DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

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COINCIDENCE POINTS OF WEAKLY INWARD f-CONTRACTIONS

Abstract. Some coincidence point theorems for weakly inward multivalued f-contractions are proved. Thus several related results in the literature are either extended or improved.

1. Introduction and preliminaries

Let $X = (X, d)$ be a metric space and S a nonempty subset of X. We denote by $CD(S)$ the family of nonempty closed subsets of S, by $CB(S)$ the family of nonempty closed bounded subsets of S, and by $K(X)$ the family of nonempty compact subsets of S. If d is not bounded, we can replace d by a bounded equivalent metric, say, $max\{d, 1\}$. Let H be the Hausdorff metric on $CD(X)$ and $T : S \longrightarrow CD(X)$ be a multivalued map. Then T is said to be a contraction if there exists $0 \leq k < 1$ such that $H(T(x), T(y)) \leq kd(x, y)$ for $x, y \in S$. If $k = 1$, then T is called nonexpansive. A point x^* is a fixed point of T if $x^* \in T(x^*)$. Let $f : S \longrightarrow X$ be a map and $R(f)$ denote the range of f. Then T is called an f-contraction if there exists $0 \leq k < 1$ such that $H(T(x), T(y)) \leq kd(f(x), f(y))$ for $x, y \in S$. If $k = 1$, then T is called an f-nonexpansive map. A point x^* is a coincidence point of f and T if $f(x^*) \in T(x^*)$.

Let S be a subset of a Banach space X. Then the set S is called star-shaped if there exists a fixed element $q \in S$ such that $(1 - k)x + kq \in S$ for any $x \in S$ and $0 < k < 1$. A multivalued map $T : S \longrightarrow CD(X)$ is said to be (1) weakly inward if $T(x) \subset clI_S(x)$ (the closure of $I_S(x)$) for all $x \in S$, where $I_S(x) = \{x + \lambda(z - x) : z \in S, \lambda \geq 1\}$; (2) demiclosed if $\{x_n\} \subset S$ and $y_n \in T(x_n)$ are sequences such that $\{x_n\}$ converges weakly to x_0 and $\{y_n\}$

Key words and phrases: coincidence point, weak inwardness condition, f-contraction, Banach space.
1991 *Mathematics Subject Classification*: 47H10, 54H25.

converges to y_0 in X, then $x_0 \in S$ and $y_0 \in T(x_0)$. For any set $A \subset X$, ∂A stands for boundary of A.

The well-known Banach contraction principle was extended to a multivalued contraction by Nadler [12] in 1969. A proper generalization of Nadler's result was obtained by Kaneko [5]. Afterwards, Daffer and Kaneko [2] extended a coincidence point result of Keneko [5] to include the class of multivalued f-contractive maps. On the other hand, several fixed point theorems for multivalued nonself-mappings have been obtained by a number of authors, see, for instance, the work of Assad and Kirk [1], Downing and Kirk [3], Kirk and Massa [6], Lim [9, 10], Xu [13], Yi and Zhao [14], Zhang [15], Zhong, Zhu and Zhao [16], etc. Recently, Latif and Tweddle [8] obtained several coincidence point theorems for nonself f-contraction and f-nonexpansive maps without commutativity assumptions, which unify and extend various known fixed point theorems. Most of their results involve compact-valued maps. It is natural to ask the question whether the conclusions of their results remain valid if the maps are closed-valued. In this paper, we prove some coincidence point theorems for weakly inward f-contraction and f-nonexpansive maps which only takes closed values. We also consider a new class of f-contraction maps with a non-constant contractive coefficient. Thus, the results of Itoh and Takahashi [4], Lami Dozo [7], Latif and Tweddle [8], Martinez-Yanez [11], and Zhang [15] are either extended or improved.

2. Main results

THEOREM 2.1. *Let S be a nonempty subset of a complete metrically convex metric space X. Let $f : S \longrightarrow X$ be any map such that $R(f)$ is closed and let $T : S \longrightarrow CD(X)$ be an f-contraction. Assume that for any $t \in R(f)$, $T(x) \subset cl\hat{I}_{R(f)}(t)$ for $x \in f^{-1}(t)$, where $\hat{I}_{R(f)}(t) = \{u \in X : \exists s \in R(f)$ s.t. $s \in [t, u]\}$. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.*

Proof. The proof follows ideas of Latif and Tweddle [8]. Let $F : R(f) \longrightarrow CD(X)$ be a mapping defined by $F(t) = \cup_{x \in f^{-1}(t)} T(x)$. Since T is constant on $f^{-1}(t)$, it follows that $F(t) = T(x)$ for all $x \in f^{-1}(t)$. Furthermore, F is an f-contraction on $R(f)$. Indeed, fix any $s, t \in R(f)$. Then $H(F(s), F(t)) = H(T(x), T(y))$ for all $(x, y) \in f^{-1}(s) \times f^{-1}(t)$. It further implies that $H(F(s), F(t)) \leq hd(s, t)$. Also F is weakly inward in the sense that for any $t \in R(f)$, $F(t) \subset cl\hat{I}_{R(f)}(t)$. By Theorem 2 of Lim [9], F has a fixed point, that is, there exists $t \in R(f)$ such that $t \in F(t)$. Hence, taking $x^* \in f^{-1}(t)$, we obtain $f(x^*) \in T(x^*)$.

