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SETS OF SEQUENCES THAT ARE STRONGLY τ-BOUNDED AND MATRIX TRANSFORMATIONS BETWEEN THESE SETS

Abstract. In this paper we are giving some new properties of the operator of first-difference mapping a space s_τ into itself and we are dealing with the spaces $s_\tau(\Delta^h)$ and $s_\tau((\Delta^+)^h)$. Next are given some properties of the spaces $\widetilde{w_\tau}(\lambda,\mu)$, $\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu')$ generalizing the space $w_\infty(\lambda)$ and the space $c_\infty(\lambda)$ of sequences that are strongly bounded. Then are given some properties of matrix transformations between these spaces.

1. Introduction, notations and preliminary results

In this work, we shall use the infinite linear system of the form:

$$(1) \qquad \sum_{m=1}^{+\infty} a_{nm}x_m = b_n \qquad n = 1, 2, \dots.$$

where the sequences $(a_{nm})_{n,m\geq 1}$ and $(b_n)_{n\geq 1}$ are given, $(x_n)_{n\geq 1}$ being the unknown sequence. This system is equivalent to the single matrix equation

$$(2) \qquad AX = B,$$

where $A = (a_{nm})_{n,m\geq 1}$, n being the index of the n-th row, m the one of the m-th column, n and m being integers greater than 1; $X = (x_n)_{n\geq 1}$ and $B = (b_n)_{n\geq 1}$ are one-column matrices.

A Banach space E of complex sequences with the norm $\|\cdot\|_E$ is a BK space if each projection $P_n : X \to P_nX$ is continuous. Let s is the set of all sequences. For any sequence $\tau = (\tau_n)_{n\geq 1}$ such that $\tau_n > 0$ for every n, we denote by s_τ the Banach space

$$(3) \qquad s_\tau = \{(x_n)_{n\geq 1} \in s \mid x_n = O(\tau_n) \quad n \to \infty\},$$

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normed by

(4) $$\|X\|_{s_\tau} = \sup_{n\geq 1}\left(\frac{|x_n|}{\tau_n}\right).$$

We denote by S_τ the Banach algebra

(5) $$S_\tau = \left\{A = (a_{nm})_{n,m\geq 1} \mid \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\tau_m}{\tau_n}\right) < \infty\right\},$$

normed by

(6) $$\|A\|_{S_\tau} = \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\tau_m}{\tau_n}\right).$$

If $\|I - A\|_{S_\tau} < 1$, we shall say that $A \in \Gamma_\tau$. S_τ being a unital algebra, we have the useful result: if $A \in \Gamma_\tau$, A is invertible in the space S_τ, and for every $B \in s_\tau$, equation (2) admits a unique solution in s_τ, given by

(7) $$X = \sum_{i=0}^{\infty}(I - A)^i B.$$

If $\tau = (r^n)_{n\geq 1}$, Γ_τ, S_τ and s_τ are replaced by Γ_r, S_r and s_r respectively (see [2], [4–8]). When $r = 1$, we obtain the space of all bounded sequences $l^\infty = s_1$.

For any subset E of s, we put

(8) $$AE = \{Y \in s \ / \ \exists X \in E \quad Y = AX\}.$$

If F is a subset of s, we shall denote

(9) $$F(A) = F_A = \{X \in s \ / \ Y = AX \in F\}.$$

We can see that $F(A) = A^{-1}F$. If A maps E into F, we write that $A \in (E, F)$, see [3]. Remark that $A \in (s_\tau, s_\tau)$ iff $A \in S_\tau$.

2. Some properties of the operator Δ^h for h real

2.1. Properties of Δ^h relatively to s_r

The well-known operator $\Delta^{(h)}: s \to s$ where h is an integer ≥ 1, is represented by the infinite lower triangular matrix Δ^h where $\Delta = \begin{pmatrix} 1 & & O \\ -1 & 1 & \\ O & . & . \end{pmatrix}$.

This definition can be generalized to the case when h is a real, see [1, 17]. For this, recall that we can associate to any power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ defined in the open disk $|z| < R$ the upper triangular infinite matrix

$A = \varphi(f) \in \bigcup_{0<r<R} S_r$ defined by

$$\varphi(f) = \begin{pmatrix} a_0 & a_1 & a_2 & . \\ & a_0 & a_1 & . \\ O & & a_0 & . \\ & & & . \end{pmatrix},$$

(see [7], [13]). Practically we shall write $\varphi[f(z)]$ instead of $\varphi(f)$. We have

LEMMA 1. *i) The map $\varphi : f \to A$ is an isomorphism from the algebra of the power series defined in $|z| < R$, into the algebra of the corresponding matrices $\bar{A}$.*

ii) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_0 \neq 0$, and assume that $\frac{1}{f(z)} = \sum_{k=0}^{\infty} a'_k z^k$ admits $R' > 0$ as radius of convergence. We then have

$$\varphi\left(\frac{1}{f}\right) = [\varphi(f)]^{-1} \in \bigcup_{0<r<R'} S_r.$$

So, if $h \in R - N$, we denote

$$\begin{cases} \begin{pmatrix} -h+k-1 \\ k \end{pmatrix} = \dfrac{-h(-h+1)\dots(-h+k-1)}{k!} & \text{if } k > 0, \\ \begin{pmatrix} -h+k-1 \\ k \end{pmatrix} = 1 & \text{if } k = 0. \end{cases}$$

If we write $\Delta^t = \Delta^+$, we have for any $h \in R$

$$(\Delta^+)^h = \varphi[(1-z)^h] = \varphi\left[\sum_{k=0}^{\infty} \begin{pmatrix} -h+k-1 \\ k \end{pmatrix} z^k\right] \quad \text{for } |z| < 1.$$

We deduce that if $\Delta^h = (\tau_{nm})_{n,m\geq 1}$,

$$\tau_{nm} = \begin{cases} \begin{pmatrix} -h+n-m-1 \\ n-m \end{pmatrix} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{10}$$

Using the isomorphism φ, we get:

PROPOSITION 1. ([7]) *i) The operator represented by Δ is bijective from s_r into itself, for every $r > 1$ and Δ^+ is bijective from s_r into itself, for all r, $0 < r < 1$.*

ii) Δ^+ is surjective and not injective from s_r into itself, for all $r > 1$.

iii) $\forall r \neq 1$ and for every integer $h \geq 1$ $(\Delta^+)^h s_r = s_r$.

iv) We have successively

α) If h is a real > 0 and $h \notin N$, then Δ^h maps s_r into itself when $r \geq 1$ but not for $0 < r < 1$.

