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CONVOLUTIONS OF ERLANG AND OF PASCAL DISTRIBUTIONS WITH APPLICATIONS TO RELIABILITY

Abstract. The main aim of the paper is to give a generalization of the results given by Sen and Balakrishnan in [7]. The results obtained are used to calculate the reliability of some class of systems.

1. Introduction

The distribution of the sum of n independent exponentially distributed random variables with different parameters μ_i $(i = 1, 2, \ldots, n)$ is given in [2], [3], [4] and [7]. In this paper, we give the distribution of this sum without the assumption that all the parameters μ_i are different. We use the fact that the sum of k independent identically distributed exponential random variables with parameter μ has an Erlang distribution with k degrees of freedom and parameter μ, i.e. $Erl\,(k, \mu)$. Thus, grouping the components of the sum which have the same parameter μ, the problem reduces to one of finding the distribution of the sum of independent random variables having Erlang distributions. This problem will be considered in Section 2.

All of the above problems are the special cases of the general case of a sum of independent random variables with gamma distributions. A formula for such a sum was provided by Mathai [5] in 1982. This formula is significantly complicated even in the case when random variables are exponentially distributed. Sen and Balakrishnan [7] describe it as "substantially messy" then derive it again in the particular case, when all random variables are exponentially distributed and intensities are all distinct from each other. For completeness, the formulae provided by Mathai will be presented in Section 3.

A similar problem for a sum of random variables with geometric distribution, where all the parameteres are different is considered in [7]. In this paper, we give the distribution of the sum of n independent random variables

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having geometric distributions with not necessarily different parameters q_i $(i = 1, 2, \ldots, n)$. It is well known that the sum of k independent random variables from the same geometric distribution with parameter q has a Pascal distribution with parameters k and q i.e. $Pasc\,(k, q)$ (see [1] for example). Hence a suitable grouping of the components of the sum which have the same parameter q_i reduces this problem to one of finding the distribution of the sum of independent random variables having Pascal distributions. This problem will be considered in Section 4.

In Section 5, using the results obtained in the previous sections, we give some examples of applications to reliability.

In the next sections we use the formula for the nth derivative of a product of m functions. This is the well known Leibniz formula:

$$(1)\qquad (v_1 v_2 \ldots v_m)^{(n)} = \sum_{\substack{k_1+\ldots+k_m=n \\ k_1,\ldots,k_m \geq 0}} \frac{n!}{k_1!k_2!\ldots k_m!} v_1^{(k_1)} v_2^{(k_2)} \ldots v_m^{(k_m)} .$$

The Laplace transform of a positive random variable X with probability density function (pdf) $f_X(t)$ is defined by

$$(2)\qquad \hat{f}(s) = T\left(f_X\left(t\right)\right) = \mathrm{E}\,\mathrm{e}^{-sX} = \int_0^\infty \mathrm{e}^{-st} f\left(t\right)\, dt .$$

We denote the inverse transform by

$$(3)\qquad T^{-1}\left(\hat{f}\left(s\right)\right) = f_X\left(t\right).$$

The probability generating function (pgf) for a discrete random variable X defined on the nonnegative integers is given by

$$(4)\qquad P_X\left(s\right) = \mathrm{E}\,s^X = \sum_{k=0}^\infty s^k f_X\left(k\right),$$

where $f_X\left(k\right) = \Pr\left(X = k\right)$ and $k = 0, 1, \ldots,$.

The probabilities $f_X\left(k\right)$ are obtained by differentiating the pgf k times, setting $s = 0$ and dividing by $k!$

$$(5)\qquad f_X\left(k\right) = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}s^k} P_X\left(s\right)\Big|_{s=0} .$$

