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AN ESTIMATE OF THE RATE OF CONVERGENCE OF A BÉZIER VARIANT OF THE BASKAKOV-KANTOROVICH OPERATORS FOR BOUNDED VARIATION FUNCTIONS

Abstract. In the present paper we introduce a Bézier variant of the Baskakov–Kantorovich operators and study the rate of convergence for functions of bounded variation. Furthermore, we present the complete asymptotic expansion for the Baskakov–Kantorovich operators.

1. Introduction

Let $W(0,\infty)$ be the class of functions f which are locally integrable on $(0,\infty)$ and are of polynomial growth as $t \to \infty$, i.e., for some positive r, there holds $f(t) = O(t^r)$ as $t \to \infty$. The Kantorovich variant V_n^* of the Baskakov operators [3, Eq. (9.2.3), p. 115] associates to each function $f \in W(0,\infty)$ the series

$$V_n^*(f;x) = n \sum_{k=0}^{\infty} v_{n,k}(x) \int_{I_k} f(t)\, dt, \qquad x \in [0,\infty), \tag{1}$$

where $I_k = [k/n, (k+1)/n]$ and

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

The operators V_n^* result from the ordinary Baskakov operators V_n given by

$$V_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x)\; f\left(\frac{k}{n}\right), \qquad x \in [0,\infty)$$

by replacing $f\left(\frac{k}{n}\right)$ by $\int_{I_k} f(t)\, dt$ in order to approximate integrable functions.

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In the present paper we introduce the Bézier variants of the operators (1). For each function $f \in W(0,\infty)$ and $\alpha \geq 1$, we introduce the Bézier type Baskakov–Kantorovich operators $V_{n,\alpha}^*$ as

$$V_{n,\alpha}^*(f;x) = n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{I_k} f(t)\, dt, \tag{2}$$

where

$$Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$$

and

$$J_{n,k}(x) = \sum_{j=k}^{\infty} v_{n,j}(x)$$

is the Baskakov–Bézier basis function. It is obvious that $V_{n,\alpha}^*$ are positive linear operators and $V_{n,\alpha}^*(1;x) = 1$. In the special case $\alpha = 1$, the operators $V_{n,\alpha}^*$ reduce to the operators $V_n^* \equiv V_{n,1}^*$. Some basic properties of $J_{n,k}$ are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = v_{n,k}(x) \qquad (k = 0,1,2,\ldots);$
(ii) $J_{n,k}'(x) = n v_{n+1,k-1}(x) \qquad (k = 1,2,3,\ldots);$
(iii) $J_{n,k}(x) = n \int_0^x v_{n+1,k-1}(t)\, dt \qquad (k = 1,2,3,\ldots);$
(iv) $0 < \ldots < J_{n,k+1}(x) < J_{n,k}(x) < \ldots < J_{n,1}(x) < J_{n,0}(x) \equiv 1$ $(x > 0);$
(v) $J_{n,k}$ is strictly increasing on $[0,\infty)$.

Rates of convergence on functions of bounded variation, for different Bézier type operators, were studied in several papers, e.g., [7], [8], [9]. In the present paper we estimate the rate of convergence by the Bézier–Baskakov–Kantorovich operators (2).

Furthermore, we find the limit of the sequence $V_{n,\alpha}^*(f;x)$ for bounded locally integrable functions f having a discontinuity of the first kind in $x \in (0,\infty)$.

The last section presents the complete asymptotic expansion for the Baskakov–Kantorovich operators (1).

2. The main results

As main result we derive the following estimate on the rate of convergence.

THEOREM 1. *Assume that $f \in W(0,\infty)$ is a function of bounded variation on every finite subinterval of $(0,\infty)$. Furthermore, let $\alpha \geq 1$, $x \in (0,\infty)$ and $\lambda > 1$ be given. Then, for each $r \in \mathbb{N}$, there exists a constant $M(f,\alpha,r,x)$, such that, for sufficiently large n, the Bézier type Baskakov–Kantorovich* *operators* $V_{n,\alpha}^*$ *satisfy the estimate*

$$\left|V_{n,\alpha}^*(f;x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-)\right]\right| \tag{3}$$

$$\leq \frac{2\alpha\lambda(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x)$$

$$+ \frac{7\alpha\sqrt{1+x}}{2\sqrt{nx}} |f(x+) - f(x-)| + \frac{M(f,\alpha,r,x)}{n^r},$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-) & (0 \leq t < x), \\ 0 & (t = x), \\ f(t) - f(x+) & (x < t < \infty), \end{cases} \tag{4}$$

and $\bigvee_a^b (g_x)$ *is the total variation of* g_x *on* $[a,b]$.

