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B.-Y. CHEN'S INEQUALITY FOR CR-SUBMANIFOLDS OF LOCALLY CONFORMAL KAEHLER SPACE FORMS

Abstract. In this article, we investigate sharp inequalities involving δ-invariant for CR-submanifolds in locally conformal Kaehler space forms of constant holomorphic sectional curvature with arbitrary codimension.

1. Introduction

Riemannian invariants are the intrinsic characteristics of the Riemainnian manifold. Among all Riemannian invariants, curvature is "the N^o1Riemannian invariant and the most natural" according to M. Berger in [1]. Classically, among the Riemannian curvature invariants, geometers have been studying sectional, scalar and Ricci curvature.

Recently, in [3] B.-Y. Chen introduced new types of curvature invariants, defining two strings of scalar-valued Riemannian curvature functions, namely $\delta(n_1, \ldots, n_k)$ and $\hat{\delta}(n_1, \ldots, n_k)$ for every $(n_1, \ldots, n_k)$ satisfying $n_1 < n, n_j \geq 2$ and $n_1 + \ldots + n_k \leq n$. For these two strings of Riemannian curvature invariants, one always has trivially $\delta(n_1, \ldots, n_k) \geq \hat{\delta}(n_1, \ldots, n_k)$. We simply called these invariants the *δ-invariants*. The first string of δ-invariants, $\delta(n_1, \ldots, n_k)$, extend naturally the Riemannain invariant introduced in [2]. In [3] B.-Y. Chen studied δ-invariants for submanifolds in Riemannian space forms with arbitrary codimension. Also, in [10] A. Oiaga and I. Mihai investigated δ-invarants for slant submanifolds in complex space forms.

In this paper, we study submanifolds of locally conformal Kaehler space forms of constant holomorphic sectional curvature with arbitrary codimension and establish δ-invariants for CR-submanifolds in locally conformal Kaehler space forms.

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2. Preliminaries

Let $\tilde{M}$ be a Hermitian manifold with almost complex structure J and a Hermitian metric g. A Hermitian manifold $\tilde{M}$ is called a *locally conformal Kaehler manifold* if each point $p \in \tilde{M}$ has an open neighborhood U with a differentiable map $\phi : U \longrightarrow \mathbb{R}$ such that

$$g^* = e^{-2\phi} g|_U \tag{2.1}$$

is Kaehler metric on U (See [6, 9, 11]). On the other hand, the fundamental 2-form w of $\tilde{M}$ is defined by

$$w(X,Y) = g(JX,Y) \tag{2.2}$$

for any tangent vectors X, Y on $\tilde{M}$.

PROPOSITION 2.1 ([6]). *A Hermitian manifold $\tilde{M}$ is a locally conformal Kaehler manifold if and only if there exists a global closed 1-form α satisfying*

$$\begin{aligned}(\tilde{\nabla}_Z w)(X,Y) = \beta(Y)g(X,Z) - \beta(X)g(Y,Z) \\ + \alpha(Y)w(X,Z) - \alpha(X)w(Y,Z)\end{aligned} \tag{2.3}$$

for any tangent vectors X, Y, Z on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to g and the 1-form β is given by $\beta(X) = -\alpha(JX)$.

The 1-form α is called *Lee form* and its dual vector field is *Lee vector field.* A locally conformal Kaehler manifold having parallel Lee form is said to be a *generalized Hopf manifold.* On a locally conformal Kaehler manifold, a symmetric $(0,2)$-tensor P is defined by

$$P(X,Y) = -(\tilde{\nabla}_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}||\alpha||^2 g(X,Y), \tag{2.4}$$

and another $(0,2)$-tensor $\tilde{P}$ by $\tilde{P}(X,Y) = P(JX,Y)$, where $||\alpha||$ is the norm of α with respect to g.

