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ON SOME NATURAL OPERATORS IN VECTOR FIELDS

Abstract. Given natural numbers m, n, r, s, q with $s \geq r \leq q$ there are two vector bundle functors $T^{r,s,q*} = J^{(r,s,q)}(., \mathbf{R}^{1,1})_0$ and $T^{r,s*} = J^{(r,s)}(., \mathbf{R})_0$ on the category $\mathcal{FM}_{m,n}$ of (m, n)-dimensional fibered manifolds. In the present paper we prove that for natural numbers m, n, r, s, q with $s \geq r \leq q$ and $m \geq 2$ the space of natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ lifting a projectable vector field X on Y into a 1-form $A(X)$ on $T^{r,s,q*}Y$ is a $2(q+r)$-dimensional module over $C^\infty(\mathbf{R}^{q+r})$ and we construct explicitely the basis of this module. We prove also that for natural numbers m, n, r, s with $s \geq r$ and $m \geq 2$ the space of all natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ lifting a projectable vector field X on Y into a 1-form $A(X)$ on $T^{r,s*}Y$ is a $2r$-dimensional module over $C^\infty(\mathbf{R}^r)$ and we construct explicitely the basis of this module.

Introduction

In this paper we consider the following categories over manifolds: the category $\mathcal{M}f_m$ of m-dimensional manifolds and embeddings, the category $\mathcal{FM}$ of fibered manifolds and fibered maps, the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and fibered embeddings, the category $\mathcal{VB}$ of all vector bundles and vector bundle maps.

The notions of bundle functors and natural operators can be found in the fundamental monograph [4].

In [13], we studied the problem how a vector field X on an m-manifold M induces a 1-form $A(X)$ on the r-cotangent bundle $T^{r*}M = J^r(M, \mathbf{R})_0$ of M. This problem is reflected in the concept of natural operators $A : T_{|\mathcal{M}f_m} \rightsquigarrow T^*T^{r*}$. We proved that for natural numbers $m \geq 2$ and r all natural operators $A : T_{|\mathcal{M}f_m} \rightsquigarrow T^*T^{r*}$ form a $2r$-dimensional module over $C^\infty(\mathbf{R}^r)$. We constructed the basis of this module.

In the present paper we try to extend the result of [13] on fibered manifolds. We study the problem how a projectable vector field X on an (m, n)-

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dimensional fibered manifold Y induces a 1-form $A(X)$ on the (r,s,q)-cotangent bundle $T^{r,s,q*}Y = J^{(r,s,q)}(Y,\mathbf{R}^{1,1})_0$ of Y. This problem is reflected in the concept of natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$. We prove that for natural numbers m,n,r,s,q with $s \geq r \leq q$ and $m \geq 2$ all natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ form a $2(q+r)$-dimensional module over $C^\infty(\mathbf{R}^{q+r})$. We construct the basis of this module.

In similar way we study the problem how a projectable vector field X on an (m,n)-dimensional fibered manifold Y induces a 1-form $A(X)$ on the (r,s)-cotangent bundle $T^{r,s*}Y = J^{(r,s)}(Y,\mathbf{R})_0$ of Y. This problem is reflected in the concept of natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$. We prove that for natural numbers m,n,r,s with $s \geq r$ and $m \geq 2$ all natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ form a $2r$-dimensional module over $C^\infty(\mathbf{R}^r)$. We construct the basis of this module.

Natural operators lifting functions, vector fields and 1-form to some bundle functors were used practically in all papers in which problem of prolongations of geometric structures was studied, e.g. [14]. Such natural operators in the case of the (higher order) cotangent bundle functor were studied in [1]—[4], [6]—[11], [13], e.t.c.

From now on the usual coordinates on $\mathbf{R}^{m,n}$, the trivial bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$, are denoted by $x^1,\dots,x^m,y^1,\dots,y^n$.

All manifolds are assumed to be finite dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

1. The natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$

1.1. The (r,s,q)-cotangent bundle $T^{r,s,q*}$

Let r,s,q,m,n be natural numbers with $s \geq r \leq q$.

