
00-950 WARSZAWA, POLAND

Tomasz Rychlik
MATHEMATICAL INSTITUTE
THE POLISH ACADEMY OF SCIENCES
ul. Chopina 12
87-100 TORUŃ, POLAND

Dominik Szynal
MATHEMATICAL INSTITUTE
MARIE CURIE-SKŁODOWSKA UNIVERSITY
20-031 LUBLIN, POLAND

Received April 11st., 2002.