REMARK 1. The exponent r in the last term of Eq. (3) can be chosen arbitrary large.

As an immediate consequence of Theorem 1 we obtain in the special case $\alpha = 1$ the following estimate.

COROLLARY 2. *Under the assumptions of Theorem* 1 *the following estimate, for sufficiently large* n,

$$\left|V_n^*(f;x) - \frac{1}{2}[f(x+) + f(x-)]\right| \leq \frac{2\lambda(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x)$$

$$+ \frac{7\sqrt{1+x}}{2\sqrt{nx}} |f(x+) - f(x-)| + \frac{M(f,1,r,x)}{n^r},$$

holds, where g_x *is as defined in Theorem* 1.

We mention that Aniol [2, Theorem 1] studied Kantorovich–type operators from a more general point of view. In the case of the operators V_n^* she used the crucial estimate (see [2, page 13])

$$|V_n^*(\mathrm{sign}_x(t);x)| \leq 10\left(8x^2 + 5x + 1\right) / \sqrt{nx(1+x)^3},$$

while our Eq. (11) yields, for $\alpha = 1$,

$$|V_n^*(\mathrm{sign}_x(t);x)| < 7\sqrt{(1+x)/(nx)}.$$

THEOREM 3. *Let* $x \in (0,\infty)$. *If* $f \in L(0,\infty)$ *has a discontinuity of the first* *kind in x, then we have*

$$\lim_{n\to\infty} V_{n,\alpha}^*(f;x) = \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-).$$

3. Auxiliary results

In order to prove our main result we shall need the following lemmas. Throughout the paper let e_r denote the monomials $e_r(t) = t^r$ $(r = 0, 1, 2, \ldots)$ and, for each real x, put $\psi_x(t) = t - x$.

LEMMA 4 ([10, Lemma 1]). *For all $x > 0$ and $n, k \in \mathbb{N}$, is satisfied the inequality*

$$Q_{n,k}^{(\alpha)}(x) \le \alpha\, v_{n,k}(x) < \alpha\sqrt{\frac{1+x}{2enx}}.$$

By direct calculation (cf. Lemma 9) we find

$$V_n^*(e_0;x) = 1, \qquad V_n^*(e_1;x) = x + \frac{1}{2n},$$
$$V_n^*(e_2;x) = x^2 + \frac{x(x+2)}{n} + \frac{1}{3n^2},$$
$$V_n^*(\psi_x^2;x) = \frac{x(1+x)}{n} + \frac{1}{3n^2}.$$

REMARK 2. Note that, given any $\lambda > 1$ and any $x > 0$, for all n sufficiently large, we have the estimate

$$V_n^*(\psi_x^2;x) < \frac{\lambda x(1+x)}{n}.$$

As we shall show in the last section (Lemma 10 and Remark 4), for each fixed $x \in [0,\infty)$ and $s \in \mathbb{N}_0$, the central moments $V_n^*(\psi_x^s;x)$ of the Baskakov–Kantorovich operators (1) satisfy

$$V_n^*(\psi_x^s;x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n\to\infty). \tag{5}$$

