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ON ASYMPTOTICALLY STATISTICAL EQUIVALENT SEQUENCES

Abstract. This paper presents the following definition which is a natural combination of the definition for Asymptotically equivalent and Statistically limit. Two nonnegative sequences $[x]$ and $[y]$ are said to be **asymptotically statistical equivalents of multiple** L provided that for every $\epsilon > 0$, $\lim_n \frac{1}{n}\{$the number of $k \leq n : |\frac{x_k}{y_k} - L| \geq \epsilon\} = 0$ (denoted by $x \overset{S_L}{\sim} y$), and simply **asymptotically statistical equivalent** if $L = 1$. In addition, there are also statistical analogs of theorems of Poyvanents in [5].

1. Introduction

In 1980 Poyvanents presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. This paper presents statistical analogs of these definitions. In addition to these definitions there are also statistical analogs of theorems of Poyvanents in [5].

2. Definitions, and notations

Let $l' = \{x_k : \sum_{k=1}^{\infty} |x_k| < \infty.\}$, $d_A = \{x_k : \lim_n \sum_{k=1}^{\infty} a_{n,k}x_k = \text{exists}.\}$, $P_\delta = \{$The set of all real number sequences such that $x_k \geq \delta > 0$ for all $k\}$, and $P_0 = \{$The set of all nonnegative sequences which have at most a finite number of zero entries.$\}$.

DEFINITION 2.1 (Fridy, [1]). For each $[x]$ in l' the "*remainder sequence*" $[Rx]$ is the sequence whose n-th term is

$$R_n x := \sum_{k \geq n} |x_k|.$$

DEFINITION 2.2 (Marouf, [3]). Two nonnegative sequences $[x]$, and $[y]$ are said to be *asymptotically equivalent* if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

DEFINITION 2.3 (Fridy, [2]). The sequence $[x]$ has *statistic limit* L, denoted by $st - \lim s = L$ provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n} \{\text{the number of } k \leq n : |x_k - L| \geq \epsilon\} = 0.$$

The next definition is natural combination of definition (2.2) and (2.3).

DEFINITION 2.4. Two nonnegative sequences $[x]$ and $[y]$ are said to be *asymptotically statistical equivalents of multiple L* provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n} \left\{ \text{the number of } k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} = 0$$

(denoted by $x \overset{S_L}{\sim} y$), and simply *asymptotically statistical equivalent* if $L = 1$.

DEFINITION 2.5. A summability matrix A is *asymptotically statistical regular* provided that $Ax \overset{S_L}{\sim} Ay$ whenever $x \overset{S_L}{\sim} y$, $[x] \in P_0$, and $[y] \in P_\delta$ for some $\delta > 0$.

3. Main result

The following theorem presents necessary and sufficient conditions on the entries of a summability matrix to ensure that the matrix transformation will preserve asymptotically statistical equivalents of multiple L of a given sequence.

THEOREM 3.1. *If A is a nonnegative summability matrix that maps bounded sequences $[x]$ into l' then the following statements are equivalent:*

1. *If $[x]$ and $[y]$ are sequences such that $x \overset{S_L}{\sim} y$, $[x] \in P_0$, and $[y] \in P_\delta$ for some $\delta > 0$ then*

$$R_n(Ax) \overset{S_L}{\sim} R_n(Ay).$$

2.

$$\lim_n \frac{1}{n} \left\{ \text{the number of } k \leq n : \left| \frac{\sum_{i=k}^{\infty} a_{i,m}}{\sum_{i=k}^{\infty} \sum_{p=1}^{\infty} a_{i,p}} \right| \geq \epsilon \text{ for each } m \& \epsilon > 0 \right\} = 0.$$

Proof. The definition for asymptotically statistical equivalents of multiple L can be interpreted as the following

$$\left| \frac{x_i}{y_i} - L \right| < \epsilon \text{ for almost all } k \text{ (denoted } a.a. \ k).$$

This implies that

(3.1) $$(L - \epsilon)y_i \leq x_i \leq (L + \epsilon)y_i, \ a.a. \ i.$$

Let us consider the following

$$\frac{R_n(Ax)}{R_n(Ay)} \leq \frac{\sum_{j=1}^{J-1} \sum_{i=n}^{\infty} \max_{0 \leq j \leq J-1}\{a_{i,j}\}}{\sum_{i=n}^{\infty} \sum_{j=1}^{\infty} a_{i,j} y_j}$$

$$+ \frac{\sum_{i=n}^{\infty} \sum_{j=J}^{\infty} a_{i,j} x_j}{\sum_{i=n}^{\infty} \sum_{j=1}^{\infty} a_{i,j} y_j}.$$

Using Equation 3.1 we obtain the following

$$\frac{R_n(Ax)}{R_n(Ay)} \leq \frac{\sum_{j=1}^{J-1} \sum_{i=n}^{\infty} \max_{0 \leq j \leq J-1}\{a_{i,j}\}}{\delta \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} a_{i,j}} + (L + \epsilon), \ a.a. \ n.$$