The concept of r-jets can be generalized as follows, see [4]. Let $Y \to M$ and $Z \to N$ be fibered manifolds. We recall that two fibered maps $f,g : Y \to Z$ with base maps $\underline{f},\underline{g} : M \to N$ determine the same (r,s,q)-jet $j_y^{(r,s,q)}f = j_y^{(r,s,q)}g$ at $y \in Y_x$, $x \in M$, if $j_y^r f = j_y^r g$, $j_y^s(f|Y_x) = j_y^s(g|Y_x)$ and $j_x^q \underline{f} = j_x^q \underline{g}$. The space of all (r,s,q)-jets of Y into Z is denoted by $J^{(r,s,q)}(Y,Z)$. The composition of fibered maps induces the composition of (r,s,q)-jets, [4], p. 126.

The vector r-cotangent bundle functor $T^{r*} = J^r(.,\mathbf{R})_0 : \mathcal{M}f_m \to \mathcal{VB}$ can be generalized as follows, see [4], [12]. Let $\mathbf{R}^{1,1} = \mathbf{R} \times \mathbf{R}$ be the trivial bundle over $\mathbf{R}$. The space $T^{r,s,q*} = J^{(r,s,q)}(Y,\mathbf{R}^{1,1})_0$, $0 \in \mathbf{R}^2$, has an induced structure of a vector bundle over Y. Every $\mathcal{FM}_{m,n}$-map $f : Y \to Z$ induces a vector bundle map $T^{r,s,q*}f : T^{r,s,q*}Y \to T^{r,s,q*}Z$ covering f, $T^{r,s,q*}f(j_y^{(r,s,q)}\gamma) = j_{f(y)}^{(r,s,q)}(\gamma \circ f^{-1})$, $\gamma : Y \to \mathbf{R}^{1,1}$, $\gamma(y) = 0$. The correspondence $T^{r,s,q*}: \mathcal{FM}_{m,n} \to \mathcal{VB}$ is a vector bundle functor in the sense of [4]. We call it the (r,s,q)-cotangent bundle functor.

1.2. Examples of natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$

We recall that a projectable vector field on a fibered manifold Y over M is a vector field X on Y such that there exists an underlying vector field $\underline{X}$ on M which is p-related with X, where $p: Y \to M$ is the bundle projection. The flow of a projectable vector field is formed by $\mathcal{FM}$-morphisms.

We are going to study the problem how a projectable vector field X on an (m,n)-dimensional fibered manifold Y induces canonically a 1-form $A(X)$ on $T^{r,s,q*}Y$. This problem is reflected in the concept of natural operators $A: T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ in the sense of [4].

EXAMPLE 1. Let X be a projectable vector field on an (m,n)- dimensional fibered manifold Y. For every $k = 1,\dots,q$ we have map $\overset{(k)}{X}: T^{r,s,q*}Y \to \mathbf{R}$, $\overset{(k)}{X}(j_y^{(r,s,q)}\gamma) := (X^k\gamma_1)(y), \gamma = (\gamma_1,\gamma_2): Y \to \mathbf{R}\times\mathbf{R}, y \in Y, \gamma(y) = 0$, where $X^k = X \circ \dots \circ X$ (k-times). It is well-defined because if $j_y^{(r,s,q)}\gamma = j^{(r,s,q)}\tilde{\gamma}$ then $j_y^q\gamma_1 = j_y^q\tilde{\gamma}_1$. Then for every $k = 1,\dots,q$ we have 1-form $d\overset{(k)}{X}$ on $T^{r,s,q*}Y$. The correspondence $\overset{(k)}{A}: T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$, $X \to d\overset{(k)}{X}$, is a natural operator.