Throughout the paper let

$$K_{n,\alpha}(x,t) = n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\ \chi_{n,k}(t),$$

where $\chi_{n,k}$ denotes the characteristic function of the interval $[k/n, (k+1)/n]$ with respect to $[0,\infty)$. With this definition, for each function $f \in W(0,\infty)$, we have for all sufficiently large n, the relation

$$V_{n,\alpha}^*(f;x) = \int_0^\infty K_{n,\alpha}(x,t)\ f(t)\,dt. \tag{6}$$

Furthermore, put

$$\lambda_{n,\alpha}(x,y)=\int_0^y K_{n,\alpha}(x,t)\,dt. \tag{7}$$

Note that, in particular,

$$\lambda_{n,\alpha}(x,\infty)=\int_0^\infty K_{n,\alpha}(x,u)\,du=1.$$

LEMMA 5. *Let $x\in(0,\infty)$. For each $\lambda>1$, and for all sufficiently large n, we have,*

$$\lambda_{n,\alpha}(x,y)=\int_0^y K_{n,\alpha}(x,t)\,dt\le\frac{\lambda\alpha x(1+x)}{n(x-y)^2}\qquad(0\le y<x), \tag{8}$$

and

$$1-\lambda_{n,\alpha}(x,z)=\int_z^\infty K_{n,\alpha}(x,t)\,dt\le\frac{\lambda\alpha x(1+x)}{n(z-x)^2}\qquad(x<z<\infty). \tag{9}$$

Proof. We first prove Eq. (8). Notice that

$$\begin{aligned}\int_0^y K_{n,\alpha}(x,t)\,dt&\le\int_0^y K_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}\,dt\\&\le(x-y)^{-2}V_{n,\alpha}^*(\psi_x^2;x)\le\alpha(x-y)^{-2}V_{n,1}^*(\psi_x^2;x),\end{aligned}$$

where we applied Lemma 4. Now Eq. (8) is a consequence of Remark 2. The proof of Eq. (9) is similar. ∎

LEMMA 6 ([10, Lemma 5]). *For all $x\in(0,\infty)$, the inequality*

$$\left|\sum_{k>nx}v_{n,k}(x)-1/2\right|\le\frac{3\sqrt{1+x}}{\sqrt{nx}}$$

holds.

4. Proof of the main results

Proof of Theorem 1. Our starting point is the identity

$$\begin{aligned}f(t)=\frac{1}{2^\alpha}f(x+)+\left(1-\frac{1}{2^\alpha}\right)f(x-)+\frac{f(x+)-f(x-)}{2^\alpha}\operatorname{sign}_x(t)\\+g_x(t)+\delta_x(t)\left[f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right],\end{aligned}$$

where

$$\operatorname{sign}_x(t)=\begin{cases}2^\alpha-1 & (t>x),\\ 0 & (t=x),\\ -1 & (t<x),\end{cases}$$

$\delta_x(t)=1$ $(t=x)$ and $\delta_x(t)=0$ $(t\neq x)$. Since $V_{n,\alpha}^*(\delta_x;x)=0$, we conclude

$$(10)\qquad \left|V_{n,\alpha}^{*}(f;x)-\left[\frac{1}{2^{\alpha}}f\left(x+\right)+\left(1-\frac{1}{2^{\alpha}}\right)f\left(x-\right)\right]\right|$$
$$\leq\frac{1}{2^{\alpha}}\left|f\left(x+\right)-f\left(x-\right)\right|\left|V_{n,\alpha}^{*}(\mathrm{sign}_{x}\left(t\right);x)\right|+\left|V_{n,\alpha}^{*}(g_{x};x)\right|.$$

First, we estimate $V_{n,\alpha}^{*}(\mathrm{sign}_{x}\left(t\right);x)$ as follows. Choose k' such that $x\in\left[k'/n,\left(k'+1\right)/n\right)$. Hence,

$$\begin{aligned}V_{n,\alpha}^{*}(\mathrm{sign}_{x}(t);x)&=\sum_{k=0}^{k'-1}(-1)Q_{n,k}^{(\alpha)}(x)+nQ_{n,k'}^{(\alpha)}(x)\\&\quad\times\Big(\int_{k'/n}^{x}(-1)\,dt+\int_{x}^{(k'+1)/n}(2^{\alpha}-1)\,dt\Big)\\&\quad+\sum_{k=k'+1}^{\infty}(2^{\alpha}-1)Q_{n,k}^{(\alpha)}(x)\\&=2^{\alpha}\sum_{k=k'+1}^{\infty}Q_{n,k}^{(\alpha)}(x)+nQ_{n,k'}^{(\alpha)}(x)\int_{x}^{(k'+1)/n}2^{\alpha}dt-1,\end{aligned}$$