If $-1 < h < 0$, *then* Δ^h *maps* s_r *into itself when* $r > 1$ *but not for* $r = 1$.

β) *If* $h > 0$ *and* $h \notin N$, *then* $(\Delta^+)^h$ *maps* s_r *into itself when* $0 < r \leq 1$ *but not if* $r > 1$.

If $-1 < h < 0$, *then* $(\Delta^+)^h$ *maps* s_r *into itself for* $0 < r < 1$ *but not for* $r = 1$.

v) For a given integer $h \geq 1$, *we have successively*

$$\begin{cases} \forall r > 1 : A \in (s_r(\Delta^h), s_r) \Leftrightarrow \ \sup_{n\geq 1}(\sum_{m=1}^{\infty} |a_{nm}| \, r^{m-n}) < \infty, \\ \forall r \in]0,1[: A \in (s_r((\Delta^+)^h), s_r) \Leftrightarrow \sup_{n\geq 1}(\sum_{m=1}^{\infty} |a_{nm}| \, r^{m-n}) < \infty. \end{cases}$$

vi) For every integer $h \geq 1$

$$s_1 \subset s_1(\Delta^h) \subset s_{(n^h)_{n\geq 1}} \subset \bigcap_{r>1} s_r.$$

2.2. Properties of well-known operators mapping s_τ into itself

In this subsection we shall consider the well-known operators $C(\lambda)$ and $\Delta(\lambda)$, see [15], [11]. We shall see that these operators are obtained from Δ and $\Delta^{-1} = \Sigma$. Then we establish some properties of the spaces $s_\tau(\Delta^h)$, $l^\infty(C(\tau))$ and of the sequence $C(\tau)\tau$.

Put $U = \{(u_n)_{n\geq 1}/ \quad u_n \neq 0 \ \forall n\}$. If $\lambda = (\lambda_n)_{n\geq 1} \in U$, we have

$$C(\lambda) = \begin{pmatrix} 1/\lambda_1 & & & O & \\ 1/\lambda_2 & 1/\lambda_2 & & & \\ . & . & . & & \\ 1/\lambda_n & . & 1/\lambda_n & 1/\lambda_n & \\ & & & . & . \end{pmatrix}.$$

If $\xi = (\xi_n)_{n\geq 1}$ is a given sequence, we put $D_\xi = (\xi_n \delta_{nm})_{n,m\geq 1}$, where $\delta_{nm} = 0$ for $n \neq m$, and $\delta_{nn} = 1$ for all n. One gets $C(\lambda) = D_{1/\lambda}\Sigma$. Since $\Sigma^t = \varphi(\sum_{n=0}^{\infty} z^n) = \varphi(\frac{1}{1-z})$ with $|z| < 1$, Σ is the lower infinite triangular matrix all of whose entries below the main diagonal are equal to 1. Further, $(C(\lambda))^{-1} = \Sigma^{-1} D_\lambda = \Delta D_\lambda$. If we let $\Delta(\lambda) = \Delta D_\lambda$, then

$$\Delta(\lambda) = \begin{pmatrix} \lambda_1 & & & O & \\ -\lambda_1 & \lambda_2 & & & \\ & . & . & & \\ O & & -\lambda_{n-1} & \lambda_n & \\ & & & . & . \end{pmatrix},$$

is the inverse of $C(\lambda)$.

Further, we shall say that $\xi = (\xi_n)_{n\geq 1} \in \Gamma$ if

$$\overline{\lim_{n\to\infty}}\left(\left|\frac{\xi_{n-1}}{\xi_n}\right|\right) < 1.$$

Note that $\Delta \in \Gamma_{|\xi|}$ implies $\xi \in \Gamma$. Now we can express the following

PROPOSITION 2. *i) $\tau \in \Gamma$ if and only if there is an integer $q \geq 1$ such that*

$$\gamma_q(\tau) = \sup_{n\geq q+1}\left(\frac{\tau_{n-1}}{\tau_n}\right) < 1.$$

ii) If $\tau \in \Gamma$, we deduce the following properties

α) Δ is bijective from s_τ into itself.

β) There is an integer $q \geq 1$, and two reals $M > 0$ and $\kappa > 1$, for which $\tau_n \geq M\kappa^n$ for all $n \geq q+1$.

γ) $l^\infty(C(\tau)) = s_\tau$.

δ) $C(\tau)\tau \in l^\infty$.

ε) For any positive integer h, we have $s_\tau(\Delta^h) = s_\tau$.

iii) We get successively

α)

$$s_\tau(\Delta) = s_\tau \Leftrightarrow l^\infty(C(\tau)) = s_\tau \Leftrightarrow C(\tau)\tau \in l^\infty.$$

β) Let h be any fixed integer ≥ 1 and assume that $(\frac{\tau_{n-1}}{\tau_n})_{n\geq 2} \in l^\infty$, then

$$s_\tau(\Delta^h) = s_\tau \Leftrightarrow l^\infty(C(\tau)) = s_\tau \Leftrightarrow C(\tau)\tau \in l^\infty.$$

Proof. i) If $\tau \in \Gamma$, $l = \overline{\lim}_{n\to\infty}(\frac{\tau_{n-1}}{\tau_n}) < 1$ and $\inf_{q\geq 1}(\gamma_q(\tau)) < 1$. Hence there exists ε_0, $0 < \varepsilon_0 < 1 - l$ and an integer q_0 such that $l \leq \gamma_{q_0}(\tau) \leq l + \varepsilon_0$, which proves the necessary condition. Conversely, if there exists $q \geq 1$ such that $\gamma_q(\tau) < 1$, then

$$\inf_{q\geq 1}(\gamma_q(\tau)) = \overline{\lim_{n\to\infty}}\left(\frac{\tau_{n-1}}{\tau_n}\right) < 1.$$

Assertion ii) First let us prove that $\tau \in \Gamma$ implies that Δ is bijective from s_τ into itself. Denote for any integer $q \geq 1$ by $\Sigma^{(q)}$ the infinite matrix

$$\begin{pmatrix} [\Delta^{(q)}]^{-1} & & O \\ & & 1 & \\ & O & & . \end{pmatrix},$$

where $\Delta^{(q)}$ is the finite matrix whose elements are those of the q first rows and of the q first columns of Δ. We get $\Sigma^{(q)}\Delta = (a_{nm})_{n,m\geq 1}$, with $a_{nn} = 1$ for all n, $a_{nn-1} = -1$ for all $n \geq q+1$, and $a_{nm} = 0$ otherwise. We see that if $\tau \in \Gamma$, there exists an integer $q \geq 1$ such that

$$\|I - \Sigma^{(q)}\Delta\|_{s_\tau} = \gamma_q(\tau) < 1.$$

For all $B \in s_\tau$, we see that $\Sigma^{(q)}B \in s_\tau$. Then equation $\Delta X = B$ being equivalent to $(\Sigma^{(q)}\Delta)X = \Sigma^{(q)}B$ admits only one solution in s_τ for all $B \in s_\tau$. This proves that Δ is bijective from s_τ into itself.