EXAMPLE 2. Let X be a projectable vector field on an (m,n)- dimensional fibered manifold Y. For every $l = 1,\dots,r$ we have map $\overset{((l))}{X}: T^{r,s,q*}Y \to \mathbf{R}$, $\overset{((l))}{X}(j_y^{(r,s,q)}\gamma) := (X^l\gamma_2)(y)$, $\gamma = (\gamma_1,\gamma_2): Y \to \mathbf{R}\times\mathbf{R}$, $y \in Y$, $\gamma(y) = 0$. It is well-defined because if $j_y^{(r,s,q)}\gamma = j^{(r,s,q)}\tilde{\gamma}$ then $j_y^r\gamma_2 = j_y^r\tilde{\gamma}_2$. Then for every $l = 1,\dots,r$ we have 1-form $d\overset{((l))}{X}$ on $T^{r,s,q*}Y$. The correspondence $\overset{((l))}{A}: T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$, $X \to d\overset{((l))}{X}$, is a natural operator.

EXAMPLE 3. Let X be a vector field on an (m,n)-dimensional fibered manifold Y. For every $k = 1,\dots,q$ we have 1-form $\overset{<k>}{X}: TT^{r,s,q*}Y \to \mathbf{R}$ on $T^{r,s,q*}Y$, $\overset{<k>}{X}(v) = \langle d_y(X^{k-1}\gamma_1), T\pi(v)\rangle$, $v \in (TT^{r,s,q*})_yY$, $y \in Y$, $\gamma = (\gamma_1,\gamma_2): Y \to \mathbf{R}\times\mathbf{R}$, $\gamma(y) = 0$, $p^T(v) = j_y^{(r,s,q)}\gamma$, $p^T: TT^{r,s,q*}Y \to T^{r,s,q*}Y$ is the tangent bundle projection, $\pi: T^{r,s,q*}Y \to Y$ is the bundle projection. The correspondence $\overset{<k>}{A}: T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$, $X \to \overset{<k>}{X}$, is a natural operator.

EXAMPLE 4. Let X be a vector field on an (m,n)-dimensional fibered manifold Y. For every $l = 1,\dots,r$ we have 1-form $\overset{<<l>>}{X} : TT^{r,s,q*}Y \to \mathbf{R}$ on $T^{r,s,q*}Y$, $\overset{<<l>>}{X}(v) = < d_y(X^{l-1}\gamma_2), T\pi(v) >$, $v \in (TT^{r,s,q*})_y Y$, $y \in Y$, $\gamma = (\gamma_1,\gamma_2) : Y \to \mathbf{R}\times\mathbf{R}$, $\gamma(y) = 0$, $p^T(v) = j_y^{(r,s,q)}\gamma$. The correspondence $\overset{<<l>>}{A} : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$, $X \to \overset{<<l>>}{X}$, is a natural operator.

1.3. The $\mathcal{C}^\infty(\mathbf{R}^{q+r})$-module of natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$

The set of all natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ is a module over the algebra $\mathcal{C}^\infty(\mathbf{R}^{q+r})$. Actually, if $f \in \mathcal{C}^\infty(\mathbf{R}^{q+r})$ and $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ is a natural operator, then $fA : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ is given by $(fA)(X) = f(\overset{(1)}{X},\dots,\overset{(q)}{X},\overset{((1))}{X},\dots,\overset{((r))}{X})A(X)$, $X \in \mathcal{X}_{proj}(Y)$, $Y \in Obj(\mathcal{FM}_{m,n})$.

1.4. The classification theorem

The first main result of this paper is the following classification theorem.

THEOREM 1. *For natural numbers r,s,q,m,n with $s \geq r \leq q$ and $m \geq 2$ the $\mathcal{C}^\infty(\mathbf{R}^{q+r})$-module of all natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$ is free and $2(q+r)$-dimensional. The natural operators $\overset{(k)}{A}$, $\overset{((l))}{A}$, $\overset{<k>}{A}$ and $\overset{<<l>>}{A}$ for $k = 1,\dots,q$ and $l = 1,\dots,r$ form the basis over $\mathcal{C}^\infty(\mathbf{R}^{q+r})$ of this module.*

The proof of Theorem 1 will occupy Subsections 1.5 and 1.6.

1.5. Some preparations

Let us consider a natural operator $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s,q*}$.