2. Convolution of Erlang distributions

Let Y_i be independent random variables with exponential distributions, i.e. $\Pr\left(Y_i > t\right) = \mathrm{e}^{-\mu_i t}$, $i = 1, 2, \ldots, r$. The parameters μ_i do not have to be different. We would like to obtain the distribution of the random variable

$$(6)\qquad X = Y_1 + Y_2 + \ldots + Y_r .$$

Assume, that there are n different parameters from among $\mu_1, \mu_2, \ldots, \mu_r$, where $1 < n < r$. Without loss of generality, we can assume that these different parameters are $\mu_1, \mu_2, \ldots, \mu_n$. The components of the sum (6) are grouped with respect to the parameter μ_i of the distribution. Let k_i denote the number of the components with the same parameter μ_i, where $k_1 + k_2 + \ldots + k_n = r$. The random variable X_i is defined as the sum of components having the same parameter μ_i. It is known, that the random variable X_i has an Erlang distribution $Erl\,(k_i, \mu_i)$, i.e. its density is of the form

$$f_{X_i}(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ \dfrac{\mu_i^{k_i} t^{k_i - 1}}{(k_i - 1)!} \mathrm{e}^{-\mu_i t} & \text{for } t > 0. \end{cases} \tag{7}$$

Then one can write the sum (6) as a sum of independent random variables having $Erl\,(k_i, \mu_i)$ distributions ($i = 1, 2, \ldots, n$) i.e.

$$X = X_1 + X_2 + \ldots + X_n, \tag{8}$$

where $\mu_i \neq \mu_j$ for $i \neq j$.

THEOREM 1. *The pdf of the random variable X defined by* (8) *is given by*

$$\begin{aligned} f_X(t) = {} & \sum_{i=1}^{n} \mu_i^{k_i} \mathrm{e}^{-\mu_i t} \sum_{j=1}^{k_i} \frac{(-1)^{k_i - j}}{(j-1)!} t^{j-1} \\ & \times \sum_{\substack{n_1 + \ldots + n_n = k_i - j \\ n_i = 0}} \prod_{\substack{l=1 \\ l \neq i}}^{n} \binom{k_l + n_l - 1}{n_l} \frac{\mu_l^{k_l}}{(\mu_l - \mu_i)^{k_l + n_l}}, \end{aligned} \tag{9}$$

for $t > 0$ and $f_X(t) = 0$ for $t \leq 0$.

Proof. The Laplace transform of the random variable X_i is of the form

$$T\left(f_{X_i}(t)\right) = \left(\frac{\mu_i}{s + \mu_i}\right)^{k_i}.$$

Since the X_i are independent, then

$$\hat{f}_X(s) = T\left(f_X(t)\right) = \prod_{i=1}^{n} \mu_i^{k_i} \prod_{i=1}^{n} (s + \mu_i)^{-k_i}. \tag{10}$$

The function $\hat{f}_X(s)$ is a proper rational function. This function can be represented as a sum of partial fractions in the following manner:

$$\hat{f}_X(s) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{c_{ij}}{(s + \mu_i)^j}, \tag{11}$$

where the constants c_{ij} are given by the formula

$$c_{ij} = \frac{1}{(k_i - j)!} \lim_{s \to -\mu_i} \frac{\mathrm{d}^{k_i - j}}{\mathrm{d}s^{k_i - j}} \left(\hat{f}_X(s)\,(s + \mu_i)^{k_i} \right) , \tag{12}$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k_i$. Now, we calculate the constants c_{ij}. Substuting (10) into (12) we can write

$$c_{ij} = \frac{\prod_{i=1}^{n} \mu_i^{k_i}}{(k_i - j)!} \lim_{s \to -\mu_i} \frac{\mathrm{d}^{k_i - j}}{\mathrm{d}s^{k_i - j}} \prod_{\substack{l=1 \\ l \neq i}}^{n} (s + \mu_l)^{-k_l} . \tag{13}$$

Since

$$\frac{\mathrm{d}^{n_l}\,(s + \mu_l)^{-k_l}}{\mathrm{d}s^{n_l}} = (-1)^{n_l}\, n_l! \binom{k_l + n_l - 1}{n_l} \frac{1}{(s + \mu_l)^{k_l + n_l}},$$

then using the Leibniz formula (1), we obtain

$$\begin{aligned} &\frac{\mathrm{d}^{k_i - j}}{\mathrm{d}s^{k_i - j}} \prod_{\substack{l=1 \\ l \neq i}}^{n} (s + \mu_l)^{-k_l} \\ &= (k_i - j)! \sum_{\substack{n_1 + \ldots + n_n = k_i - j \\ n_i = 0}} (-1)^{k_i - j} \prod_{\substack{l=1 \\ l \neq i}}^{n} \binom{k_l + n_l - 1}{n_l} \frac{1}{(s + \mu_l)^{k_l + n_l}} . \end{aligned} \tag{14}$$