Thus by (2)

$$\limsup_n \frac{R_n(Ax)}{R_n(Ay)} \leq (L + \epsilon), \ a.a. \ n.$$

Inequality (3.1) can be used in a similar manner to obtain the following

$$\liminf_n \frac{R_n(Ax)}{R_n(Ay)} \geq (L - \epsilon), \ a.a. \ n.$$

Thus

$$R_n(Ax) \overset{S_L}{\sim} R_n(Ay).$$

For the second part of this theorem let us consider the following two sequences,

$$x_p := \begin{cases} 0, & \text{if } p \leq \bar{K} \\ 1, & \text{otherwise,} \end{cases}$$

where $\bar{K}$ is a positive integer and $y_p = 1$ for all p. These two sequences imply the following:

$$R_n(Ax) = \sum_{k=n}^{\infty} (Ax)_k$$

$$= \sum_{k=n}^{\infty} \sum_{p=\bar{K}+1}^{\infty} a_{k,p}$$

$$= \sum_{k=n}^{\infty} \sum_{p=1}^{\infty} a_{k,p} - \sum_{k=n}^{\infty} \sum_{p=\bar{K}}^{\infty} a_{k,p}.$$

Therefore

$$\liminf_n \frac{R_n(Ax)}{R_n(Ay)} \leq 1 - \limsup_n \frac{\sum_{i=n}^{\infty} a_{i,\bar{K}}}{\sum_{i=n}^{\infty} \sum_{p=1}^{\infty} a_{i,p}}.$$

Since each nonconstant element of the last inequality has statistically limit zero we obtain the following

$$\lim_n \frac{R_n(Ax)}{R_n(Ay)} = 1, \ a.a. \ n.$$

This completes the proof of this theorem.   □

In 1980 Poyvanents presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Using these definitions he also presented a Silverman Toeplitz type characterization for asymptotic equivalent sequences. In similar manner I have presented definitions for asymptotically statistical equivalent sequences and asymptotically statistical regular matrices. These definitions are also used to present the following matrix characterization of asymptotically statistical equivalent sequences.

THEOREM 3.2. *In order for a summability matrix A to be asymptotically statistical regular it is necessary and sufficient that for each fixed positive integer k_0*

1. $$\sum_{i=1}^{k_0} a_{n,i} \ \textit{is bounded for each } n,$$

2. $$\lim_n \frac{1}{n} \left\{ \textit{ the number of } k \leq n \colon \left| \frac{\sum_{i=1}^{k_0} a_{n,i}}{\sum_{i=1}^{\infty} a_{n,i}} \right| \geq \epsilon, \textit{ for fixed } k_0 \ \& \ \epsilon > 0 \right\} = 0.$$

P r o o f. The necessary part of this theorem is established in manner similar to that necessary part of the last theorem. To establish the sufficient part of this theorem, let $\epsilon > 0$, $x \overset{S_L}{\sim} y$, $[x] \in P_0$, and $[y] \in P_\delta$ for some $\delta > 0$. These conditions imply that

(3.2) $(L - \epsilon)y_{i+\alpha} \leq x_{i+\alpha} \leq (L + \epsilon)y_{i+\alpha}, \ a.a. \ i, \text{ for } \alpha = 1, 2, \ldots .$

Let us consider the following

$$\frac{(Ax)_n}{(Ay)_n} = \frac{\sum_{i=1}^{\alpha} a_{n,i}x_i + \sum_{i=\alpha+1}^{\infty} a_{n,i}x_i}{\sum_{i=1}^{\alpha} a_{n,i}y_i + \sum_{i=\alpha+1}^{\infty} a_{n,i}y_i}$$

$$= \frac{\dfrac{\sum_{i=1}^{\alpha} a_{n,i}x_i}{\sum_{i=\alpha+1}^{\infty} a_{n,i}y_i} + \dfrac{\sum_{i=\alpha+1}^{\infty} a_{n,i}x_i}{\sum_{i=\alpha+1}^{\infty} a_{n,i}y_i}}{\dfrac{\sum_{i=1}^{\alpha} a_{n,i}y_i}{\sum_{i=\alpha+1}^{\infty} a_{n,i}y_i} + 1}.$$

Inequality 3.2 implies that

$$\lim_n \frac{\sum_{i=\alpha+1}^{\infty} a_{n,i}x_i}{\sum_{i=\alpha+1}^{\infty} a_{n,i}y_i} = L, \ a.a. \ n.$$

Since $[x] \in P_0$, $[y] \in P_\delta$, and condition (2) holds we obtain the following

$$\lim_n \frac{\sum_{i=1}^{\alpha} a_{n,i} x_i}{\sum_{i=\alpha+1}^{\infty} a_{n,i} y_i} = 0, \ a.a. \ n,$$

and

$$\lim_n \frac{\sum_{i=1}^{\alpha} a_{n,i} y_i}{\sum_{i=\alpha+1}^{\infty} a_{n,i} y_i} = 0, \ a.a. \ n.$$

Thus

$$\lim_n \frac{(Ax)_n}{(Ay)_n} = L, \ a.a. \ n.$$

This implies that $Ax \overset{S_L}{\sim} Ay$ whenever $x \overset{S_L}{\sim} y$, $[x] \in P_0$, and $[y] \in P_\delta$ for some $\delta > 0$. This completes the proof of this theorem. □

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