Now, from i) we deduce that if $\tau \in \Gamma$ there exists an integer $q \geq 1$ and a real $\varkappa$, with $0 < \varkappa < 1$, for which $n \geq q+1$ implies $\frac{\tau_{n-1}}{\tau_n} \leq \varkappa$. Then $\tau_n \geq \frac{1}{\varkappa^n}\tau_q\varkappa^q$ for $n \geq q+1$, and we obtain β), in which $\kappa = 1/\varkappa$ and $M = \tau_q\varkappa^q$.

Let us prove that $\tau \in \Gamma$ implies γ). First we have $(C(\tau))^{-1} = \Delta D_\tau$. It is obvious that D_τ is bijective from l^∞ into s_τ and as we have seen above Δ is bijective from s_τ into itself. Then $l^\infty(C(\tau)) = \Delta D_\tau l^\infty = \Delta s_\tau = s_\tau$.

$\tau \in \Gamma$ implies $C(\tau)\tau \in l^\infty$. Indeed, since $\tau \in \Gamma$ equation

$$\Delta X = \tau,$$

(where $\tau \in s_\tau$) admits in s_τ the unique solution $X = \Sigma.\tau$. This means that $\frac{\tau_1+\ldots+\tau_n}{\tau_n} = O(1)$ as $n \to \infty$ and we conclude that $C(\tau)\tau \in l^\infty$.

Since $\tau \in \Gamma$ implies α), we deduce that Δ^h is bijective from s_τ into itself. Then $s_\tau(\Delta^h) = s_\tau$ and $\tau \in \Gamma$ implies ε).

Assertion iii) α). Since the matrix Δ is lower triangular, we deduce that $s_\tau(\Delta) = s_\tau$ iff Δ is bijective from s_τ to s_τ. From the identities $s_\tau = D_\tau l^\infty$ and $(C(\tau))^{-1} = \Delta(\tau)$, we deduce that

$$s_\tau(\Delta) = s_\tau \Leftrightarrow \Delta s_\tau = s_\tau \Leftrightarrow \Delta(\tau)l^\infty = l^\infty(C(\tau)) = s_\tau.$$

Now let us show that $C(\tau)\tau \in l^\infty$ iff $l^\infty(C(\tau)) = s_\tau$. First prove that if $C(\tau)\tau \in l^\infty$ then $l^\infty(C(\tau)) = s_\tau$. Take $X = (x_n)_n \in l^\infty(C(\tau))$. Then $C(\tau)X \in l^\infty$, i.e. $X \in \Delta(\tau)l^\infty$. Hence there exists $(b_n)_n \in l^\infty$ such that $X = (\tau_n b_n - \tau_{n-1}b_{n-1})_{n\geq 1}$, with the convention $b_0 = 0$. And since $C(\tau)\tau \in l^\infty$ implies that $\frac{\tau_{n-1}}{\tau_n} = O(1)$ $(n \to \infty)$ then

$$\frac{x_n}{\tau_n} = b_n - \frac{\tau_{n-1}}{\tau_n}b_{n-1} = O(1) \quad n \to \infty$$

and $X \in s_\tau$. This proves that $l^\infty(C(\tau)) \subset s_\tau$.

Now let us show that $s_\tau \subset l^\infty(C(\tau))$. Let $X \in s_\tau$. If we put $Y = (y_n)_n = C(\tau)X$ and if $[C(\tau)\tau]_n$ denotes the n-th coordinate of $C(\tau)\tau$, there is a real M such that

$$|y_n| = \left|\frac{1}{\tau_n}\sum_{k=1}^{n} x_k\right| \leq \frac{M}{\tau_n}\sum_{k=1}^{n}\tau_k = M[C(\tau)\tau]_n \quad \forall n.$$

We conclude that $Y \in l^\infty$ since $C(\tau)\tau \in l^\infty$ and $X \in l^\infty(C(\tau))$. Finally, if $l^\infty(C(\tau)) = s_\tau$ then $\tau \in l^\infty(C(\tau))$ and $C(\tau)\tau \in l^\infty$. This achieves the proof of iii) α).

Assertion iii) β). First, note that if $(\frac{\tau_{n-1}}{\tau_n})_{n\geq 2} \in l^\infty$, then $\Delta \in S_\tau$ and $\Delta \in (s_\tau, s_\tau)$. From iii) α) we see that it is enough to prove that $s_\tau(\Delta^h) = s_\tau$ iff $s_\tau(\Delta) = s_\tau$. If $s_\tau(\Delta) = s_\tau$ we deduce that Δ is bijective from s_τ to s_τ since the map $X \to \Delta X$ is injective. This implies that Δ^h is bijective from s_τ to s_τ and $s_\tau(\Delta^h) = s_\tau$. Conversely, suppose that $s_\tau(\Delta^h) = s_\tau$. First, we see that $\Delta^{h-1}s_\tau \subset s_\tau$, since $(\frac{\tau_{n-1}}{\tau_n})_{n\geq 2} \in l^\infty$ implies that $\Delta s_\tau \subset s_\tau$. On the other hand for any given $B \in s_\tau$ the equation $\Delta^h X = B$ admits a unique solution in s_τ and

$$B = \Delta(\Delta^{h-1}X) \in \Delta s_\tau. \tag{11}$$

This proves that $s_\tau \subset \Delta s_\tau$. Finally, $\Delta s_\tau = s_\tau$ and Δ is bijective from s_τ to s_τ, which gives the conclusion.