Operators $\overset{(1)}{A},\dots,\overset{(q)}{A},\overset{((1))}{A},\dots,\overset{((r))}{A},\overset{<1>}{A},\dots,\overset{<q>}{A},\overset{<<1>>}{A},\dots,\overset{<<r>>}{A}$ are $\mathcal{C}^\infty(\mathbf{R}^{q+r})$-linearly independent. So, we prove only that A is a linear combination of $\overset{(1)}{A},\dots,\overset{(q)}{A},\overset{((1))}{A},\dots,\overset{((r))}{A},\overset{<1>}{A},\dots,\overset{<q>}{A},\overset{<<1>>}{A},\dots,\overset{<<r>>}{A}$ with $\mathcal{C}^\infty(\mathbf{R}^{q+r})$-coefficients.

The following lemma shows that A is uniquely determined by the restriction $A(\frac{\partial}{\partial x^1})|(TT^{r,s,q*})_0\mathbf{R}^{m,n}$.

LEMMA 1. *If $A(\frac{\partial}{\partial x^1})|(TT^{r,s,q*})_0\mathbf{R}^{m,n} = 0$, then $A = 0$.*

Proof. The proof is standard. We use the naturality of A and the fact that any projectable vector field with non-vanishing underline vector field is locally $\frac{\partial}{\partial x^1}$ in some fiber manifold coordinates. □

So, we will study the restriction $A(\frac{\partial}{\partial x^1})|(TT^{r,s,q*})_0\mathbf{R}^{m,n}$.

LEMMA 2. *There are* $f_{(1)},\dots,f_{(q)}\in C^\infty(\mathbf{R}^{q+r})$ *and* $f_{((1))},\dots,f_{((r))}\in C^\infty(\mathbf{R}^{q+r})$ *such that*

$$(A-\sum_{k=1}^{q}f_{(k)}\overset{(k)}{A}-\sum_{l=1}^{r}f_{((l))}\overset{((l))}{A})(\frac{\partial}{\partial x^1})|(VT^{r,s,q*})_0\mathbf{R}^{m,n}=0,$$

where $VT^{r,s,q*}Y\subset TT^{r,s,q*}Y$ *denotes the vertical subbundle (with respect to the bundle projection* $\pi: T^{r,s,q*}Y\to Y$*).*

Proof. We have $(VT^{r,s,q*})_0\mathbf{R}^{m,n}\tilde{=}T_0^{r,s,q*}\mathbf{R}^{m,n}\times T_0^{r,s,q*}\mathbf{R}^{m,n}$,

$$\frac{d}{dt}_{|t=0}(u+tw)\tilde{=}(u,w),\ u,w\in T_0^{r,s,q*}\mathbf{R}^{m,n}.$$

For $k=1,\dots,q$ we define $f_{(k)}:\mathbf{R}^{q+r}\to\mathbf{R}$,

$$f_{(k)}(a,b)$$
$$=A\Big(\frac{\partial}{\partial x^1}\Big)\Big(j_0^{(r,s,q)}\Big(\sum_{\tilde{k}=1}^{q}\frac{1}{\tilde{k}!}a_{\tilde{k}}(x^1)^{\tilde{k}},\sum_{\tilde{l}=1}^{r}\frac{1}{\tilde{l}!}b_{\tilde{l}}(x^1)^{\tilde{l}}\Big),j_0^{(r,s,q)}\Big(\frac{1}{k!}(x^1)^k,0\Big)\Big),$$

$a=(a_1,\dots,a_q)\in\mathbf{R}^q$, $b=(b_1,\dots,b_r)\in\mathbf{R}^r$.

