DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

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DIRECT AND CONVERSE THEOREMS FOR INTEGRAL-TYPE OPERATORS

Abstract. We establish direct and converse results for integral—type operators defined with the aid of Szász-Mirakjan operator and Baskakov operator, respectively. Moreover, some applications will be given.

1. Introduction

We denote by $C_B[0,\infty)$ the set of all bounded continuous functions on $[0,\infty)$ with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. By $L^1[0,\infty)$ we shall denote the set of all Lebesgue integrable functions on $[0,\infty)$.

Let $g : [0,\infty) \times (t_0,\infty) \times [0,\infty) \to [0,\infty)$ be a function such that $g(x,t,\cdot) \in C[0,\infty) \cap L^1[0,\infty)$ for all $(x,t) \in [0,\infty) \times (t_0,\infty)$ and $t_0 \geq 0$. If $G : (0,\infty) \times (t_0,\infty) \to (0,\infty)$,

$$G(x,t) = \int_0^\infty g(x,t,\theta)d\theta \tag{1}$$

then we assume that

$$\frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \cdot (\theta - x)d\theta = 0, \tag{2}$$

$$\frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \cdot (\theta - x)^2 d\theta \leq \beta(t)\varphi^2(x) \tag{3}$$

and

$$\frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \cdot (\theta - x)^2 \cdot \frac{d\theta}{1+\theta} \leq \beta(t)x \tag{4}$$

1991 *Mathematics Subject Classification*: 41A10, 41A27, 41A35, 41A36.

Key words and phrases: Szász-Mirakjan operator, Baskakov operator, the second modulus of smoothness of Ditzian-Totik, integral operators.

for every $(x,t) \in (0,\infty) \times (t_0,\infty)$, where $\beta : (t_0,\infty) \to (0,\infty)$ is a given function and φ is a weight function of the following form: $\varphi(x) = \sqrt{x}$ $(x \geq 0)$ or $\varphi(x) = \sqrt{x(1+x)}$ $(x \geq 0)$ corresponding to the well-known Szász-Mirakjan operator

$$S_n(f,x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

and to the Baskakov operator

$$V_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right),$$

respectively. In what follows we shall denote by L_n the operators S_n and V_n.

Now we can define a new sequence of positive linear integral-type operator $L_n^t : C_B[0,\infty) \to C[0,\infty)$ by means of the functions g and G and the sequence $\{L_n\}$ as follows

$$L_n^t(f,x) = \frac{1}{G(x,t)} \cdot \int_0^{\infty} g(x,t,\theta) \cdot L_n(f,\theta)\, d\theta$$

and $L_n^t(f,0) = f(0)$, where the parameter t may be depend only on the natural number n, $n \in \mathbb{N} := \{1,2,\ldots\}$.

The aim of this paper is to study the global approximation properties of L_n^t, establishing direct and converse theorems for L_n^t using the second modulus of smoothness of Ditzian - Totik defined by

$$\omega_\varphi^2(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|.$$

We need also the following $K-$ functional

$$K_{2,\varphi}(f,\delta) = \inf_{h' \in A.C._{loc}} \{\|f-h\| + \delta\|\varphi^2 h''\|\}.$$

It is well-known (see [2], Theorem 2.1.1.) that

$$\omega_\varphi^2(f,\sqrt{\delta}) \sim K_{2,\varphi}(f,\delta). \tag{5}$$

Throughout this paper C_0 and C denote absolute constants and not necessarily the same at each occurrence.

The next section contains the formulation of our results and their proofs will be given in the third section. The last section is devoted to the applications.

2. Main results

The theorems in question are the following:

THEOREM 1. *If $f \in C_B[0,\infty)$, $t \in (t_0,\infty)$ and $n \in \mathbb{N}$ then*

$$\|L_n^t(f) - f\| \leq C\omega_\varphi^2(f,(n^{-1}+\beta(t))^{1/2}).$$

COROLLARY 2. *Let* $f \in C_B[0,\infty)$ *and* $t = t(n)$, $\beta(t) \le Cn^{-1}$, $n \in \mathbb{N}$. *Then*

(*i*) $\|L_n^t(f) - f\| \le C\ \omega_\varphi^2(f, n^{-1/2})$;
(*ii*) $\|L_n^t(f) - f\| \le C\ \|L_n(f) - f\|$.