Substituting (14) into (13) we have

$$\begin{aligned} c_{ij} &= \prod_{i=1}^{n} \mu_i^{k_i} (-1)^{k_i - j} \\ &\times \sum_{\substack{n_1 + \ldots + n_n = k_i - j \\ n_i = 0}} \prod_{\substack{l=1 \\ l \neq i}}^{n} \binom{k_l + n_l - 1}{n_l} \frac{1}{(\mu_l - \mu_i)^{k_l + n_l}} . \end{aligned} \tag{15}$$

Since T^{-1} is a linear operator and

$$T^{-1}\left(\left(\frac{1}{s + \mu_i} \right)^j \right) = \frac{t^{j-1}}{(j-1)!} \mathrm{e}^{-\mu_i t} ,$$

then

$$f_X(t) = T^{-1}\left(\hat{f}(s) \right) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} c_{ij} \frac{t^{j-1}}{(j-1)!} \mathrm{e}^{-\mu_i t}. \tag{16}$$

Substituting (15) into (16), we obtain the assertion. □

If $k_1 = k_2 = \ldots = k_n = 1$ i.e. all the parameters μ_i are different, then we obtain formula (9) from paper [7].

3. Results of Mathai

In [5], Mathai gives formulae for the pdf of the random variable $X = Y_1+Y_2\ldots+Y_n$, where Y_i are independent gamma random variables in general and some particular cases. Since in the previous section we considered only the case of Erlang distribution, we shall give below the Mathai's formulae in such case only.

THEOREM 2 (Mathai[5]). *The pdf of random variable X defined by (8) is given by*

$$f_X(t) = \Big(\prod_{i=1}^{n}(-\mu_i)^{k_i}\Big)\sum_{i=1}^{n}\mathrm{e}^{-\mu_i t}\sum_{j=1}^{k_i}\frac{(-1)^j\, b_{ij}t^{j-1}}{(j-1)!} \tag{17}$$

for $t > 0$ and $f_X(t) = 0$ for $t \le 0$. Coefficients b_{ij} are given by

$$b_{ij} = \Bigg(\sum_{l_1=1}^{k_i-j-1}\binom{k_i-j-1}{l_1}A_i^{(k_i-j-l_1-1)}\sum_{l_2}^{l_1-1}\binom{l_1-1}{l_2}A_i^{(l_1-1-l_2)}\ldots\Bigg)\frac{\Delta_i}{(k_i-j)!},$$

where

$$\Delta_i = \prod_{\substack{l=1\\ l\neq i}}^{n}(\mu_i-\mu_l)^{-k_l},$$

$$A_i^{(s)} = (-1)^{s+1}\, s!\sum_{\substack{l=1\\ l\neq i}}^{n}k_l\,(\mu_i-\mu_l)^{-(s+1)}.$$

The formula (17) is very complex, thus Sen and Balakrishnan in [7] derive the pdf directly, not from Theorem 2. In Section 2, we proceed similarly, Theorem 1 is derived not from [5] as a particular case but directly as in [7], using however quite different methods to obtain a more general result and a relatively simple formula.

4. Convolution of Pascal distributions

Let $Y_1, Y_2, \ldots, Y_r$ be independent geometric random variables with pf

$$f_{Y_i}(k) = \Pr(Y_i = k) = (1-q_i)^k\, q_i, \quad k = 0,1,2,\ldots$$

The parameters q_i do not have to be different. We would like to obtain the distribution of the random variable

$$X = Y_1 + Y_2 + \ldots + Y_r. \tag{18}$$

Assume, that there are n different parameters from among $q_1, q_2, \ldots, q_r$, where $1 < n < r$. Without loss of generality, we can assume that these different parameters are $q_1, q_2, \ldots, q_n$. The components of the sum (18) are grouped with respect to the parameter q_i of the distribution. Let r_i denote the number of the components in the sum (18) with the same parameter q_i, where $r_1 + r_2 + \ldots + r_n = r$ $(r_i \geq 1)$. We denote the sum of r_i components with geometric distribution $Geo(q_i)$ by X_i. It is known, that the random variable X_i has a Pascal distribution $Pasc(r_i, q_i)$, i.e. its pf is of the form

$$(19) \qquad f_{X_i}(k) = \binom{r_i + k - 1}{r_i - 1} (1 - q_i)^k q_i^{r_i}, \quad k = 0, 1, 2, \ldots$$

(see [1] for example). Then one can represent the sum (18) as a sum of independent random variables having $Pasc(r_i, q_i)$ distributions $(i = 1, 2, \ldots, n)$, i.e.

$$(20) \qquad X = X_1 + X_2 + \ldots + X_n,$$

where $q_i \neq q_j$ for $i \neq j$.