For $l=1,\dots,r$ we define $f_{((l))}:\mathbf{R}^{q+r}\to\mathbf{R}$,

$$f_{((l))}(a,b)$$
$$=A\Big(\frac{\partial}{\partial x^1}\Big)\Big(j_0^{(r,s,q)}\Big(\sum_{\tilde{k}=1}^{q}\frac{1}{\tilde{k}!}a_{\tilde{k}}(x^1)^{\tilde{k}},\sum_{\tilde{l}=1}^{r}\frac{1}{\tilde{l}!}b_{\tilde{l}}(x^1)^{\tilde{l}}\Big),j_0^{(r,s,q)}\Big(0,\frac{1}{l!}(x^1)^l\Big)\Big),$$

$a=(a_1,\dots,a_q)\in\mathbf{R}^q$, $b=(b_1,\dots,b_r)\in\mathbf{R}^r$.

We prove the assertion of the lemma.

For simplicity denote $\tilde{A}=A-\sum_{k=1}^q f_{(k)}\overset{(k)}{A}-\sum_{l=1}^r f_{((l))}\overset{((l))}{A}$.

Consider $\mathcal{FM}$-morphisms $\gamma,\eta:\mathbf{R}^{m,n}\to\mathbf{R}^{1,1}$, $\gamma(0)=\eta(0)=0$. Define $a=(a_1,\dots,a_q)\in\mathbf{R}^q$ by

$$j_0^{(r,q,s)}(\gamma_1(x^1,0,\dots,0),0)=j_0^{(r,s,q)}\Big(\sum_{k=1}^{q}\frac{1}{k!}a_k(x^1)^k,0\Big).$$

Define $b=(b_1,\dots,b_r)\in\mathbf{R}^r$ by

$$j_0^{(r,q,s)}(0,\gamma_2(x^1,0,\dots,0))=j_0^{(r,s,q)}\Big(0,\sum_{l=1}^{r}\frac{1}{l!}b_l(x^1)^l\Big).$$

Define $\tilde{a}=(\tilde{a}_1,\dots,\tilde{a}_q)\in\mathbf{R}^q$ by

$$j_0^{(r,q,s)}(\eta_1(x^1,0,\dots,0),0)=j_0^{(r,s,q)}\Big(\sum_{k=1}^{q}\frac{1}{k!}\tilde{a}_k(x^1)^k,0\Big).$$

Define $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_r) \in \mathbf{R}^r$ by

$$j_0^{(r,q,s)}(0, \eta_2(x^1, 0, \dots, 0)) = j_0^{(r,s,q)}\Big(0, \sum_{l=1}^{r} \frac{1}{l!}\tilde{b}_l (x^1)^l\Big).$$

Using the naturality of $\tilde{A}$ with respect to the homotheties $(x^1, tx^2, \dots, tx^m, ty^1, \dots, ty^n))$ for $t \neq 0$ and putting $t \to 0$ we obtain

$$\tilde{A}\Big(\frac{\partial}{\partial x^1}\Big)(j_0^{(r,s,q)}\gamma, j_0^{(r,s,q)}\eta) = \tilde{A}\Big(\frac{\partial}{\partial x^1}\Big)(j_0^{(r,s,q)}(\gamma(x^1, 0, \dots, 0)), j_0^{(r,s,q)}(\eta(x^1, 0, \dots, 0))).$$

Then we have

$$\tilde{A}\Big(\frac{\partial}{\partial x^1}\Big)(j_0^{(r,s,q)}\gamma, j_0^{(r,s,q)}\eta) = \sum_{k=1}^{q} \tilde{a}_k f_{(k)}(a,b) + \sum_{l=1}^{r} \tilde{b}_l f_{((l))}(a,b) - \\ - \sum_{k=1}^{q} f_{(k)}(a,b)\tilde{a}_k - \sum_{l=1}^{r} f_{((l))}(a,b)\tilde{b}_l = 0.$$

The proof of Lemma 2 is complete. □

1.6. Proof of Theorem 1

Replacing A by $A - \sum_{k=1}^{q} f_{(k)} \overset{(k)}{A} - \sum_{l=1}^{r} f_{((l))} \overset{((l))}{A}$ we can assume that

$$A(\frac{\partial}{\partial x^1})|(VT^{r,s,q*})_0\mathbf{R}^{m,n} = 0.$$

It remains to show that there exist $g_{<1>}, \dots, g_{<q>}, g_{<<1>>}, \dots, g_{<<r>>} \in \mathcal{C}^\infty(\mathbf{R}^{q+r})$ with $A = \sum_{k=1}^{q} g_{<k>} \overset{<k>}{A} + \sum_{l=1}^{r} g_{<<l>>} \overset{<<l>>}{A}$.