THEOREM 3. *Let* $f \in C_B[0,\infty)$ *and* $t = t(n)$, $2C_0\beta(t) \le \gamma n^{-1}$, $n \in \mathbb{N}$, *where* $C_0 > 0$ *is the absolute constant of Lemma below and* $0 < \gamma < 1$. *Then*

$$(1-\gamma)\ \|L_n(f) - f\| \le \|L_n^t(f) - f\|\ \le (1+\gamma)\|L_n(f) - f\|.$$

3. Proofs

At first we state the following lemma, which was established in [5], (2.7) for $L_n = S_n$ and in [5] for $L_n = V_n$, respectively:

LEMMA. *Let* $f \in C_B[0,\infty)$. *Then*

$$\frac{1}{n}\|\varphi^2 L_n''(f)\| \le C_0\ \|L_n(f) - f\|, \tag{6}$$

where C_0 *is an absolute constant and* $n \in \mathbb{N}$.

Proof of Theorem 1. Let $x > 0$. By Taylor's formula we have

$$L_n(f,\theta) = L_n(f,x) + L_n'(f,x)(\theta - x) + \int_x^\theta (\theta - v)L_n''(f,v)dv.$$

Hence, in view of (1) and (2), we obtain

$$\begin{aligned}
|L_n^t(f,x) - L_n(f,x)| &= \left| \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \left\{ \int_x^\theta (\theta - v)\ L_n''(f,v)dv \right\} d\theta \right| \\
&\le \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \cdot \left| \int_x^\theta (\theta - v)L_n''(f,v)dv \right| d\theta \qquad (7)\\
&\le \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \cdot \left| \int_x^\theta \frac{|\theta - v|}{\varphi^2(v)}dv \right| d\theta \cdot \|\varphi^2 L_n''(f)\|.
\end{aligned}$$

Using [2], (9.6.1) and (9.6.2) we obtain

$$\left| \int_x^\theta \frac{|\theta - v|}{\varphi^2(v)}dv \right| \le \frac{(\theta - x)^2}{\varphi^2(x)} \quad \text{if} \quad \varphi(x) = \sqrt{x}$$

and

$$\left| \int_x^\theta \frac{|\theta - v|}{\varphi^2(v)}dv \right| \le \frac{(\theta - x)^2}{x} \cdot \left(\frac{1}{1+x} + \frac{1}{1+\theta} \right) \quad \text{if} \quad \varphi(x) = \sqrt{x(1+x)}.$$

Hence, by (7) and (3) we have

$$(8)\quad |L_n^t(f,x) - L_n(f,x)| \le \frac{\|\varphi^2 L_n''(f)\|}{\varphi^2(x)} \cdot \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta)\cdot(\theta-x)^2 d\theta \le \beta(t)\cdot\|\varphi^2 L_n''(f)\|$$

for $L_n = S_n$ and, in view of (7), (3) and (4), we have

$$(9)\quad \begin{aligned} |L_n^t(f,x) - L_n(f,x)| &\le \frac{\|\varphi^2 L_n''(f)\|}{\varphi^2(x)} \cdot \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta)\cdot(\theta-x)^2\, d\theta \\ &+ \frac{\|\varphi^2 L_n''(f)\|}{x} \cdot \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta)\cdot(\theta-x)^2 \cdot \frac{d\theta}{1+\theta} \\ &\le 2\beta(t)\|\varphi^2 L_n''(f)\| \end{aligned}$$

for $L_n = V_n$, respectively. In conclusion, from (8), (9) and $L_n^t(f,0) = f(0)$ we get

$$(10)\qquad \|L_n^t(f) - L_n(f)\| \le 2\beta(t)\ \|\varphi^2 L_n''(f)\|.$$

Furthermore, L_n^t is a bounded operator. Indeed, by (1) and [2], (9.3.4) we have

$$|L_n^t(f,x)| \le \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta)|L_n(f,\theta)|d\theta \le \|L_n(f)\| \le \|f\|,$$

i.e.