REMARK 1. The converse of the property "$\tau \in \Gamma$ implies $C(\tau)\tau \in l^\infty$" in the previous proposition is false. Let us show that there is $\tau \notin \Gamma$ such that $C(\tau)\tau \in l^\infty$. Indeed, let $\zeta > 1$ and consider the sequence $(\tau_n)_{n\geq 1}$ defined by

$$\tau_n = \begin{cases} \zeta^k & \text{if } n = 2k, \\ \zeta^k & \text{if } n = 2k+1. \end{cases}$$

We see that $\sup_{n\geq q+1}(\frac{\tau_{n-1}}{\tau_n}) = 1$ for all $q \geq 1$. On the other hand we get

$$\sum_{i=1}^{2p} \tau_i = \sum_{i=1}^{p} \tau_{2i-1} + \sum_{i=1}^{p} \tau_{2i} = \frac{\zeta^{p+1} + \zeta^p - \zeta - 1}{\zeta - 1} \quad \text{for all } p.$$

Then

$$\frac{1}{\tau_{2p}} \sum_{i=1}^{2p} \tau_i \leq \frac{\zeta + 1}{\zeta - 1}.$$

Doing analogous calculations, we get

$$\frac{1}{\tau_{2p+1}} \sum_{i=1}^{2p+1} \tau_i \leq \frac{\zeta + 1}{\zeta - 1}.$$

This proves that $C(\tau)\tau \in l^\infty$.

From iii) in Proposition 2 it can be easily seen that if there is $\tau \notin \Gamma$ such that $C(\tau)\tau \in l^\infty$, then $s_\tau(\Delta) = l^\infty(C(\tau)) = s_\tau$.

REMARK 2. Note that the sequence used in the previous remark satisfies β in ii). It is enough to take $\zeta > \kappa^2$.

2.3. Other properties of $(\Delta^+)^h$ relatively to s_τ

In order to express other usefull results on the operators Δ^h and $(\Delta^+)^h$ for h real, we need to recall the next result, see [2], [8].

Let $A(t_1, t_2, \dots t_p)$, $p \in N$, be the matrix obtained from A by addition of the following rows:

$$t_1 = (t_{1m})_{m\geq 1},\ t_2 = (t_{2m})_{m\geq 1}, \dots t_p = (t_{pm})_{m\geq 1}$$

with $t_{kk} \neq 0$ $(k = 1, 2, .., p)$, $t_{ij} \in R$, that is

$$A(t_1, t_2, \dots t_p) = \begin{bmatrix} t_{1,1} & . & . & t_{1,m} & . & . \\ . & . & . & . & . & . \\ t_{p,1} & . & . & t_{p,m} & . & . \\ a_{1,1} & . & . & a_{1,m} & . & . \\ . & . & . & . & . & . \\ a_{n,1} & . & . & a_{n,m} & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Similarly put

$$B(u_1, u_2, \dots u_p) = (u_1, \dots u_p, b_1, b_2, \dots)^t$$

and

$$\eta(u_1, u_2, \dots u_p) = (u_1, \dots u_p, 0, 0, \dots)^t.$$

We shall use the matrix $D_{a'} = D_{(a'^{-1}_{nn})_n}$, where a'^{-1}_{nn} are the inverses of the diagonal elements of the matrix $A(t_1, t_2, \dots t_p)$. Then we get:

PROPOSITION 3. ([8]) *Let $\tau = (\tau_n)$, $\tau_n > 0$ $\forall n$, be a sequence such that*

(12) $\quad D_{a'}A(t_1, t_2, \dots, t_p) \in \Gamma_\tau$ *and* $D_{a'}B(u_1, u_2, \dots, u_p) \in s_\tau$.

Then

i) the solutions of $AX = B$ in the space s_τ are:

$$X = [D_{a'}A(t_1, t_2, \dots t_p)]^{-1} D_{a'}B(u_1, u_2, \dots u_p) \quad u_1, u_2, \dots .u_p \in C.$$

ii) The linear space $KerA \cap s_\tau$ of the solutions of $AX = 0$ in the space s_τ is of dimension p and is given by

$$KerA \cap s_\tau = span(X_1, X_2, \dots ..X_p)$$

where

$$X_k = [A(t_1, t_2, \dots t_p)]^{-1} \eta(0, 0, \dots, 1, 0, \dots 0),\ k = 1, 2, \dots .p,$$

1 being the k-th term of the p-tuple.

In the following we shall use the sequences $e_n = (0, .., 1, \dots)$, (where 1 is in the n-th position), $e = (1, 1, \dots)$ and

$$\begin{cases} V_1 = e^t,\ V_2 = (A_1^1, A_2^1, \dots)^t,\ V_3 = (0, A_2^2, A_3^2, \dots)^t, \dots \\ V_n = (0, 0, \dots, A_{n-1}^{n-1}, A_n^{n-1}, \dots, A_n^{n-1}, \dots)^t, \dots \end{cases}$$

$A_i^j = \frac{i!}{(i-j)!}$, with $0 \leq j \leq i$, being the number of permutations of i things taken j at a time. We obtain:

THEOREM 1. *i) Let h be a real. For every $r > r_h = \frac{2^{\frac{1}{|h|}}}{2^{\frac{1}{|h|}}-1}$, we have*

$$s_r(\Delta^h) = s_r.$$

ii) We get successively

α) Let h be an integer ≥ 1 and take $r > r_h^+ = \frac{1}{2^{\frac{1}{h}}-1}$. Then

$$s_r((\Delta^+)^h) = s_r.$$

Furthermore for any $B \in s_r$, the equation

$$(\Delta^+)^h X = B \tag{13}$$

admits in s_r infinitely many solutions given by

$$X = Z_0 + \sum_{i=1}^{h} u_i V_i \qquad u_1, u_2, \dots, u_h \text{ being arbitrary scalars,}$$

where

$$Z_0 = \left[((\Delta^+)^h)((-1)^h e_1, \dots, (-1)^h e_\mu)\right]^{-1} B(0, \dots, 0).$$

β) Let h be a real > 0 and denote by q the greatest integer strictly less than $h+1$. Then $\dim Ker((\Delta^+)^h) = q$ and

$$Ker((\Delta^+)^h) = span(V_1, \dots, V_q).$$

Proof. Assertion i). We deduce from (10) that

$$\begin{aligned}\|I - \Delta^h\|_{S_r} &= \sup_{n\geq 2}\left(\sum_{m=1}^{n-1} \frac{|h(h-1)\dots(h-n+m+1)|}{(n-m)!}\frac{1}{r^{n-m}}\right)\\ &= \sum_{k=1}^{\infty}\frac{|h(h-1)\dots(h-k+1)|}{k!}\frac{1}{r^k}\\ &\leq \sum_{k=1}^{\infty}\frac{|h|\,(|h|+1)\dots(|h|+k-1)}{k!}\frac{1}{r^k}.\end{aligned}$$