since $\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha)}(x)=1$. Noting that

$$0\leq nQ_{n,k'}^{(\alpha)}(x)\int_{x}^{(k'+1)/n}2^{\alpha}\,dt\leq 2^{\alpha}Q_{n,k'}^{(\alpha)}(x)$$

we conclude that

$$\begin{aligned}|V_{n,\alpha}^{*}(\mathrm{sign}_{x}(t);x)|&\leq\left|2^{\alpha}\sum_{k=k'+1}^{\infty}Q_{n,k}^{(\alpha)}(x)-1\right|+2^{\alpha}\,Q_{n,k'}^{(\alpha)}(x)\\&=|2^{\alpha}J_{n,k'+1}^{\alpha}(xt)-1|+2^{\alpha}\,Q_{n,k'}^{(\alpha)}(x).\end{aligned}$$

Application of the inequality $|a^{\alpha}-b^{\alpha}|\leq\alpha\,|a-b|$, for $0\leq a,b\leq 1$, and $\alpha\geq 1$, yields

$$\begin{aligned}\left|2^{\alpha}J_{n,k'+1}^{\alpha}\left(x\right)-1\right|&\leq\alpha 2^{\alpha}\left|J_{n,k'+1}\left(x\right)-1/2\right|\\&=\alpha 2^{\alpha}\left|\sum_{k=k'+1}^{\infty}v_{n,k}\left(x\right)-1/2\right|=\alpha 2^{\alpha}\left|\sum_{k>nx}v_{n,k}(x)-1/2\right|.\end{aligned}$$

Therefore, by Lemma 6 and Lemma 4, we obtain

$$(11)\quad |V_{n,\alpha}^{*}(\mathrm{sign}_{x}(t);x)|\leq\alpha 2^{\alpha}\frac{3\sqrt{1+x}}{\sqrt{nx}}+2^{\alpha}\frac{\alpha\sqrt{1+x}}{\sqrt{2enx}}<\frac{7\alpha\cdot 2^{\alpha-1}\sqrt{1+x}}{\sqrt{nx}}.$$

In order to complete the proof of the theorem we need an estimate of $V_{n,\alpha}^{*}(g_{x};x)$. We use the integral representation (6) and decompose $[0,\infty)$

into three parts as follows

$$(12)\qquad V^*_{n,\alpha}(g_x;x) = \Big(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{\infty}\Big) K_{n,\alpha}(x,t) g_x(t)dt$$
$$= I_1 + I_2 + I_3, \text{ say.}$$

We start with I_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|g_x(t)| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x)$$

and thus

$$(13)\qquad |I_2| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x) \le \frac{1}{n} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x).$$

Next we estimate I_1. Put $y = x - x/\sqrt{n}$. Using integration by parts with Eq. (7) we have

$$I_1 = \int_0^y g_x(t)\ d_t\lambda_{n,\alpha}(x,t) = g_x(y)\lambda_{n,\alpha}(x,y) - \int_0^y \lambda_{n,\alpha}(x,t)\ d_t g_x(t).$$

Since $|g_x(y)| = |g_x(y) - g_x(x)| \le \bigvee_y^x (g_x)$, we conclude that

$$|I_1| \le \bigvee_y^x (g_x)\ \lambda_{n,\alpha}(x,y) + \int_0^y \lambda_{n,\alpha}(x,t)\ d_t\Big(-\bigvee_t^x (g_x)\Big).$$

Since $y = x - x/\sqrt{n} \le x$, Eq. (8) of Lemma 5 implies, for each $\lambda > 1$ and n sufficiently large, that

$$|I_1| \le \frac{\alpha\lambda x(1+x)}{n(x-y)^2} \bigvee_y^x (g_x) + \frac{\alpha\lambda x(1+x)}{n} \int_0^y \frac{1}{(x-t)^2}\ d_t\Big(-\bigvee_t^x (g_x)\Big).$$