Let M be an n-dimensional submainfold of an m-dimensional locally conformal Kaehler manifold $\tilde{M}$. Let ∇ be the induced Levi-Civita connection of M. Then the Gauss and Weingarten formulas given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\tilde{\nabla}_X V = -A_V X + D_X V \tag{2.6}$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{2.7}$$

We also use g for the induced Riemannian metric on M as well as the locally conformal Kaehler manifold $\tilde{M}$. Moreover, the mean curvature vector H on M is defined by $H = \frac{1}{n}\mathrm{trace}h$.

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For any orthornormal basis $e_1, \cdots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined by to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.8}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \cdots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{2.9}$$

Given an orthonormal basis $\{e_1, \cdots, e_n\}$ of the tangent space T_pM, we simply denote by $\tau_{1\cdots r}$ the scalar curvature of the r-plane section spanned by $e_1, \cdots, e_r$. The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p, and if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L. Geometrically, $\tau(L)$ is nothing but the scalar curvature of the image $\exp_p(L)$ of L at p under the exponential map at p. For an integer $k \geq 0$ denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \cdots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of unordered k-tuples with $k \geq 0$ for a fixed n. For each k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$ the two sequences of Riemannian invariants $\mathcal{S}(n_1, \cdots, n_k)(p)$ and $\hat{\mathcal{S}}(n_1, \cdots, n_k)(p)$ are defined respectively by

$$\begin{aligned} \mathcal{S}(n_1, \cdots, n_k)(p) &= \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \\ \hat{\mathcal{S}}(n_1, \cdots, n_k)(p) &= \sup\{\tau(L_1) + \cdots + \tau(L_k)\}, \end{aligned} \tag{2.10}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1, \ldots, k$. The two strings of Riemannian curvature invariants $\delta(n_1, \cdots, n_k)(p)$ and $\hat{\delta}(n_1, \cdots, n_k)(p)$ introduced by B.-Y. Chen in [3] are given by

$$\begin{aligned} \delta(n_1, \cdots, n_k)(p) &= \tau(p) - \mathcal{S}(n_1, \cdots, n_k)(p), \\ \hat{\delta}(n_1, \cdots, n_k)(p) &= \tau(p) - \hat{\mathcal{S}}(n_1, \cdots, n_k)(p). \end{aligned} \tag{2.11}$$

In terms of these δ-invariants, the scalar curvature τ is nothing but $\delta(\emptyset) = \hat{\delta}(\emptyset)$ (with $k = 0$); moreover, the invariant δ_M introduced in [2] is nothing but the invariant $\delta(2)$ (with $k = 1, n_1 = 2$). Obviously, one has $\delta(n_1, \cdots, n_k) \geq \hat{\delta}(n_1, \cdots, n_k)$ for any k-tuple $(n_1, \cdots, n_k)$ in $\mathcal{S}(n)$.

3. δ-invariants of CR-submanifolds

Let M be an n-dimensional submanifold isometrically immersed in an m-dimensional locally conformal Kaehler manifold $\tilde{M}$. A locally conformal Kaehler manifold $\tilde{M}$ is said to be a *locally conformal Kaehler space form* if the holomorphic sectional curvature is a real constant c along $\tilde{M}$. A locally conformal Kaehler space form will be denoted by $\tilde{M}(c)$. Then, the Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}(c)$ is given by

$$\begin{aligned}(3.1)\qquad \tilde{R}(X,Y,Z,W) = \frac{c}{4}\{&g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\\ &+ g(JX,W)g(JY,Z) - g(JX,Z)g(JY,W)\\ &- 2g(JX,Y)g(JZ,W)\}\\ &+ \frac{3}{4}\{P(X,W)g(Y,Z) - P(X,Z)g(Y,W)\\ &+ g(X,W)P(Y,Z) - g(X,Z)P(Y,W)\}\\ &+ \frac{1}{4}\{P(X,JW)g(JY,Z) - P(X,JZ)g(JY,W)\\ &+ g(JX,W)P(Y,JZ) - g(JX,Z)P(Y,JW)\\ &- 2P(X,JY)g(JZ,W) - 2P(Z,JW)g(JX,Y)\},\end{aligned}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$ and

$$P(X,Y) = P(Y,X), \quad P(X,JY) = -P(JX,Y), \quad P(JX,JY) = P(X,Y).$$

The equation of Gauss is given by

$$R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),$$

where R is the Riemannian curvature tensor of M.