For $k = 1, \dots, q$ define $g_{<k>} : \mathbf{R}^{q+r} \to \mathbf{R}$,

$$g_{<k>}(a,b) = A\Big(\frac{\partial}{\partial x^1}\Big)\Big(T^{r,s,q*}\Big(\frac{\partial}{\partial x^2}\Big)\Big(j_0^{(r,s,q)}\Big(\sum_{\tilde{k}=1}^{q} \frac{1}{\tilde{k}!} a_{\tilde{k}} (x^1)^{\tilde{k}} \\ + \frac{1}{(k-1)!}(x^1)^{k-1}x^2, \sum_{\tilde{l}=1}^{r} \frac{1}{\tilde{l}} b_{\tilde{l}} (x^1)^{\tilde{l}}\Big)\Big)\Big),$$

$a = (a_1, \dots, a_q) \in \mathbf{R}^q$, $b = (b_1, \dots, b_r) \in \mathbf{R}^r$, where $T^{r,s,q*}X$ denotes the complete lifting (flow prolongation) of a projectable vector field $X \in \mathcal{X}_{proj}(Y)$ to $T^{r,s,q*}Y$.

For $l = 1, \dots, r$ define $g_{<<l>>} : \mathbf{R}^{q+r} \to \mathbf{R}$,

$$g_{<<l>>}(a,b) = A\left(\frac{\partial}{\partial x^1}\right)\left(T^{r,s,q*}\left(\frac{\partial}{\partial x^2}\right)\left(j_0^{(r,s,q)}\left(\sum_{\tilde{k}=1}^{q}\frac{1}{\tilde{k}!}a_{\tilde{k}}(x^1)^{\tilde{k}},\right.\right.\right.$$
$$\left.\left.\left.\sum_{\tilde{l}=1}^{r}\frac{1}{\tilde{l}}b_{\tilde{l}}(x^1)^{\tilde{l}} + \frac{1}{(l-1)!}(x^1)^{l-1}x^2\right)\right)\right),$$

$a = (a_1, \dots, a_q) \in \mathbf{R}^q$, $b = (b_1, \dots, b_r) \in \mathbf{R}^r$.

We prove $A = \sum_{k=1}^{q} g_{<k>} \overset{<k>}{A} + \sum_{l=1}^{r} g_{<<l>>} \overset{<<l>>}{A}$.

By Lemma 1 and $A(\frac{\partial}{\partial x^1})|(VT^{r,s,q*})_0\mathbf{R}^{m,n} = 0$ it is sufficient to show

$$A\left(\frac{\partial}{\partial x^1}\right)(T^{r,s,q*}(\partial)(j_0^{(r,s,q)}\gamma)) =$$
$$= \left(\sum_{k=1}^{q} g_{<k>} \overset{<k>}{A} + \sum_{l=1}^{r} g_{<<l>>} \overset{<<l>>}{A}\right)\left(\frac{\partial}{\partial x^1}\right)(T^{r,s,q*}(\partial)(j_0^{(r,s,q)}\gamma))$$

for any $\mathcal{FM}$-morphism $\gamma : \mathbf{R}^{m,n} \to \mathbf{R}^{1,1}$, $\gamma(0) = 0$ and any constant projectable vector field ∂ on $\mathbf{R}^{m,n}$. Using the regularity and the naturality of A and $\sum_{k=1}^{q} g_{<k>} \overset{<k>}{A} + \sum_{l=1}^{r} g_{<<l>>} \overset{<<l>>}{A}$ with respect to linear $\mathcal{FM}_{m,n}$-morphisms $\mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ preserving $\frac{\partial}{\partial x^1}$ we can assume $\partial = \frac{\partial}{\partial x^2}$. For simplicity denote $\tilde{A} = \sum_{k=1}^{q} g_{<k>} \overset{<k>}{A} + \sum_{l=1}^{r} g_{<<l>>} \overset{<<l>>}{A}$.