Then, for $r > r_h$, $\ln(1 - \frac{1}{r}) > -\frac{\ln 2}{|h|}$ and

$$\|I - \Delta^h\|_{S_r} \leq \left(1 - \frac{1}{r}\right)^{-|h|} - 1 < 1.$$

We conclude that Δ^h is bijective from s_r into s_r and $s_r(\Delta^h) = s_r$.

ii) α) Let

$$\Delta'_h = [(\Delta^+)^h]((-1)^h e_1, \dots, (-1)^h e_h).$$

Δ'_h is lower triangular with non-zero coefficients on the main diagonal. For any real $r > r_h^+$, we have

$$\|I - (-1)^h \Delta'_h\|_{S_r} = \sum_{k=1}^{h} \frac{1}{r^k}\binom{h}{k} = \left(1 + \frac{1}{r}\right)^h - 1 < 1.$$

From Proposition 3 we deduce that

$$\dim Ker((\Delta^+)^h) \bigcap s_r = h.$$

Further, we can easily verify that the h vectors V_1, V_2 ,.., V_h are linearly independent and belong to $Ker((\Delta^+)^h) \bigcap s_r$. Hence V_1, V_2 ,.., V_h form a basis of this space. Moreover

$$Z_0 = (\Delta'_h)^{-1} B(0, 0, \ldots, 0) \in s_r$$

is a particular solution of equation (13). Then the solutions of this equation are given by

$$X = (\Delta'_h)^{-1} B(u_1, u_2, \ldots, u_h) = Z_0 + \sum_{i=1}^{h} u_i V_i \quad u_1, u_2, \ldots .u_h \in R.$$

ii) β) It is well-known [1] that $Ker((\Delta^+)^h)$ is the set of all the sequences $(P_{q-1}(n))_{n \geq 1}$, P_{q-1} being an arbitrary polynomial of degree less than $q - 1$. Then $\dim Ker((\Delta^+)^h) = q$ and since V_1, V_2,..., $V_q \in Ker((\Delta^+)^h)$ are linearly independent, we conclude that $Ker((\Delta^+)^h) = span(V_1, V_2, \ldots, V_q)$.

3. Generalization of the sets of sequences that are strongly τ-bounded

In this section we recall some properties of the sequence spaces that are strongly τ-bounded. Next we give definitions of the matrices $\widehat{\Delta}(\lambda, \mu)$ and $\widehat{C}(\lambda, \mu)$ generalizing the matrices $\Delta(\lambda)$ and $C(\lambda)$ and we deal with the spaces $\widetilde{w_\tau}(\lambda, \mu)$, $\widetilde{c_\tau}(\lambda, \mu, \lambda', \mu')$ generalizing the well-known space of sequences that are strongly bounded.

3.1. Sequence spaces that are strongly τ-bounded

For every sequence $X = (x_n)_n$, we define $|X| = (|x_n|)_{n \geq 1}$. For $\lambda \in U$, put

(14) $$w_\tau(\lambda) = \{X \in s \ / \ C(\lambda)(|X|) \in s_\tau\}.$$

If there exist A and $B > 0$, such that $A < \tau_n < B$ for all n, we get the well-known space $w_\tau(\lambda) = w_\infty(\lambda)$, (see [15]). If s_τ is replaced by c_0, it is written that $w_\tau(\lambda) = w_0(\lambda)$.

Consider now for $\lambda \in U$, $\lambda' \in s$ the space

(15) $$c_\tau(\lambda, \lambda') = \{X \in s \ / \ C(\lambda)(|\Delta(\lambda')X|) \in s_\tau\}.$$

When $\lambda = \lambda'$ we shall write $c_\tau(\lambda)$ for short. $c_\tau(\lambda)$ is called the set of sequences that are strongly τ-bounded [9]. If s_τ is replaced by c_0, $c_\tau(\lambda)$ is written $c_0(\lambda)$ and is called the set of sequences that are strongly convergent to 0. Similarly when s_τ is replaced by l^∞, $c_\alpha(\lambda) = c_\infty(\lambda)$ is called the set of sequences that are strongly bounded [11], [15], [14]. We can also define the set $c(\lambda)$ of sequences that are strongly convergent by

$$c(\lambda) = \{X \in s \ / \ X - le^t \in c_0(\lambda) \text{ for some } l \in C\}.$$

We have the next result, where λ, $\lambda' \in U$.

THEOREM 2. ([9]) *i) Assume that* $\tau\lambda \in \Gamma$. *Then* $c_\tau(\lambda, \lambda')$ *is a BK space with respect to* $\| \|_{s_{\tau \frac{|\lambda|}{|\lambda'|}}}$. *We have*

$$c_\tau(\lambda, \lambda') = s_\tau(C(\lambda)\Delta(\lambda')) = s_{\tau \frac{|\lambda|}{|\lambda'|}} \quad \textit{and} \quad c_\tau(\lambda) = s_\tau \ . \tag{16}$$

ii) Assume that $\chi = \sup_{n\geq 2} (\frac{1}{\tau_{n-1}|\lambda_{n-1}|}) < \infty$ *and define the sequence* $\xi = (\xi_n)_n$ *by* $0 < \xi_1 < 1/\chi$ *and* $\xi_n = \tau_{n-1} |\lambda_{n-1}|$ *for all* $n \geq 2$. *Then*

$$s_\tau(C(\lambda)\Delta^+) = s_\xi.$$

Moreover for any $B \in s_\tau$ *equation* $(C(\lambda)\Delta^+)X = B$, *admits in* s_ξ *infinitely many solutions given by*

$$X = [(D_\lambda C(\lambda)\Delta^+)(-e_1)]^{-1}(D_\lambda B)(u), \quad \textit{for all scalars } u.$$

REMARK 3. Note that ii) in the previous theorem follows from Proposition 3 in Section 2.

We can deduce from the preceding the next results [9].