A submanifold M isometrically immersed in $\tilde{M}(c)$ is called *totally real* if the almost complex structure J of $\tilde{M}(c)$ carries each tangent space of M into its corresponding normal space. For a totally real submanifold M on $\tilde{M}(c)$ we have $w(X,Y) = 0$ for vector fields X, Y tangent to M. On the other hand, a submanifold M of a locally conformal Kaehler space form $\tilde{M}(c)$ is called a *CR-submanifold* if there exists a differentiable distribution $D : x \to D_x \subset T_xM$ on M satisfying the following conditions : (i) D is holomorphic i.e., $JD_x = D_x$ for each $x \in M$ and (ii) the complementary orthogonal distribution $D^\perp : x \to D_x^\perp \subset T_xM$ is totally real, i.e., $JD_x^\perp \subset T_x^\perp M$ for each $x \in M$. For any vector field X tangent to M, we put

$$X = TX + FX,$$

where TX and FX belong to the distribution D and $D^\perp$, respectively.

Let M be a CR-submanifold of a locally conformal Kaehler space form $\tilde{M}(c)$. Then, the equation of Gauss on M is given by

$$
\begin{aligned}
(3.2)\quad g(R(X,Y)Z,W) = &\frac{c}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\\
&+ g(JTX,W)g(JTY,Z) - g(JTX,Z)g(JTY,W)\\
&- 2g(JTX,Y)g(JTZ,W)\}\\
&+ \frac{3}{4}\{g(X,W)P(Y,Z) - g(Y,W)P(X,Z)\\
&+ P(X,W)g(Y,Z) - P(Y,W)g(X,Z)\}\\
&+ \frac{1}{4}\{P(X,JTW)g(JTY,Z) - P(X,JTZ)g(JTY,W)\\
&+ g(JTX,W)P(Y,JTZ) - g(JTX,Z)P(Y,JTW)\\
&- 2g(JTZ,W)P(X,JTY) - 2P(Z,JTW)g(JTX,Y)\}\\
&+ g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))
\end{aligned}
$$

for any vector fields X,Y,Z,W tangent to M.

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis for T_pM such that $\{e_1, \cdots, e_d, e_{d+1} = Je_1, \cdots, e_{2d} = Je_d\}$ is the basis of D and $\{e_{2d+1}, \cdots, e_n\}$ is the basis of $D^{\perp}$. Then, the scalar curvature τ of M at p is obtained by

$$
(3.3)\quad 2\tau(p) = n^2||H||^2 - ||h||^2 + \frac{1}{4}n(n-1)(c+6\sigma) + \frac{3}{2}d(c-2\sigma),
$$

where $||H||^2$ and $||h||^2$ are the squared mean curvature and the squared norm of the second fundamental form, and we put $\sigma = \frac{1}{n}\sum_{i=1}^{n} P(e_i, e_i)$ and $2d = \dim D$.

We give the following lemma for later use.

LEMMA 3.1 ([2]). *Let $a_1, \cdots, a_n, b$ be $n+1$ $(n \geq 2)$ real numbers such that*

$$
\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big).
$$

Then, $2a_1a_2 \geq b$, with the equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

For each $(n_1, \cdots, n_k) \in \mathcal{S}(n)$, let $c(n_1, \cdots, n_k)$ and $b(n_1, \cdots, n_k)$ denote the positive constants given by

$$
(3.4)\quad c(n_1, \cdots, n_k) = \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)},
$$

$$
(3.5)\quad b(n_1, \cdots, n_k) = \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\Big).
$$

THEOREM 3.2. *Let M be an n-dimensional CR-submanifold of an m-dimensional locally conformal Kaehler space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. Then, for any point $p \in M$ and any plane section $\pi \subset T_pM$, we have*

$$(3.6)\quad \delta_M \le \frac{n-2}{2}\left(\frac{n^2}{n-1}||H||^2 + \frac{1}{4}(n+1)(c+6\sigma)\right) + \frac{3}{4}d(c-2\sigma).$$

The equality holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ for T_pM and an orthonormal basis $\{e_{n+1}, \cdots, e_{2m}\}$ for $T_p^{\perp}M$ such that (a) $\pi = Span\{e_1, e_2\}$ (b) the shape operator $A_r = A_{e_r}, r = n+1, \cdots, 2m$, take the following forms :

$$(3.7)\qquad A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix},$$

$$(3.8)\qquad A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $a + b = c$ and $c_r, d_r \in \mathbb{R}$.