$$(11)\qquad \|L_n^t(f)\| \le \|f\|.$$

Now, let $h \in C_B[0,\infty)$ such that $\varphi^2 h'' \in C_B[0,\infty)$. Then, by (10) and [2], (9.3.7) we have

$$\|L_n^t(h) - h\| \le C\beta(t)\|\varphi^2 h''\|.$$

Hence, in view of (11) and [2], (9.3.4), we get

$$(12)\quad \begin{aligned} \|L_n^t(f) - L_n(f)\| &\le \|L_n^t(f-h) - L_n(f-h)\| + \|L_n^t(h) - L_n(h)\| \\ &\le 2\|f-h\| + C\beta(t)\|\varphi^2 h''\| \\ &\le C\{\|f-h\| + \beta(t)\|\varphi^2 h''\|\}. \end{aligned}$$

Because

$$\|L_n(f) - f\| \le C\cdot\omega_\varphi^2(f, n^{-1/2})$$

(see [2], (9.3.1)), we get, by (5) and definition of $K_{2,\varphi}(f,n^{-1})$ that

$$(13)\qquad \|L_n(f) - f\| \le C\left\{\|f-h\| + \frac{1}{n}\ \|\varphi^2 h''\|\right\}.$$

Then (12) and (13) imply

$$\begin{aligned}\|L_n^t(f)-f\| &\le \|L_n^t(f)-L_n(f)\|+\|L_n(f)-f\|\\ &\le C\left\{\|f-h\|+\left(\beta(t)+\frac{1}{n}\right)\cdot\|\varphi^2h''\|\right\}\end{aligned}$$

or

$$\|L_n^t(f)-f\|\le C\cdot K_{2,\varphi}\left(f,\frac{1}{n}+\beta(t)\right).$$

Using again (5), we get the assertion of the theorem.

Proof of Corollary 2. (*i*) It is a direct consequence of Theorem 1. (*ii*) It follows from (*i*) and $\|L_n(f)-f\|\sim\omega_\varphi^2(f,n^{-1/2})$ (see [5], Theorem 1.2 and Theorem 3.2).

Proof of Theorem 3. Using (10) and (6), we obtain

$$\begin{aligned}\|L_n(f)-f\| &\le \|L_n^t(f)-f\|+\|L_n^t(f)-L_n(f)\|\\ &\le \|L_n^t(f)-f\|+2\beta(t)\|\varphi^2L_n''(f)\|\\ &\le \|L_n^t(f)-f\|+2\beta(t)\cdot C_0n\|L_n(f)-f\|\\ &\le \|L_n^t(f)-f\|+\gamma\|L_n(f)-f\|.\end{aligned}$$

Thus

$$(1-\gamma)\|L_n(f)-f\|\ \le\|L_n^t(f)-f\|. \tag{14}$$

In similar way we obtain

$$\|L_n^t(f)-f\|\le(1+\gamma)\|L_n(f)-f\|. \tag{15}$$

Then (14) and (15) complete the proof of the theorem.

REMARK. Under the assumptions of Theorem 3 we have

$$C^{-1}\omega_\varphi^2(f,n^{-1/2})\le\|L_n^t(f)-f\|\le C\omega_\varphi^2(f,n^{-1/2}).$$