COROLLARY 1. *i) Assume that* $\tau\lambda \in \Gamma$.

a- If $\lambda \in U$, $\lambda' \in U$ *with* $(\lambda'_n)_n \nrightarrow \infty$, *then*

$$c_\tau(\lambda, \lambda') \neq l^\infty.$$

b- If $\lambda \in s_{|\lambda'|}$ *and* $\lambda' \in s_{|\lambda|}$, *then* $c_\tau(\lambda, \lambda') = s_\tau$.

ii) Suppose that $(|\lambda_n|)_n$ *is increasing and* $\tau \in \Gamma$, *then*

$$c_\tau(\lambda, \lambda') = s_{\tau \frac{|\lambda|}{|\lambda'|}} \quad \textit{and} \quad c_\tau(\lambda) = s_\tau.$$

iii) Let R *be a real strictly greater than* 1. *If we suppose that* $\lambda \in \Gamma$, *then*

$$c_R(\lambda, \lambda') = s_R(C(\lambda)\Delta(\lambda')) = s_{(R^n |\frac{\lambda_n}{\lambda'_n}|)_n}.$$

iv) If $0 < \xi_1 < [\sup_{n\geq 2} (\frac{1}{|\lambda_{n-1}|})]^{-1}$, *then* $l^\infty(C(\lambda)\Delta^+) = s_\xi$.

REMARK 4. If we assume that $\tau \in l^\infty$ the space $c_\tau(\lambda, \lambda')$ cannot be written in the form s_ζ, $\zeta = (\zeta_n)_n$ being a sequence satisfying $\zeta_n > 0$ $\forall n$. However, it is the case for $l^\infty(C(\lambda)\Delta^+)$.

We can assert the following result concerning matrix transformations, ξ being the sequence defined in ii) of Theorem 2. Using the property,

$$A \in (s_\tau, s_v) \Leftrightarrow \sup_n \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\tau_m}{v_n} \right) < \infty, \tag{17}$$

where $\tau = (\tau_n)_n$, $v = (v_n)_n$ with v_n and $\tau_n > 0$ for all n, (see [9]), we get $(s_\tau, s_v) = S_{\tau,v}$, where

$$S_{\tau,v} = \left\{ A = (a_{nm})_{n,m\geq 1} / \sup_{n\geq 1} \left(\frac{1}{v_n} \left(\sum_{m=1}^{\infty} |a_{nm}| \tau_m \right) \right) < \infty \right\}.$$

So, we obtain

Corollary 2. ([9])

$$A \in (s_\tau(C(\lambda)\Delta^+), s_v) \Leftrightarrow \sup_{n\geq 1} \left(|a_{n1}| \frac{\xi_1}{v_n} + \sum_{m=2}^{\infty} |a_{nm}| \frac{\tau_{m-1} |\lambda_{m-1}|}{v_n} \right) < \infty.$$

3.2. Generalization to the spaces $\widetilde{w_\tau}(\lambda, \mu)$ and $\widetilde{c_\tau}(\lambda, \mu, \lambda', \mu')$

Let $\lambda = (\lambda_n)_{n\geq 1} \in U$ and $\mu = (\mu_n)_{n\geq 1}$. We write

$$\widehat{\Delta}(\lambda, \mu) = \begin{pmatrix} \lambda_1 & & & O & \\ -\mu_1 & \lambda_2 & & & \\ & . & . & & \\ O & & -\mu_{n-1} & \lambda_n & \\ & & & . & . \end{pmatrix}.$$

If we put $\rho_k = \frac{\mu_k}{\lambda_k}$, it can be easily seen that if

$$\widehat{C}(\lambda, \mu) = (\widehat{\Delta}(\lambda, \mu))^{-1} = (c_{nm})_{n,m\geq 1},$$

we get

$$c_{nm} = \begin{cases} \frac{1}{\lambda_n} & \text{if } m = n, \\ \frac{1}{\lambda_n} \prod_{k=m}^{n-1} \rho_k & \text{if } m < n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\widehat{\Delta}(\lambda, \lambda) = \Delta(\lambda)$ and $\widehat{C}(\lambda, \lambda) = C(\lambda)$. We can define the set

$$\widetilde{w_\tau}(\lambda, \mu) = \{X \in s \ / \ \widehat{C}(\lambda, \mu)(|X|) \in s_\tau\},$$

that is

$$\widetilde{w_\tau}(\lambda, \mu) = \left\{ X \in s \ / \ \frac{1}{\lambda_n} \left[|x_n| + \sum_{m=1}^{n-1} \left(\prod_{k=m}^{n-1} \rho_k \right) |x_m| \right] = O(\tau_n) \quad n \to \infty \right\}.$$

For any sequences λ, $\lambda' = (\lambda'_n)_{n\geq 1} \in U$, μ and $\mu' = (\mu'_n)_{n\geq 1}$ we define also the set

$$\widetilde{c_\tau}(\lambda, \mu, \lambda', \mu') = \{X \in s \;/\quad \widehat{C}(\lambda, \mu)(\left|\widehat{\Delta}(\lambda', \mu')X\right|) \in s_\tau\}$$

and using the convention $\mu'_0 = 0$ we obtain $X \in \widetilde{c_\tau}(\lambda, \mu, \lambda', \mu')$ iff

$$x_n = \frac{1}{\lambda_n}\left[|\lambda'_n x_n - \mu'_{n-1}x_{n-1}| + \sum_{m=1}^{n-1}\left(\prod_{k=m}^{n-1}\rho_k\right)|\lambda'_m x_m - \mu'_{m-1}x_{m-1}|\right] = O(\tau_n) \quad n \to \infty.$$

It can be easily seen that $\widetilde{w_\tau}(\lambda, \lambda) = w_\tau(\lambda)$, $\widetilde{c_\tau}(\lambda, \lambda, \lambda', \lambda') = c_\tau(\lambda, \lambda')$ and $c_\tau(\lambda, \lambda) = c_\tau(\lambda)$, see [9].