Proof. Let p be a point of M and let π be a plane section contained in the tangent space T_pM of M at p. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ for T_pM such that $\{e_1, \cdots, e_d, e_{d+1} = Je_1, \cdots, e_{2d} = Je_d\}$ is the basis of D and $\{e_{2d+1}, \cdots, e_n\}$ is the basis of $D^{\perp}$ and $\{e_{n+1}, \cdots, e_{2m}\}$ for the normal space $T_p^{\perp}M$ at p such that e_1 and e_2 generate the plane section π and the normal vector e_{n+1} is in the direction of the mean curvature vector H. Then the equation of Gauss gives

$$(3.9)\qquad K(\pi) = K(e_1 \wedge e_2) = \frac{c}{4} + \frac{3}{2}\sigma + h_{11}^{n+1}h_{22}^{n+1} + \sum_{r=n+2} h_{11}^r h_{22}^r - (h_{12}^{n+1})^2 - \sum_{r=n+2} (h_{12}^r)^2.$$

We put

$$(3.10)\quad \eta = 2\tau - \frac{n^2(n-2)}{n-1}||H||^2 - \frac{1}{4}n(n-1)(c+6\sigma) - \frac{3}{2}d(c-2\sigma).$$

Substituting (3.3) into (3.10), we have

$$(3.11)\qquad n^2||H||^2 = (n-1)(||h||^2 + \eta),$$

in other words,

$$(3.12)\quad \Big(\sum_{i=1}^{n}(h_{ii}^{n+1})\Big)^2=(n-1)\Big(\sum_{i=1}^{n}(h_{ii}^{n+1})^2+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2+\eta\Big).$$

Thus, we have

$$(3.13)\quad K(\pi)\geq\frac{c}{4}+\frac{3}{2}\sigma+\frac{\eta}{2}+\sum_{r=n+1}^{2m}\sum_{j>2}\{(h_{1j}^r)^2+(h_{2j}^r)^2\}+\frac{1}{2}\sum_{i\neq j>2}(h_{ij}^{n+1})^2$$
$$+\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}(h_{ij}^r)^2+\frac{1}{2}\sum_{r=n+2}^{2m}(h_{11}^r+h_{22}^r)^2.$$

Making use of (3.10), we get (3.6).

Suppose the equality of (3.6) holds. Then, the terms involving h_{ij}^r's in (3.13) vanish at the same time and thus

$$h_{1j}^{n+1}=h_{2j}^{n+1}=0,\quad h_{ij}^{n+1}=0,\quad i\neq j>2,$$
$$h_{1j}^{r}=h_{2j}^{r}=h_{ij}^{r}=0,\quad r=n+2,\cdots,2m;i,j\geq 3,$$
$$h_{11}^{r}+h_{22}^{r}=0,\quad r=n+2,\cdots,2m.$$

Moreover, we may choose e_1 and e_2 such that $h_{12}^{n+1}=0$. Also, Lemma 3.1 implies that

$$h_{11}^{n+1}+h_{22}^{n+1}=h_{33}^{n+1}=\cdots=h_{nn}^{n+1}.$$

Therefore, the shape operator A_r ($r=n+1,\cdots,2m$) take the form (3.7) and (3.8). The converse is obvious. □

COROLLARY 3.3 ([9]). *Let M be an n-dimensional totally real submanifold of an m-dimensional locally conformal Kaehler space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. Then we have*

$$\delta_M\leq\frac{n-2}{2}\Big(\frac{n^2}{n-1}||H||^2+\frac{1}{4}(n+1)(c+6\sigma)\Big).$$

The equality holds at a point $p\in M$ if and only if there exists an orthonormal basis $\{e_1,\cdots,e_n\}$ for T_pM and an orthonormal basis $\{e_{n+1},\cdots,e_{2m}\}$ for $T_p^{\perp}M$ such that (a) $\pi=Span\{e_1,e_2\}$ (b) the shape operator $A_r=A_{e_r},r=n+1,\cdots,2m$, take the following forms :

$$A_{n+1}=\begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix},$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $a+b=c$ *and* $c_r, d_r \in \mathbb{R}$.