4. Applications

1) *The generalized Szász-Mirakjan operator* has the form

$$S_n^t(f,x)=\left(1+\frac{n}{t}\right)^{-tx}\cdot\sum_{k=0}^{\infty}\left(1+\frac{t}{n}\right)^{-k}\cdot\frac{tx(tx+1)\dots(tx+k-1)}{k!}\cdot f\left(\frac{k}{n}\right),$$

where $f\in C_B[0,\infty)$, $x\ge0$ and $t>0$. For $t=1/\alpha$, $\alpha>0$ we receive back the operator considered in [3]. We have, by [3], Theorem 2.8: $S_n^t(f,x)=\mathcal{G}_t(S_n(f),x)$, where

$$\mathcal{G}_t(f,x)=\frac{1}{\Gamma(tx)}\cdot\int_0^\infty e^{-\theta}\theta^{tx-1}f\left(\frac{\theta}{t}\right)d\theta,\ x>0,\ t>0$$

is a Gamma-type operator. Here $g(x,t,\theta) = e^{-t\theta}\theta^{tx-1}$, $\theta > 0$ and

$$\frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} \cdot \theta d\theta = x,$$

$$\frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} \cdot (\theta - x)^2 d\theta = \frac{x}{t},$$

$$\frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} \cdot (\theta - x)^2 \cdot \frac{d\theta}{1+\theta} d\theta \le \frac{x}{t}.$$

Therefore we can consider $\beta(t) = 1/t$ and $\varphi(x) = \sqrt{x}$. Now we can establish the following results:

$$\|S_n^t(f) - f\| \le C\, \omega_\varphi^2(f, (n^{-1} + t^{-1})^{1/2});$$

$$(1-\gamma)\|S_n(f) - f\| \le \|S_n^t(f) - f\| \le (1+\gamma)\|S_n(f) - f\|$$
$$\text{if } t = t(n) \text{ and } 2C_0 \cdot \frac{1}{t} \cdot n \le \gamma < 1,\ n \in \mathbb{N}.$$

2) Furthermore, we can consider the following new integral-type operator

$$\bar{S}_n^t(f,x) = \frac{1}{\Gamma(tx)} \cdot \int_0^\infty e^{-\theta}\, \theta^{tx-1} V_n\left(f, \frac{\theta}{t}\right) d\theta, \qquad \bar{S}_n^t(f,0) = f(0),$$

where $f \in C_B[0,\infty)$, $x \ge 0$ and $t > 0$. Hence we have

$$\bar{S}_n^t(f,x) = \frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} V_n(f,\theta) d\theta.$$

Thus $g(x,t,\theta) = e^{-t\theta}\theta^{tx-1}$, $\theta > 0$ and $G(x,t) = t^{-tx}\Gamma(tx)$. Using the same computations as in the first case, we get

$$\frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} \cdot \theta d\theta = x,$$

$$\frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} \cdot (\theta - x)^2 d\theta = \frac{x}{t} \le \frac{x(1+x)}{t},$$

$$\frac{t^{tx}}{\Gamma(tx)} \cdot \int_0^\infty e^{-t\theta}\theta^{tx-1} \cdot (\theta - x)^2 \cdot \frac{d\theta}{1+\theta} \le \frac{x}{t}.$$

So we have $\beta(t) = 1/t$ and $\varphi(x) = \sqrt{x(1+x)}$. The results are the following:

$$\|\bar{S}_n^t(f) - f\| \le C\omega_\varphi^2(f, (n^{-1} + t^{-1})^{1/2});$$

$$(1-\gamma)\|V_n(f)-f\| \le \|\bar{S}_n^t(f)-f\| \le (1+\gamma)\|V_n(f)-f\|$$
$$\text{if } t=t(n) \text{ and } 2C_0\cdot\frac{1}{t}\cdot n \le \gamma < 1,\ n\in\mathbb{N}.$$

3) *The generalized Baskakov operator* is defined by

$$V_n^t(f,x)=\sum_{k=0}^{\infty} v_{n,k}(x,t) f\left(\frac{k}{n}\right),$$

where $f\in C_B[0,\infty)$, $x\ge 0$, $t>1$ and

$$v_{n,k}(x,t)=\binom{n+k-1}{k}\cdot\frac{\prod_{i=0}^{k-1}(tx+i)\prod_{j=1}^{n}(t+j)}{\prod_{r=1}^{n+k}(t(x+1)+r)}.$$