Integrating the last term by parts, we obtain

$$|I_1| \le \frac{\alpha\lambda x(1+x)}{n} \Big(x^{-2} \bigvee_0^x (g_x) + 2\int_0^y \frac{\bigvee_t^x (g_x)}{(x-t)^3}\, dt\Big).$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x)(x-t)^{-3}\, dt = \sum_{k=1}^{n-1} \int_{x/\sqrt{k+1}}^{x/\sqrt{k}} \bigvee_{x-t}^x (g_x)\, t^{-3}\, dt$$
$$\le \frac{1}{2x^2} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x} (g_x).$$

Hence

(14) $$|I_1| \le \frac{2\alpha\lambda(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x} (g_x) .$$

Finally, we estimate I_3. We put

$$\widetilde{g}_x(t) = \begin{cases} g_x(t) & (0 \le t \le 2x), \\ g_x(2x) & (2x < t < \infty), \end{cases}$$

and divide $I_3 = I_{31} + I_{32}$, where

$$I_{31} = \int_{x+x/\sqrt{n}}^{\infty} K_{n,\alpha}(x,t)\ \widetilde{g}_x(t)\ dt \qquad \text{and}$$

$$I_{32} = \int_{2x}^{\infty} K_{n,\alpha}(x,t)\ [g_x(t) - g_x(2x)]\ dt.$$

With $y = x + x/\sqrt{n}$ the first integral can be written in the form

$$I_{31} = \lim_{R\to+\infty} \Big\{ g_x(y)[1 - \lambda_{n,\alpha}(x,y)] + \widetilde{g}_x(R)[\lambda_{n,\alpha}(x,R) - 1] + \int_y^R [1 - \lambda_{n,\alpha}(x,t)]\ d_t\widetilde{g}_x(t) \Big\}.$$

By Eq. (9) of Lemma 5, we conclude, for each $\lambda > 1$ and n sufficiently great, that

$$|I_{31}| \le \frac{\alpha\lambda x(1+x)}{n} \lim_{R\to+\infty} \left\{ \frac{\bigvee_x^y(g_x)}{(y-x)^2} + \frac{|\widetilde{g}_x(R)|}{(R-x)^2} + \int_y^R \frac{1}{(t-x)^2} d_t\Big(\bigvee_x^t(\widetilde{g}_x)\Big) \right\}$$
$$= \frac{\alpha\lambda x(1+x)}{n} \left\{ \frac{\bigvee_x^y(g_x)}{(y-x)^2} + \int_y^{2x} \frac{1}{(t-x)^2}\, d_t\left(\bigvee_x^t(g_x)\right) \right\}.$$

In a similar way as above we obtain

$$\int_y^{2x} \frac{1}{(t-x)^2}\, d_t\Big(\bigvee_x^t(g_x)\Big) \le x^{-2}\bigvee_x^{2x}(g_x) - \frac{\bigvee_x^y(g_x)}{(y-x)^2} + x^{-2}\sum_{k=1}^{n-1}\bigvee_x^{x+x/\sqrt{k}}(g_x)$$

which implies the estimate

(15) $$|I_{31}| \le \frac{2\alpha\lambda(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x}^{x+x/\sqrt{k}} (g_x) .$$

Lastly, we estimate I_{32}. By assumption, there exists an integer r, such that $f(t) = O(t^{2r})$ as $t \to \infty$. Thus, for a certain constant $M > 0$ depending only on f, x and r, we have

$$|I_{32}| \le Mn \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \chi_{n,k}(t)\, t^{2r}\, dt$$
$$\le \alpha Mn \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} \chi_{n,k}(t)\, t^{2r}\, dt,$$

where we used Lemma 4. Obviously, $t \ge 2x$ implies that $t \le 2(t-x)$ and it follows that

$$|I_{32}| \le 2^{2r} \alpha M V_n^*(\psi_x^{2r}; x).$$

By Eq. 5, the central moments of the Baskakov–Kantorovich operators (1) satisfy $V_n^*(\psi_x^{2r}; x) = O(n^{-r})$ $(n \to \infty)$, and we arrive at

$$I_{32} = O(n^{-r}) \qquad (n \to \infty). \tag{16}$$

Collecting the estimates (13), (14), (15), and (16), we obtain with regard to Eq. (12)

$$|V_{n,\alpha}^*(g_x; x)| \le \frac{2\alpha\lambda(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(n^{-r}) \quad (n \to \infty). \tag{17}$$