Now, in order to generalize Theorem 2, we define for any sequence τ, the set

$$\Phi_\tau = \left\{X = (x_n)_{n\geq 1}/\quad \overline{\lim_{n\to\infty}}\left(|x_n|\frac{\tau_{n-1}}{\tau_n}\right) < 1\right\}.$$

Observe that $e \in \Phi_\tau$ implies that $\tau \in \Gamma$. Now, for λ, $\lambda' \in U$, μ and $\mu' \in s$, consider the following conditions:

$$\left(\frac{\mu_{n-1}}{\lambda_n}\right)_{n\geq 2} \in \Phi_\tau, \tag{18}$$

$$\left(\frac{\mu'_{n-1}}{\lambda'_{n-1}}\right)_{n\geq 2} \in \Phi_{|\lambda|\tau}. \tag{19}$$

THEOREM 3. *i) For any given $\lambda \in U$, $\mu \in s$ such that* (18) *holds, we have*

$$\widetilde{w_\tau}(\lambda, \mu) = s_{|\lambda|\tau}.$$

ii) Assume that (18) *and* (19) *hold for any given λ, $\lambda' \in U$, μ and $\mu' \in s$. Then $\widetilde{c_\tau}(\lambda, \mu, \lambda', \mu')$ is a BK space with respect to the norm $\|\|_{s_{|\frac{\lambda}{\lambda'}|\tau}}$ and*

$$\widetilde{c_\tau}(\lambda, \mu, \lambda', \mu') = s_{|\frac{\lambda}{\lambda'}|\tau}.$$

Proof. i) Put $\Delta''(\lambda, \mu) = \widehat{\Delta}(\lambda, \mu)D_{1/\lambda}$ in order to obtain all the diagonal entries of $\Delta''(\lambda, \mu)$ equal to 1. We see that the map $X \to D_\lambda X$ is bijective from s_τ into $s_{|\lambda|\tau}$. Consider now the infinite matrix

$$[\Sigma''(\lambda, \mu)]^{(q)} = \begin{pmatrix} [[\Delta''(\lambda, \mu)]^{(q)}]^{-1} & & O \\ & & 1 & \\ O & & & . \end{pmatrix},$$

where $[\Delta''(\lambda, \mu)]^{(q)}$ is the finite matrix whose entries are those of the q first rows and of the q first columns of $\Delta''(\lambda, \mu)$. We get $[\Sigma''(\lambda, \mu)]^{(q)}\,\widehat{\Delta}(\lambda, \mu) = (a_{nm})_{n,m\geq 1}$, with $a_{nn} = 1$ for all n, $a_{nn-1} = -\rho_{n-1}$ for $n \geq q+1$, and

$a_{nm} = 0$ otherwise. From (18) we deduce that there is $q \geq 1$ such that

$$\|I - [\Sigma''(\lambda,\mu)]^{(q)}\Delta''(\lambda,\mu)\|_{s_{|\lambda|\tau}} = \sup_{n\geq q+1}\left(|\rho_{n-1}|\left|\frac{\lambda_{n-1}}{\lambda_n}\right|\frac{\tau_{n-1}}{\tau_n}\right) = \sup_{n\geq q+1}\left(\left|\frac{\mu_{n-1}}{\lambda_n}\right|\frac{\tau_{n-1}}{\tau_n}\right) < 1. \tag{20}$$

For every $B \in s_{|\lambda|\tau}$, we see that $[\Sigma''(\lambda,\mu)]^{(q)} B \in s_{|\lambda|\tau}$. Furthermore, equation $\Delta''(\lambda,\mu)X = B$ being equivalent to

$$[\Sigma''(\lambda,\mu)]^{(q)}\,\Delta''(\lambda,\mu)X = [\Sigma''(\lambda,\mu)]^{(q)}\,B,$$

we deduce that the equation $\Delta''(\lambda,\mu)X = B$ admits only one solution in $s_{|\lambda|\tau}$ for all $B \in s_{|\lambda|\tau}$. This proves that $\Delta''(\lambda,\mu)$ is bijective from $s_{|\lambda|\tau}$ into itself. We conclude that $\widehat{\Delta}(\lambda,\mu) = \Delta''(\lambda,\mu)D_\lambda$ is bijective from s_τ into $s_{|\lambda|\tau}$ and

$$\widetilde{w_\tau}(\lambda,\mu) = \{X \in s \;/\; |X| \in \widehat{\Delta}(\lambda,\mu)s_\tau = s_{|\lambda|\tau}\} = s_{|\lambda|\tau}.$$

ii) We see that $X \in \widetilde{c_\tau}(\lambda,\mu,\lambda',\mu')$ iff $\left|\widehat{\Delta}(\lambda',\mu')X\right| \in \widehat{\Delta}(\lambda,\mu)s_\tau$. And since (18) holds we deduce from i) that $\widehat{\Delta}(\lambda,\mu)s_\tau = s_{|\lambda|\tau}$. Reasoning as above we see that the map $X \to D_{\lambda'}X$ is bijective from $s_{|\frac{\lambda}{\lambda'}|\tau}$ into $s_{|\lambda|\tau}$ and from (19) there exists $q \geq 1$ such that

$$\|I - [\Sigma''(\lambda',\mu')]^{(q)}\,\Delta''(\lambda',\mu')\|_{s_{|\lambda|\tau}} = \sup_{n\geq q+1}\left(\left|\frac{\mu'_{n-1}}{\lambda'_{n-1}}\right|\left|\frac{\lambda_{n-1}}{\lambda_n}\right|\frac{\tau_{n-1}}{\tau_n}\right) < 1. \tag{21}$$

Then, $\widehat{\Delta}(\lambda',\mu')$ is bijective from $s_{|\frac{\lambda}{\lambda'}|\tau}$ into $s_{|\lambda|\tau}$. Finally, for any given X, we get

$$X \in \widetilde{c_\tau}(\lambda,\mu,\lambda',\mu') \Leftrightarrow |\widehat{\Delta}(\lambda',\mu')X| \in \widehat{\Delta}(\lambda,\mu)s_\tau$$

equivalent to

$$X \in [\widehat{\Delta}(\lambda',\mu')]^{-1}\widehat{\Delta}(\lambda,\mu)s_\tau = [\widehat{\Delta}(\lambda',\mu')]^{-1}s_{|\lambda|\tau} = s_{|\frac{\lambda}{\lambda'}|\tau}.$$

This concludes the proof.

We deduce the following

3.3. Matrix transformations

Here we give some properties of matrix transformations mapping E into F where E is either one of the spaces s_u or $\widetilde{w_\tau}(\lambda,\mu)$ or $\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu')$ and F is either one of the spaces s_v or $\widetilde{w_\alpha}(\xi,\eta)$ or $\widetilde{c_\tau}(\xi,\eta,\xi',\eta')$.