The origin of this operator can be found in [4]. Obviously $V_n^t(f,0)=f(0)$. Let

$$\mathcal{T}_t^*(f,x)=\frac{1}{B(tx,t+1)}\cdot\int_0^{\infty}\frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}}\cdot f(\theta)d\theta$$

be the modified inverse beta operator (see [1]). Because $f\in C_B[0,\infty)$ we have (see also [4], Theorem 2):

$$\begin{aligned}
V_n^t(f,x)&=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\cdot\frac{\prod_{i=0}^{k-1}(tx+i)\prod_{j=0}^{n}(t+j)}{\prod_{r=0}^{n+k}(t(x+1)+r)}\cdot(x+1)f\left(\frac{k}{n}\right)\\
&=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\cdot\frac{B(tx+k,t+n+1)}{B(tx,t)}\cdot(x+1)f\left(\frac{k}{n}\right)\\
&=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\cdot\frac{B(tx+k,t+n+1)}{B(tx,t+1)}\cdot f\left(\frac{k}{n}\right)\\
&=\frac{1}{B(tx,t+1)}\cdot\int_0^{\infty}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\cdot\frac{\theta^{tx+k-1}}{(1+\theta)^{t(x+1)+n+k+1}}\cdot f\left(\frac{k}{n}\right)d\theta\\
&=\frac{1}{B(tx,t+1)}\cdot\int_0^{\infty}\frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}}\cdot V_n(f,\theta)d\theta\\
&=\mathcal{T}_t^*(V_n(f),x).
\end{aligned}$$

Therefore

$$g(x,t,\theta)=\frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}},\ \theta>0$$

and

$$\frac{1}{B(tx,t+1)}\cdot\int_0^{\infty}\frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}}\cdot\theta d\theta=x,$$

$$\frac{1}{B(tx,t+1)} \cdot \int_0^\infty \frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}} \cdot (\theta - x)^2 d\theta = \frac{x(1+x)}{t-1},$$

$$\frac{1}{B(tx,t+1)} \cdot \int_0^\infty \frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}} \cdot (\theta - x)^2 \cdot \frac{d\theta}{1+\theta} =$$
$$= \frac{x(1+x)}{t(1+x)+1} = \frac{x}{t-1} \cdot \frac{(t-1)(1+x)}{t(1+x)+1} \le \frac{x}{t-1}$$

for every $(x,t) \in (0,\infty) \times (1,\infty)$. Hence $\beta(t) = 1/(t-1)$ and $\varphi(x) = \sqrt{x(1+x)}$. Thus we have the following results :

$$\|V_n^t(f) - f\| \le C\ \omega_\varphi^2(f, (n^{-1} + (t-1)^{-1})^{1/2});$$

$$(1-\gamma)\|V_n(f) - f\| \le \|V_n^t(f) - f\| \le (1+\gamma)\|V_n(f) - f\|$$
$$\text{if } t = t(n) \text{ and } 0 < 2C_0 \cdot \frac{1}{t-1} \cdot n \le \gamma < 1,\ n \in \mathbb{N}.$$

4) Finally, let us consider the following operators: $U_{n,k}^t : C_B[0,\infty) \to C_B[0,\infty)$,

$$U_{n,k}^t(f,0) = f(0) \quad \text{and} \quad U_{n,0}^t \equiv L_n^t,$$
$$U_{n,k+1}^t(f,x) = \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) U_{n,k}^t(f,\theta) d\theta,$$

where $k \ge 0$ is an integer. Then we have the next theorem.