Finally, combining (10), (11), (17), we obtain (3). This completes the proof of Theorem 1. ∎

P r o o f o f T h e o r e m 3. Since the function ψ_x^2 given by $\psi_x^2(t) = (t-x)^2$ is of bounded variation on every finite subinterval of $[0, \infty)$, we deduce from Theorem 1 that, for all $x \in (0, \infty)$,

$$\lim_{n \to \infty} V_{n,\alpha}^*(\psi_x^2; x) = 0.$$

If $f \in L_\infty(0, \infty)$, then g_x defined as in (4) is also bounded and is continuous at the point x. By the Korovkin theorem, we conclude

$$\lim_{n \to \infty} V_{n,\alpha}^*(g_x; x) = g_x(x) = 0.$$

Therefore, the right–hand side of Inequality (10) tends to zero as $n \to \infty$. This completes the proof of Theorem 3. ∎

5. Asymptotic expansion for the Baskakov–Kantorovich operators

Throughout this section let the numbers $Z(s,k,j)$ be given by

$$Z(s,k,j) = (-1)^{k-j} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} S_{r-j}^{r-k} \sigma_{r+1}^{r+1-j} \left(1 - \frac{j}{r+1}\right) \qquad (0 \le j \le k \le s). \tag{18}$$

The quantities S_j^i and σ_j^i in Eq. (18) denote the Stirling numbers of the first resp. second kind. The Stirling numbers are defined by

$$x^{\underline{j}} = \sum_{i=0}^{j} S_j^i x^i, \qquad x^j = \sum_{i=0}^{j} \sigma_j^i x^{\underline{i}} \qquad (j = 0, 1, \ldots), \tag{19}$$

where $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$, $x^{\underline{0}} = 1$ is the falling factorial.

For $q \in \mathbb{N}$ and $x \in (0, \infty)$, let $K[q; x]$ be the class of all functions $f \in L_\infty(0, \infty)$ which are q times differentiable at x. The following theorem is the main result of this section.

THEOREM 7. *Let $q \in \mathbb{N}$ and $x \in (0, \infty)$. For each function $f \in K[2q; x]$, the Baskakov–Kantorovich operators possess the asymptotic expansion*

$$V_n^*(f; x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} Z(s, k, j) + o(n^{-q}) \qquad (n \to \infty), \tag{20}$$

where the numbers $Z(s, k, j)$ are as defined by Eq. (18).

REMARK 3. For the convenience of the reader, we give the series explicitly, for $q = 3$:

$$\begin{aligned} V_n^*(f; x) = f(x) &+ \frac{f'(x) + x(1+x) f''(x)}{2n} \\ &+ \frac{4f^{(2)}(x) + 2x(1+x)(5+4x) f^{(3)}(x) + 3x^2(1+x)^2 f^{(4)}(x)}{24n^2} \\ &+ \frac{1}{48n^3} \Big(2f^{(3)}(x) + 2x(1+x)\left(5 + 10x + 6x^2\right) f^{(4)}(x) \\ &+ x^2(1+x)^2(7+8x) f^{(5)}(x) + x^3(1+x)^3 f^{(6)}(x)\Big). \end{aligned}$$

An immediate consequence of Theorem 7 is the following Voronovskaja–type formula.

COROLLARY 8. *Let $x \in (0, \infty)$. For each function $f \in K[2; x]$, the operators V_n^* satisfy*

$$\lim_{n \to \infty} n\left(V_n^*(f; x) - f(x)\right) = \frac{1}{2}\left(f'(x) + x(1+x) f''(x)\right). \tag{21}$$

The proof of Theorem 7 is based on the following lemmas.