Consider an $\mathcal{FM}$-morphism $\gamma = (\gamma_1, \gamma_2) : \mathbf{R}^{m,n} \to \mathbf{R}^{1,1}$, $\gamma(0) = 0$. Define $a = (a_1, \dots, a_q) \in \mathbf{R}^q$ and $b = (b_1, \dots, b_r) \in \mathbf{R}^r$ by

$$a_k = \frac{\partial^k}{\partial (x^1)^k}\gamma_1(0),\ b_l = \frac{\partial^l}{\partial (x^1)^l}\gamma_2(0)$$

for $k = 1, \dots, q$ and $l = 1, \dots, r$. Define $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_q) \in \mathbf{R}^q$ and $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_r) \in \mathbf{R}^r$ by

$$\tilde{a}_k = \frac{\partial}{\partial x^2}\frac{\partial^{k-1}}{\partial (x^1)^{k-1}}\gamma_1(0),\ \tilde{b}_l = \frac{\partial}{\partial x^2}\frac{\partial^{l-1}}{\partial (x^1)^{l-1}}\gamma_2(0)$$

for $k = 1, \dots, q$ and $l = 1, \dots, r$. Using the naturality of A with respect to the homotheties $(x^1, tx^2, \tau x^3 \dots, \tau x^m, \tau y^1, \dots, \tau y^n)$ for $t, \tau \neq 0$ we get the homogeneity condition

$$tA\left(\frac{\partial}{\partial x^1}\right)\left(T^{r,s,q*}\left(\frac{\partial}{\partial x^2}\right)(j_0^{(r,s,q)}\gamma)\right) =$$
$$= A\left(\frac{\partial}{\partial x^1}\right)\left(T^{r,s,q*}\left(\frac{\partial}{\partial x^2}\right)(j_0^{(r,s,q)}(\gamma(x^1, tx^2, \tau x^3, \dots, \tau x^m, \tau y^1, \dots, \tau y^n))\right)\right).$$

This type of homogeneity gives

$$A\left(\frac{\partial}{\partial x^1}\right)\left(T^{r,s,q*}\left(\frac{\partial}{\partial x^2}\right)(j_0^{(r,s,q)}\gamma)\right) = \sum_{k=1}^{q} g_{<k>}(a,b)\tilde{a}_k + \sum_{l=1}^{r} g_{<<l>>}(a,b)\tilde{b}_l$$

because of the homogeneous function theorem, [4]. On the other hand (it is easy to observe) we have

$$\tilde{A}\left(\frac{\partial}{\partial x^1}\right)\left(T^{r,s,q*}(\frac{\partial}{\partial x^2}\right)(j_0^{(r,s,q)}\gamma)) = \sum_{k=1}^{q} \tilde{a}_k g_{<k>}(a,b) + \sum_{l=1}^{r} \tilde{b}_l g_{<<l>>}(a,b).$$

The proof of Theorem 1 is complete. □

2. The natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$

2.1. The (r,s)-cotangent bundle $T^{r,s*}$

Let r, s, m, n be natural numbers with $s \geq r$.

The concept of r-jets can be also generalized as follows, see [4]. Let $Y \to M$ be a fibered manifold and Q be a manifold. We recall that maps $f, g : Y \to Q$ determine the same (r,s)-jet $j_y^{(r,s)} f = j_y^{(r,s)} g$ at $y \in Y_x$, $x \in M$, if $j_y^r f = j_y^r g$ and $j_y^s(f|Y_x) = j_y^s(g|Y_x)$. The space of all (r,s)-jets of Y into Q is denoted by $J^{(r,s)}(Y,Q)$.