THEOREM 3. *The pf of variable X defined by* (20) *is given by*

$$(21) \qquad f_X(k) = \sum_{k_1 + \ldots + k_n = k} \prod_{i=1}^{n} \binom{r_i + k_i - 1}{k_i} (1 - q_i)^{k_i} q_i^{r_i}.$$

Proof. The pgf of the $Pasc(r_i, q_i)$ distribution is of the form

$$P_{X_i}(s) = \left(\frac{q_i}{1 - p_i s}\right)^{r_i},$$

where $p_i = 1 - q_i$ and $s < 1/p_i$. Since the X_i are independent, then the pgf of X is given by

$$(22) \qquad P_X(s) = \prod_{i=1}^{n} q_i^{r_i} \prod_{i=1}^{n} (1 - sp_i)^{-r_i}.$$

Using formula (5), we obtain

$$f_X(k) = \frac{1}{k!} \prod_{i=1}^{n} q_i^{r_i} \lim_{s \to 0} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \prod_{i=1}^{n} (1 - sp_i)^{-r_i}.$$

From the Leibniz formula (1), we have

$$(23) \quad f_X(k) = \frac{1}{k!} \prod_{i=1}^{n} q_i^{r_i} \lim_{s \to 0} \sum_{k_1 + \ldots + k_n = k} \frac{k!}{k_1! \ldots k_n!} \prod_{i=1}^{n} \frac{\mathrm{d}^{k_i}}{\mathrm{d}s^{k_i}} (1 - sp_i)^{-r_i}.$$

Substituting

$$\frac{\mathrm{d}^{k_i}}{\mathrm{d}s^{k_i}} (1 - sp_i)^{-r_i} = \binom{r_i + k_i - 1}{k_i} \frac{k_i! p_i^{k_i}}{(1 - sp_i)^{r_i + k_i}}$$

into formula (23), we have

$$f_X(k) = \frac{1}{k!} \prod_{i=1}^{n} q_i^{r_i} \lim_{s \to 0} \sum_{k_1 + \ldots + k_n = k} \frac{k!}{k_1! \ldots k_n!} \prod_{i=1}^{n} \binom{r_i + k_i - 1}{k_i} \frac{k_i! p_i^{k_i}}{(1 - sp_i)^{r_i + k_i}}$$

and, after simple computations, we obtain the assertion. □

If $r_1 = \ldots = r_n = 1$ then $n = r$. In this case the pf of X is given in [7]. However, the geometric distribution of X_i in this paper is given by the following formula:

$$f_{X_i}(k) = (1 - q_i)^{k-1} q_i, \quad k = 1, 2, \ldots$$

5. Application to reliability

In this section, we give an application to the reliability of a simple system with replacements.

Let a system consist of n identical and independent devices:

$$E = \{e_1, e_2, \ldots, e_n\}.$$

One device of E works and other elements are in reserve.

Each device can be repaired $k-1$ times, hence we have k working periods for each element. Assume that all repairs are immediate, but after the ith repair the device is worse than the device after the $(i-1)$th repair (a new has had 0 repairs). Hence, we can assume that after the ith repair the time to the next failure has an exponential distribution with parameter μ_i and $\mu_{i+1} > \mu_i$. After the kth failure the device is replaced by a new one.

Let X_{ij} be the working time of the ith device after the $(j-1)$ repair. Hence,

$$Y_i = \sum_{j=1}^{k} X_{ij}$$

is the total working time of the ith device and

$$Z = \sum_{i=1}^{n} Y_i$$

is the working time of the system. Denoting

$$X_j = \sum_{i=1}^{n} X_{ij},$$

one can express Z as a sum of k nonidentical Erlang random variables X_j, X_j has an Erlang distribution $Erl(n, \mu_j)$, $k = 1, \ldots, n$, rather than of identical general Erlang random variables. Therefore, we can apply Theorem 1.

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