LEMMA 9. *The moments of the Baskakov–Kantorovich operators possess the representation*

$$V_n^*(e_r; x) = \sum_{k=0}^{r} n^{-k} \sum_{j=0}^{k} (-1)^{k+j} S_{r-j}^{r-k} \sigma_{r+1}^{r+1-j} \left(1 - \frac{j}{r+1}\right) x^{r-j} \qquad (r = 0, 1, 2, \ldots).$$

P r o o f o f L e m m a 9. By direct calculation, we have

$$V_n^*(e_r;x) = \frac{n^{-r}}{r+1}\sum_{k=0}^{\infty} v_{n,k}(x)\ \left((k+1)^{r+1} - k^{r+1}\right).$$

Taking advantage of the second identity of (19), we obtain

$$\left((k+1)^{r+1} - k^{r+1}\right) = \sum_{j=0}^{r+1} \sigma_{r+1}^{j} \left((k+1)^{\underline{j}} - k^{\underline{j}}\right) = \sum_{j=0}^{r} (j+1)\,\sigma_{r+1}^{j+1} k^{\underline{j}}$$

which yields

$$\begin{aligned} V_n^*(e_r;x) &= \frac{n^{-r}}{r+1}\sum_{j=0}^{r}(j+1)\,\sigma_{r+1}^{j+1}\sum_{k=0}^{\infty} v_{n,k}(x)\ k^{\underline{j}} \\ &= \frac{n^{-r}}{r+1}\sum_{j=0}^{r}(j+1)\,\sigma_{r+1}^{j+1}(n+j-1)^{\underline{j}}x^{j}. \end{aligned}$$

Using the identity

$$(n+j-1)^{\underline{j}} = (-1)^j(-n)^{\underline{j}} = (-1)^j \sum_{k=0}^{j} S_j^k(-n)^k$$

we conclude that

$$\begin{aligned} V_n^*(e_r;x) &= \frac{1}{r+1}\sum_{k=0}^{r} n^{k-r}\sum_{j=k}^{r}(-1)^{k+j}(j+1)\,S_j^k\sigma_{r+1}^{j+1}x^j \\ &= \frac{1}{r+1}\sum_{k=0}^{r} n^{-k}\sum_{j=0}^{k}(-1)^{k+j}(r-j+1)\,S_{r-j}^{r-k}\sigma_{r+1}^{r-j+1}x^{r-j} \end{aligned}$$

which completes the proof of Lemma 9. ∎

LEMMA 10. *For $s = 0,1,2,\ldots$, the central moments of the Baskakov–Kantorovich operators possess the representation*

$$V_n^*(\psi_x^s;x) = \sum_{k=\lfloor (s+1)/2\rfloor}^{s} n^{-k}\sum_{j=0}^{k} x^{s-j}Z(s,k,j),$$

where the numbers $Z(s,k,j)$ are as defined in Eq. (18).

REMARK 4. An immediate consequence of Lemma 10 is that, for $s = 0,1,$ $2,\ldots,$

$$V_n^*(\psi_x^s;x) = O\left(n^{-\lfloor (s+1)/2\rfloor}\right) \qquad (n\to\infty).$$

P r o o f o f L e m m a 10. Application of the binomial formula yields for the central moments

$$\begin{aligned}
V_n^*(\psi_x^s;x) &= \sum_{r=0}^{s}\binom{s}{r}(-x)^{s-r}V_n^*(e_r;x)\\
&= \sum_{k=0}^{s} n^{-k}\sum_{j=0}^{k}(-1)^{k+j}x^{s-j}\sum_{r=k}^{s}(-1)^{s-r}\binom{s}{r}S_{r-j}^{r-k}\sigma_{r+1}^{r+1-j}\left(1-\frac{j}{r+1}\right)\\
&= \sum_{k=0}^{s} n^{-k}\sum_{j=0}^{k}x^{s-j}Z(s,k,j)\,.
\end{aligned}$$

It remains to prove that $V_n^*(\psi_x^s;x) = O\left(n^{-\lfloor (s+1)/2\rfloor}\right)$. It is sufficient to show that, for $0 \le j \le k$, there holds $Z(s,k,j) = 0$ if $2k < s$.