The vector r-cotangent bundle functor $T^{r*} = J^r(.,\mathbf{R})_0 : \mathcal{M}f_m \to \mathcal{VB}$ can be also generalized as follows, see [12]. The space $T^{r,s*} = J^{(r,s)}(Y,\mathbf{R})_0$, $0 \in \mathbf{R}$, has an induced structure of a vector bundle over Y. Every $\mathcal{FM}_{m,n}$-map $f : Y \to Z$ induces a vector bundle map $T^{r,s*}f : T^{r,s*}Y \to T^{r,s*}Z$ covering f, $T^{r,s*}f(j_y^{(r,s)}\gamma) = j_{f(y)}^{(r,s)}(\gamma \circ f^{-1})$, $\gamma : Y \to \mathbf{R}$, $\gamma(y) = 0$. The correspondence $T^{r,s*} : \mathcal{FM}_{m,n} \to \mathcal{VB}$ is a vector bundle functor in the sense of [4]. We call it the (r,s)-cotangent bundle functor.

2.2. Examples of natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$

We are going to study the problem how a projectable vector field X on an (m,n)-dimensional fibered manifold Y induces canonically a 1-form $A(X)$ on $T^{r,s*}Y$. This problem is reflected in the concept of natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ in the sense of [4].

EXAMPLE 5. Let X be a projectable vector field on an (m,n)- dimensional fibered manifold Y. For every $k = 1, \dots, r$ we have map $\overset{(k)}{X} : T^{r,s*}Y \to \mathbf{R}$, $\overset{(k)}{X}(j_y^{(r,s)}\gamma) := (X^k\gamma)(y)$, $\gamma : Y \to \mathbf{R}$, $y \in Y$, $\gamma(y) = 0$, where $X^k = X \circ \dots \circ X$ (k-times). Then for every $k = 1, \dots, r$ we have 1-form $d\overset{(k)}{X}$ on $T^{r,s*}Y$. The correspondence $\overset{(k)}{A} : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$, $X \to d\overset{(k)}{X}$, is a natural operator.

EXAMPLE 6. Let X be a vector field on an (m, n)-dimensional fibered manifold Y. For every $k = 1, \dots, r$ we have 1-form $\overset{<k>}{X} : TT^{r,s*}Y \to \mathbf{R}$ on $T^{r,s*}Y$, $\overset{<k>}{X}(v) = < d_y(X^{k-1}\gamma), T\pi(v) >$, $v \in (TT^{r,s*})_y Y$, $y \in Y$, $\gamma : Y \to \mathbf{R}$, $\gamma(y) = 0$, $p^T(v) = j_y^{(r,s)}\gamma$, $p^T : TT^{r,s*}Y \to T^{r,s*}Y$ is the tangent bundle projection, $\pi : T^{r,s*}Y \to Y$ is the bundle projection. The correspondence $\overset{<k>}{A} : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$, $X \to \overset{<k>}{X}$, is a natural operator.

2.3. The $\mathcal{C}^\infty(\mathbf{R}^r)$-module of natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$

The set of all natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ is a module over the algebra $\mathcal{C}^\infty(\mathbf{R}^r)$. Actually, if $f \in \mathcal{C}^\infty(\mathbf{R}^r)$ and $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ is a natural operator, then $fA : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ is given by $(fA)(X) = f(\overset{(1)}{X}, \dots, \overset{(r)}{X})A(X)$, $X \in \mathcal{X}_{proj}(Y)$, $Y \in Obj(\mathcal{FM}_{m,n})$.

2.4. The classification theorem

The second main result of this paper is the following classification theorem.

THEOREM 2. *For natural numbers r, s, m, n with $s \geq r$ and $m \geq 2$ the $\mathcal{C}^\infty(\mathbf{R}^r)$-module of all natural operators $T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^*T^{r,s*}$ is free and $2r$-dimensional. The natural operators $\overset{(k)}{A}$ and $\overset{<k>}{A}$ for $k = 1, \dots, r$ form the basis over $\mathcal{C}^\infty(\mathbf{R}^r)$ of this module.*

Proof. The proof of Theorem 2 is similar to the proof of Theorem 1, but easier. We leave the details to the reader. □

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