THEOREM 3.4. *Let* M *be an* n*-dimensional CR-submanifold of an* m*-dimensional locally conformal Kaehler space form* $\tilde{M}(c)$ *of constant holomorphic sectional curvature* c. *Then we have*

$$(3.14)\quad \delta(n_1,\cdots,n_k) \leqq c(n_1,\cdots,n_k)||H||^2 + b(n_1,\cdots,n_k)\frac{c+6\sigma}{4} + \frac{3}{4}d(c-2\sigma)$$

for any k*-tuple* $(n_1,\cdots,n_k) \in \mathcal{S}(n)$.

The equality case of inequality (3.14) *holds at a point* $p \in M$ *if and only if there exists an orthonormal basis* $e_1,\cdots,e_{2m}$ *at* p *such that the shape operators of* M *in* $\tilde{M}(c)$ *at* p *take the following forms:*

$$(3.15)\qquad A_r = \begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & \mathbf{0} \\ 0 & \dots & A_k^r & \\ & \mathbf{0} & & \mu_r I \end{pmatrix}, \qquad r = n+1,\cdots,2m,$$

where I *is an identity matrix and* A_j^r *are symmetric* $n_j \times n_j$ *submatrices such that*

$$(3.16)\qquad trace(A_1^r) = \cdots = trace(A_k^r) = \mu_r.$$

Proof. Let M be a CR-submanifold of a locally conformal Kaehler space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c.

If $k=1$, this was done in Theorem 3.2. Hence, we assume $k>1$.

Let $(n_1,\cdots,n_k) \in \mathcal{S}(n)$. Put

$$(3.17)\quad \eta = 2\tau - \frac{1}{4}n(n-1)(c+6\sigma) - \frac{n^2(n+k-1-\sum n_j)}{(n+k-\sum n_j)}||H||^2 - \frac{3}{2}d(c-2\sigma).$$

Substituting (3.3) in (3.17), we have

$$(3.18)\qquad n^2||H||^2 = \gamma(\eta + ||h||^2), \quad \gamma = n+k-\sum n_j.$$

From here on, the calculations run parallel to those in the proof of Theorem 3.2 in [7]; note however that η is defined differently. Using the same notations we also arrives at the same inequality (3.17) in [7], but with η now given by (3.17) instead. This immediately leads to the inequalities (3.14). Also the conditions for the equality-case can be read off in the same way, thus finishing the proof of Theorem 3.4. This completes the proof. □

COROLLARY 3.5 ([7]). *Let M be an n-dimensional totally real submanifold of an m-dimensional locally conformal Kaehler space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. Then we have*

$$\delta(n_1, \cdots, n_k) \leqq c(n_1, \cdots, n_k)||H||^2 + b(n_1, \cdots, n_k)\frac{c + 6\sigma}{4}$$

for any k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$.

The equality holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \ldots, e_{2m}$ at p such that the shape operators of M in $\tilde{M}(c)$ at p take the following forms:

$$A_r = \begin{pmatrix} A_1^r & \ldots & 0 & \\ \vdots & \ddots & \vdots & \mathbf{0} \\ 0 & \ldots & A_k^r & \\ & \mathbf{0} & & \mu_r I \end{pmatrix}, \quad r = n+1, \cdots, 2m,$$

where I is an identity matrix and each A_j^r are symmetric $n_j \times n_j$ submatrices such that

$$trace(A_1^r) = \cdots = trace(A_k^r) = \mu_r.$$

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