Now, consider the following supplementary hypotheses:

$$\left(\frac{\eta_{n-1}}{\xi_n}\right)_{n\geq 2} \in \Phi_\alpha, \tag{22}$$

$$\left(\frac{\eta'_{n-1}}{\xi'_{n-1}}\right)_{n\geq 2} \in \Phi_{|\xi|\alpha}. \tag{23}$$

For any given sequences λ, ξ, $\lambda', \xi' \in U$, μ, μ', η, $\eta' \in s$, $\gamma = (\gamma_n)_{n\geq 1}$, $v = (v_n)_{n\geq 1}$, $\alpha = (\alpha_n)_{n\geq 1}$ and τ with γ_n, v_n, α_n and $\tau_n > 0$ $\forall n$, we obtain the following

THEOREM 4. *i) Under* (22), *we get:*

$$A \in (s_\gamma, \widetilde{w_\alpha}(\xi,\eta)) \quad \Leftrightarrow \quad \sup_{n\geq 1}\left(\frac{1}{\alpha_n\,|\xi_n|}\sum_{m=1}^{\infty}|a_{nm}|\,\gamma_m\right) < \infty;$$

ii) under (18), *we have:*

$$A \in (\widetilde{w_\tau}(\lambda,\mu), s_v) \Leftrightarrow \sup_{n\geq 1}\left(\frac{1}{v_n}\left(\sum_{m=1}^{\infty}|a_{nm}\lambda_m|\,\tau_m\right)\right) < \infty;$$

iii) under (18) *and* (22), *we have*

$$A \in (\widetilde{w_\tau}(\lambda,\mu), \widetilde{w_\alpha}(\xi,\eta)) \Leftrightarrow \sup_{n\geq 1}\left(\frac{1}{|\xi_n|\,\alpha_n}\left(\sum_{m=1}^{\infty}|a_{nm}\lambda_m|\,\tau_m\right)\right) < \infty;$$

iv) under (22) *and* (23), *we have*

$$A \in (s_\gamma, \widetilde{c_\alpha}(\xi,\eta,\xi',\eta')) \Leftrightarrow \sup_{n\geq 1}\left(\frac{|\xi'_n|}{|\xi_n|\,\alpha_n}\left(\sum_{m=1}^{\infty}|a_{nm}|\,\gamma_m\right)\right) < \infty;$$

v) under (18) *and* (19), *we obtain*

$$A \in (\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu'), s_v) \Leftrightarrow \sup_{n\geq 1}\left(\frac{1}{v_n}\left(\sum_{m=1}^{\infty}\left|a_{nm}\frac{\lambda_m}{\lambda'_m}\right|\tau_m\right)\right) < \infty;$$

vi) under (18), (22) *and* (23):

$$A \in (\widetilde{w_\tau}(\lambda,\mu), \widetilde{c_\alpha}(\xi,\eta,\xi',\eta')) \Leftrightarrow \sup_{n\geq 1}\left(\frac{|\xi'_n|}{|\xi_n|\,\alpha_n}\left(\sum_{m=1}^{\infty}|a_{nm}\lambda_m|\,\tau_m\right)\right) < \infty;$$

vii) under (18), (19) *and* (22):

$$A \in (\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu'), \widetilde{w_\alpha}(\xi,\eta)) \Leftrightarrow \sup_{n\geq 1}\left(\frac{1}{|\xi_n|\,\alpha_n}\left(\sum_{m=1}^{\infty}\left|a_{nm}\frac{\lambda_m}{\lambda'_m}\right|\tau_m\right)\right) < \infty;$$

viii) under (18), (19), (22) *and* (23):

$$A \in (\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu'), \widetilde{c_\alpha}(\xi,\eta,\xi',\eta')) \Leftrightarrow \sup_{n\geq 1}\left(\frac{|\xi'_n|}{|\xi_n|\,\alpha_n}\left(\sum_{m=1}^{\infty}\left|a_{nm}\frac{\lambda_m}{\lambda'_m}\right|\tau_m\right)\right) < \infty.$$

Proof. i) If (22) holds $\widetilde{w_\alpha}(\xi,\eta) = s_{|\xi|\alpha}$ and from (19)

$$A \in (s_\gamma, \widetilde{w_\alpha}(\xi,\eta)) \Leftrightarrow A \in S_{\gamma,|\xi|\alpha}.$$

ii) From (18) $\widetilde{w_\tau}(\lambda,\mu) = s_{|\lambda|\tau}$ and we conclude as in i). iii) Under (18), (22) and i) in Theorem 10, we get $\widetilde{w_\tau}(\lambda,\mu) = s_{|\lambda|\tau}$ and $\widetilde{w_\alpha}(\xi,\eta) = s_{|\xi|\alpha}$. Then

$$A \in (\widetilde{w_\tau}(\lambda,\mu), \widetilde{w_\alpha}(\xi,\eta)) \Leftrightarrow A \in S_{|\lambda|\tau, |\xi|\alpha},$$

hence we obtain iii).

iv) From (22), (23) and ii) in Theorem 3, we get $\widetilde{c_\alpha}(\xi,\eta,\xi',\eta') = s_{|\frac{\xi}{\xi'}|\alpha}$. Then

$$(s_\gamma, \widetilde{c_\alpha}(\xi,\eta,\xi',\eta')) = S_{\gamma, |\frac{\xi}{\xi'}|\alpha}.$$

v) As in iv) (18) and (19) imply $\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu') = s_{|\frac{\lambda}{\lambda'}|\tau}$ and

$$(\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu'), s_v) = S_{|\frac{\lambda}{\lambda'}|\tau, v}.$$

vi) (18) implies that $\widetilde{w_\tau}(\lambda,\mu) = s_{|\lambda|\tau}$, (22) and (23) imply that $\widetilde{c_\alpha}(\xi,\eta,\xi',\eta') = s_{|\frac{\xi}{\xi'}|\alpha}$. Hence

$$(\widetilde{w_\tau}(\lambda,\mu), \widetilde{c_\alpha}(\xi,\eta,\xi',\eta')) = S_{|\lambda|\tau, |\frac{\xi}{\xi'}|\alpha}.$$

Similarly we get vii).

Finally, since (18) and (19) imply $\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu') = s_{|\frac{\lambda}{\lambda'}|\tau}$ and (22) and (23) imply $\widetilde{c_\alpha}(\xi,\eta,\xi',\eta') = s_{|\frac{\xi}{\xi'}|\alpha}$ we conclude that

$$(\widetilde{c_\tau}(\lambda,\mu,\lambda',\mu'), \widetilde{c_\alpha}(\xi,\eta,\xi',\eta')) = S_{|\frac{\lambda}{\lambda'}|\tau, |\frac{\xi}{\xi'}|\alpha},$$

which gives viii).

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