THEOREM 4. *If $f \in C_B[0,\infty)$ and $t \in (t_0,\infty)$ then*
(i) $\|U_{n,k}^t(f) - f\| \le C\ \omega_\varphi^2(f, (n^{-1} + \beta(t))^{1/2})$, $\quad k \ge 0$ *and* $n \in \mathbb{N}$;
(ii) $\|U_{n,k}^t(f) - f\| \le C\ \|L_n(f) - f\|$
when $t = t(n)$ *and* $\beta(t) \le Cn^{-1}$, $n \in \mathbb{N}$, $k \ge 0$;
(iii) $(1-(k+2)\gamma)\|U_{n,k}^t(f) - f\| \le (1-(k+1)\gamma)\|U_{n,k+1}^t(f) - f\| \le (1-k\gamma)\|U_{n,k}^t(f) - f\|$
when $t = t(n)$, $2C_0\beta(t) \le \gamma n^{-1}$, $n \in \mathbb{N}$ *and* $0 < \gamma < 1/(k_0+2)$, $k_0 \in \mathbb{N}$, $k \in \{1,2,\ldots,k_0\}$.

P r o o f. *(i)* At first of all, let us observe that (11) and the definition of $U_{n,k}^t$ imply, by induction, that

$$\|U_{n,k}^t(f)\| \le \|f\|. \tag{16}$$

Furthermore, let $x \ge 0$ and $f \in C_B[0,\infty)$. Then, in view of (1), (16), Theorem 1 and $\|L_n(f) - f\| \sim \omega_\varphi^2(f, n^{-1/2})$, we obtain

$$|U_{n,1}^t(f,x) - f(x)| = \frac{1}{G(x,t)} \cdot \Big| \int_0^\infty g(x,t,\theta) \cdot \big[(L_n^t(f,\theta) - f(\theta)) - (L_n(f,\theta) - f(\theta)) + (L_n(f,\theta) - f(x))\big]\, d\theta \Big|$$

$$\leq \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta)\{|L_n^t(f,\theta) - f(\theta)| + |L_n(f,\theta) - f(\theta)|\}d\theta + |L_n^t(f,x) - f(x)|$$

$$\leq 2\|L_n^t(f) - f\| + \|L_n(f) - f\|$$

$$\leq C\{\omega_\varphi^2(f, (n^{-1} + \beta(t))^{1/2}) + \omega_\varphi^2(f, n^{-1/2})\}$$

$$\leq C\, \omega_\varphi^2(f, (n^{-1} + \beta(t))^{1/2}).$$

Thus

$$\|U_{n,1}^t(f) - f\| \leq C\omega_\varphi^2(f, (n^{-1} + \beta(t))^{1/2}).$$

The general case can be obtained by induction in the same manner using the idea of the above estimate, where $U_{n,1}^t$ and L_n^t are replaced by $U_{n,k+1}^t$ and $U_{n,k}^t$, respectively.

(*ii*) It is a direct consequence of (*i*), $\beta(t) \leq Cn^{-1}$ and $\|L_n(f) - f\| \sim \omega_\varphi^2(f, n^{-1/2})$.

(*iii*) It will be proved by mathematical induction. In view of (1) and (10) we get

$$|U_{n,1}^t(f,x) - U_{n,0}^t(f,x)| \leq \frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta) \cdot |L_n^t(f,\theta) - L_n(f,\theta)|d\theta$$

$$\leq \|L_n^t(f) - L_n(f)\| \leq 2\beta(t)\|\varphi^2 L_n''(f)\|.$$

Using (6) we obtain

$$\|U_{n,0}^t(f) - f\| \leq \|U_{n,1}^t(f) - f\| + \|U_{n,1}^t(f) - U_{n,0}^t(f)\|$$

$$\leq \|U_{n,1}^t(f) - f\| + 2\beta(t)\; C_0 n \|L_n(f) - f\|$$

$$\leq \|U_{n,1}^t(f) - f\| + \gamma\|L_n(f) - f\|.$$

But Theorem 3 implies that

$$\|U_{n,0}^t(f) - f\| \leq \|U_{n,1}^t(f) - f\| + \frac{\gamma}{1-\gamma} \cdot \|L_n^t(f) - f\|$$