Before we recall some known facts about Stirling numbers which will be useful in the sequel. The Stirling numbers of first resp. second kind possess the representation

$$S_r^{r-k} = \sum_{\mu=0}^{k} C_{k,k-\mu}\binom{r}{k+\mu}, \qquad \sigma_r^{r-k} = \sum_{\nu=0}^{k}\overline{C}_{k,k-\nu}\binom{r}{k+\nu} \tag{22}$$

$$(k = 0,\ldots,r),$$

where $C_{k,k} = \overline{C}_{k,k} = 0$, for $k \ge 1$ (see [4, p.151, Eq. (5), resp. p. 171, Eq. (7)]). The coefficients $C_{k,i}$ resp. $\overline{C}_{k,i}$ are independent of r and satisfy certain partial difference equations ([4, p. 150]). Some closed expressions for $C_{k,i}$ and $\overline{C}_{k,i}$ can be found in [1, p. 113]. We first consider the case $j \ge 1$. Taking advantage of representation (22) we obtain, for $1 \le j \le k$,

$$\begin{aligned}
S_{r-j}^{r-k}\sigma_{r+1}^{r+1-j} &= \sum_{\mu=0}^{k-j} C_{k-j,k-j-\mu}\binom{r-j}{k-j+\mu}\sum_{\nu=1}^{j}\overline{C}_{j,j-\nu}\binom{r+1}{j+\nu}\\
&= \sum_{\mu=0}^{k-j}\sum_{\nu=1}^{j}(r+1)^{\underline{k+1}}P(k,j,\mu,\nu;r)\,,
\end{aligned}$$

where

$$P(k,j,\mu,\nu;r) = \frac{C_{k-j,k-j-\mu}}{(k-j+\mu)!}\frac{\overline{C}_{j,j-\nu}}{(j+\nu)!}(r-j)^{\underline{\nu-1}}(r-k)^{\underline{\mu}}$$

is a polynomial in the variable r of degree $\le \mu+\nu-1$. Thus, we conclude

$$\begin{aligned}
Z(s,k,j) &= (-1)^{k-j}\sum_{\mu=0}^{k-j}\sum_{\nu=1}^{j}\sum_{r=k}^{s}(-1)^{s-r}\binom{s}{r}r^{\underline{k}}(r+1-j)\,P(k,j,\mu,\nu;r)\\
&= (-1)^{j}\,s^{\underline{k}}\sum_{\mu=0}^{k-j}\sum_{\nu=1}^{j}\sum_{r=0}^{s-k}(-1)^{s-r}\binom{s-k}{r}(r+k+1-j)\\
&\qquad\times P(k,j,\mu,\nu;r+k)\,.
\end{aligned}$$

Since $(r+k+1-j)\,P\,(k,j,\mu,\nu;r+k)$ is a polynomial in the variable r of degree $\leq \mu+\nu \leq k$, the inner sum vanishes if $k < s-k$, i.e., if $2k < s$.

In the case $j=0$, we have

$$\begin{aligned} Z\,(s,k,0) &= (-1)^k \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} S_r^{r-k} \\ &= (-1)^k \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} \sum_{\mu=0}^{k} C_{k,k-\mu} \binom{r}{k+\mu} \\ &= (-1)^k\, s^{\underline{k}} \sum_{\mu=0}^{k} \frac{C_{k,k-\mu}}{(k+\mu)!} \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} r^{\underline{\mu}} \end{aligned}$$

which vanishes if $k < s-k$, i.e., if $2k < s$. This completes the proof of Lemma 10. ∎

In order to derive as our main result the complete asymptotic expansion of the operators V_n^* we use a general approximation theorem for positive linear operators due to Sikkema [5, Theorem 3] (cf. [6, Theorems 1 and 2]).

LEMMA 11. *Let I be an interval. For $q \in \mathbb{N}$ and fixed $x \in I$, let $A_n : L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property*

$$(23) \qquad A_n(\psi_x^s; x) = O(n^{-\lfloor (s+1)/2 \rfloor}) \qquad (n \to \infty) \qquad (s = 0,1,\ldots,2q+2).$$

Then, we have for each $f \in L_\infty(I)$ which is $2q$ times differentiable at x the asymptotic relation

$$(24) \qquad A_n(f;x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} A_n(\psi_x^s;x) + o(n^{-q}) \qquad (n \to \infty).$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ in (24) can be replaced by $O(n^{-(q+1)})$.

Proof of Theorem 7. By Remark 4, assumption (23) in Lemma 11 is valid for the operators V_n^*. Therefore, we can apply Lemma 11 and the assertion of Theorem 7 follows after some calculations by Lemma 10. ∎

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