COROLLARY 2.2. [8, Theorem 2.1] *Let S be a nonempty subset of a complete metrically convex metric space X. Let $f : S \longrightarrow X$ be any map such that $R(f)$ is closed and let $T : S \longrightarrow CB(X)$ be an f-contraction. Assume that*

$T(x) \subset R(f)$ for all $f(x) \in \partial R(f)$. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.

In case X is a Banach space, we obtain the following result, which improves Theorem 2.2 of Latif and Tweddle [8].

THEOREM 2.3. *Let S be a nonempty subset of a Banach space X. Let $f : S \longrightarrow X$ be any map such that $R(f)$ is closed and let $T : S \longrightarrow CD(X)$ be an f-contraction. Assume that for any $t \in R(f)$, $T(x) \subset clI_{R(f)}(t)$ for $x \in f^{-1}(t)$. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.*

Proof. The proof is exactly the same as that of Theorem 2.1 we only need to notice that the corresponding map F has a fixed point by Theorem 1 of Lim [9].

REMARK 2.4. Theorem 2.3 extends Theorem 1 of Martinez-Yanez [11] to multivalued maps.

Now we prove a coincidence point theorem for a class of new multivalued f-contractions. Let $h : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous nondecreasing function satisfying $\int_0^{+\infty} \frac{dr}{1+h(r)} = +\infty$.

THEOREM 2.5. Let S be a nonempty subset of a Banach space X. Let $f : S \longrightarrow X$ be any map such that $R(f)$ is closed and let $T : S \longrightarrow K(X)$ be a mapping, $x_0 \in S$ a given point and $\sigma \in (0, 1]$ a constant such that for any $x, y \in S$

$$H(T(x), T(y)) \leq \left(1 - \frac{\sigma}{1 + h(d(f(x_0), f(x)))}\right) d(f(x), f(y)).$$

Assume that for any $t \in R(f)$, $T(x) \subset clI_{R(f)}(t)$ for $x \in f^{-1}(t)$. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.

Proof. Define $F : R(f) \longrightarrow K(X)$ by the same way as in the proof of Theorem 2.1. Then F is weakly inward. Fix any $s, t \in R(f)$. Now $H(F(s), F(t)) = H(T(x), T(y))$ for all $(x, y) \in f^{-1}(s) \times f^{-1}(t)$, which implies that

$$\begin{aligned} H(F(s), F(t)) &\leq \left(1 - \frac{\sigma}{1 + h(d(f(x_0), f(x)))}\right) d(f(x), f(y)) \\ &= \left(1 - \frac{\sigma}{1 + h(d(t_0, s))}\right) d(s, t), \end{aligned}$$

where $t_0 = f(x_0)$. By Theorem 2.5 of Zhong, Zhu and Zhao [16], F has a fixed point in $R(f)$, that is, there exists $t \in R(f)$ such that $t \in F(t)$. Hence, taking $x^* \in f^{-1}(t)$, we get $f(x^*) \in T(x^*)$.

REMARK 2.6. Theorem 2.5 can not be extended to multivalued f-nonexpansive maps; not even for the case when $f = I$, the identity map on S, see, for instance, Example 2.6 of Zhong, Zhu and Zhao [16]. It would be an interesting problem to prove Theorem 2.5 when T only takes closed-values. However, using arguments similar to those of Xu [13], it can be shown that Theorem 2.5 is still true if T is closed-valued and satisfying the condition that each $x \in f^{-1}(t)$ has a nearest point in Tx.

The following theorems are basically due to Latif and Tweddle [8], which were proved for a compact-valued map T. In view of Theorem 2.2, we observe that the conclusions of the results remain valid if T assumes only closed values. We omit their proofs.

THEOREM 2.7. *Let S be a nonempty subset of a Banach space X. Let $f : S \longrightarrow X$ be any map such that $R(f)$ is closed, bounded and starshaped and let $T : S \longrightarrow CD(X)$ be an f-nonexpansive map. Assume that $(f - T)(S)$ is closed and that for any $t \in R(f)$, $T(x) \subset clI_{R(f)}(t)$ for $x \in f^{-1}(t)$. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.*

THEOREM 2.8. *Let S be a nonempty weakly compact subset of a Banach space X. Let $f : S \longrightarrow X$ be a weakly continuous map such that $R(f)$ is starshaped and let $T : S \longrightarrow CD(X)$ be an f-nonexpansive map. Assume that $f - T$ is demiclosed and that for any $t \in R(f)$, $T(x) \subset clI_{R(f)}(t)$ for $x \in f^{-1}(t)$. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.*

In case $f = I$, we at once obtain the following result.

COROLLARY 2.9. *Let S be a nonempty weakly compact starshaped subset of a Banach space X. Let $T : S \longrightarrow CD(X)$ be a weakly inward nonexpansive map. Assume that $I - T$ is demiclosed. Then there exists $x^* \in S$ such that $f(x^*) \in T(x^*)$.*

REMARK 2.10. Corollary 2.9 improves Theorem 3.8 of Zhang [15] and contains Theorem 3.2 of Lami Dozo [7] as well as a result of Itoh and Takahashi [4] as special cases.

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Received February 8, 2002.