$$= \|U_{n,1}^t(f) - f\| + \frac{\gamma}{1-\gamma} \cdot \|U_{n,0}^t(f) - f\|.$$

Thus

$$(1-2\gamma)\|U_{n,0}^t(f) - f\| \leq (1-\gamma)\|U_{n,1}^t(f) - f\|.$$

In similar way we obtain

$$(1-\gamma)\ \|U_{n,1}^t(f)-f\| \le \|U_{n,0}^t(f)-f\|.$$

In conclusion

$$(1-2\gamma)\ \|U_{n,0}^t(f)-f\| \le (1-\gamma)\|U_{n,1}^t(f)-f\| \le \|U_{n,0}^t(f)-f\|.$$

Furthermore, we shall suppose that (iii) is true for $1,2,\dots,k$. Then, by (1) we have

$$\begin{aligned}|U_{n,k+1}^t(f,x)-U_{n,k}^t(f,x)| &\le \frac{1}{G(x,t)}\cdot\int_0^\infty g(x,t,\theta)\cdot|U_{n,k}^t(f,\theta)-U_{n,k-1}^t(f,\theta)|d\theta\\ &\le \|U_{n,k}^t(f)-U_{n,k-1}^t(f)\| \le \dots \le \|L_n^t(f)-L_n(f)\|.\end{aligned}$$

Hence, by (10) and (6) we obtain

$$\begin{aligned}\|U_{n,k+1}^t(f)-U_{m,k}^t(f)\| &\le 2\beta(t)\|\varphi^2 L_n''(f)\| \le 2\beta(t)C_0 n\|L_n(f)-f\|\\ &\le \gamma\|L_n(f)-f\|.\end{aligned}$$

Using Theorem 3 and the hypothesis of induction we get

$$\begin{aligned}\|U_{n,k+1}^t(f)-U_{n,k}^t(f)\| &\le \frac{\gamma}{1-\gamma}\cdot\|U_{n,0}^t(f)-f\|\\ &\le \frac{\gamma}{1-\gamma}\cdot\frac{1-\gamma}{1-2\gamma}\cdot\|U_{n,1}^t(f)-f\|\le\dots\\ &\le \frac{\gamma}{1-\gamma}\cdot\frac{1-\gamma}{1-2\gamma}\cdots\frac{1-k\gamma}{1-(k+1)\gamma}\cdot\|U_{n,k}^t(f)-f\|\\ &= \frac{\gamma}{1-(k+1)\gamma}\cdot\|U_{n,k}^t(f)-f\|.\end{aligned}\tag{17}$$

So

$$\begin{aligned}\|U_{n,k}^t(f)-f\| &\le \|U_{n,k+1}^t(f)-f\| + \|U_{n,k+1}^t(f)-U_{n,k}^t(f)\|\\ &\le \|U_{n,k+1}^t(f)-f\| + \frac{\gamma}{1-(k+1)\gamma}\cdot\|U_{n,k}^t(f)-f\|,\end{aligned}$$

i.e.

$$(1-(k+2)\gamma)\|U_{n,k}^t(f)-f\| \le (1-(k+1)\gamma)\|U_{n,k+1}^t(f)-f\|.$$

The another inequality follows from (17):

$$\begin{aligned}\|U_{n,k+1}^t(f)-f\| &\le \|U_{n,k}^t(f)-f\| + \|U_{n,k+1}^t(f)-U_{n,k}^t(f)\|\\ &\le \|U_{n,k}^t(f)-f\| + \frac{\gamma}{1-(k+1)\gamma}\cdot\|U_{n,k}^t(f)-f\|\\ &= \frac{1-k\gamma}{1-(k+1)\ \gamma}\cdot\|U_{n,k}^t(f)-f\|,\end{aligned}$$

i.e.

$$(1-(k+1)\gamma)\|U^t_{n,k+1}(f)-f\| \le (1-k\gamma)\|U^t_{n,k}(f)-f\|,$$

which was to be proved.

Acknowledgement. The work of the author was supported by *Domus Hungarica Scientiarium et Artium.*

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Received March 21st., 2002.