DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

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KAMENEV-TYPE OSCILLATION CRITERIA FOR HYPERBOLIC DELAY DIFFERENCE EQUATIONS

Abstract. Some new oscillation criteria and discrete Kamenev-type oscillation criteria for hyperbolic nonlinear delay difference equations are obtained.

1. Introduction

Qualitative theory for discrete dynamics systems with one dimension, i.e., ordinary difference equations which parallels the qualitative theory of differential equations, has been investigated by several authors, see e.g., the monographs [1], [2] and the references cited therein. On the other hand, the nonlinear discrete dynamics systems involving functions of two or more independent variables, i.e., partial difference equations (PDEs), are as important as difference equations, comparatively few papers have been devoted to the qualitative theory of their solutions, see, e.g., the review article of Zhang [18]. In fact, partial difference equations arise in the approximation of solutions of partial differential equations by finite difference methods, random walk problems, the study on molecular orbits, mathematical physics problems and other problems in population dynamics [3]–[6] . Hence, to further develop the qualitative theory of partial difference equations, in this paper we shall consider the following hyperbolic nonlinear delay difference equation

$$
\begin{aligned}
(1.1)\qquad & \Delta_2(p_n\Delta_2 y_{m,n}) + q_{m,n}f(y_{m,n-\sigma}) \\
& = r_n\nabla^2 y_{m-1,n+1} + \sum_{j\in J} R_{j,n}\nabla^2 y_{m-1,n+1-\gamma_j},
\end{aligned}
$$

where $\{y_{m,n}\} = \{y_{m_1,m_2,\ldots,m_l,n}\}$ which is defined in $\Omega\times\mathbb{N}_{n_0}$, $J=\{1,2,\ldots,J_0\}$, $\mathbb{N}_{n_0}=\{n_0,n_0+1,\ldots\}$ and $\Omega=\{p_1^{(1)},\ldots,p_{M_1}^{(1)}\}\times\ldots\{p_1^{(l)},\ldots,p_{M_l}^{(l)}\}$ and every $p_i^{(j)}\in\mathbb{Z}:=\{\ldots,-1,0,1,\ldots\}$.

Key words and phrases: oscillation, hyperbolic delay difference equations.
1991 *Mathematics Subject Classification*: 39A10.

We will need the following definitions which are extracted from [7].

DEFINITION 1.1. m is said to be an interior point of Ω, if m+1:={m_1+1, $\text{m}_2, \dots, m_l\}\cup\dots\cup\{\text{m}_1, \text{m}_2, \dots, m_{l-1}, m_l+1\}$ and m-1:={m_1-1, $\text{m}_2, \dots, m_l\}\cup \dots\cup\{\text{m}_1, \text{m}_2, \dots, m_{l-1}, m_l - 1\}$ are all in Ω; Ω^0 which consists of all interior points, is said to be interior of Ω.

DEFINITION 1.2. m is said to be a convex boundary point of Ω, if m$\in \Omega$ and at least l points of m$\pm$1 are in Ω; m is said to be a convex boundary point, if m, m$\pm 1 \in \Omega$ but just one of the points $\{\text{m}_1\pm1, \text{m}_2, \dots, m_l \pm 1\}$ is not in Ω, where $\{\text{m}_1\pm1, \text{m}_2, \dots, m_l \pm 1\} := \{m_1 + 1, m_2, \dots, m_l + 1\} \cup \{m_1 - 1, m_2 + 1, \dots, m_l + 1\} \cup \dots \cup \{m_1 - 1, m_2, \dots, m_l - 1\}$, $\partial\Omega$ which consists of all (convex and concave) boundary points, is said to be a boundary of Ω.

DEFINITION 1.3. Ω is said to be convex, if $\partial\Omega$ consists only of all convex points.

DEFINITION 1.4. m is said to be an exterior point, if it is neither an interior point nor a boundary point.

DEFINITION 1.5. m is said to be (admissible) point, if at least two points of m$\pm$1 are in Ω.

DEFINITION 1.6. Ω is said to be a connected net , if Ω consists only of all (admissible) points.

DEFINITION 1.7. If Ω is a rectangular solid net, then $\partial\Omega$ consists only of all convex boundary points, and Ω a convex connected solid net.

Throughout this paper we consider a convex connected solid net Ω, and assume that the following conditions are satisfied:

(h1) $r_n,\ p_n \in \mathbb{N}_{n_0} \to \mathbb{R}^+$, $\sum\limits_{n=n_0}^{\infty} \frac{1}{p_n} = \infty$, $R_{j,n} \in J \times \mathbb{N}_{n_0} \to \mathbb{R}^+$;

(h2) $q_{m,n} \in \Omega \times \mathbb{N}_{n_0} \to \mathbb{R}^+$, $q_n = \min_{m\in\Omega}\{q_{m,n}\}$, n$\in \mathbb{N}_{n_0}$, $\{q_n\}$ has a positive subsequence;

(h3) $\sigma \in \mathbb{N}_1$ and $\gamma_j \in \mathbb{N}_1$, for $j \in J$,

(h4) $f \in C(\mathbb{R}, \mathbb{R})$ is convex, $uf(u) > 0$ for $u \neq 0$, $f(u) \geq ku$.

We write ∇^2 is the discrete Laplacian operator, which is defined by $\nabla^2 y_{m-1,n+1} = \sum\limits_{i=1}^{l} \Delta^2_{m_i} y_{m_1,m_2,\dots,m_{i-1},m_i-1\dots,m_l,n+1}$, where Δ^2_i is the partial difference operator of order two i.e., $\Delta^2_i y_{m,n} = \Delta_i(\Delta_i y_{m,n})$, $\Delta_{m_i} y_{m,n} = y_{m_1,m_2,\dots,m_i+1,\dots,m_l,n} - y_{m_1,m_2,\dots,m_i,\dots,m_l,n}$ and $\Delta_2 y_{m,n} = y_{m,n+1} - y_{m,n}$.

With Eq. (1.1) we consider the boundary condition

$$\text{(B)} \qquad \Delta_N y_{m-1,n} + g_{m,n} y_{m,n} = 0, \quad \text{on} \quad \partial\Omega \times \mathbb{N}_{n_0},$$

and the initial condition (IC)

$$y_{m,s} = \mu_{m,s}, \quad \text{for } n_0 - M \le s \le n_0, \tag{1.2}$$

where $\Delta_N y_{m-1,n}$ is the normal difference at $(m,n) \in \partial\Omega \times \mathbb{N}_{n_0}$ is defined by $\Delta_N y_{m-1,n} = \sum_{all\ m\pm 1 \notin \Omega}(\Delta_1 y_{m,n} - \Delta_1 y_{m-1,n}) = \sum_{all\ m\pm 1 \notin \Omega} \Delta_1^2 y_{m,n}$, $M = \max\{\sigma, \gamma_j\colon \text{and } j \in J\}$ and $g_{m,n} \in \partial\Omega \times \mathbb{N}_{n_0} \to \mathbb{R}^+$.

By a solution of initial boundary value problem (1.1), (B), (1.2) (for short we use the notion IBVP (1.1), (B)) we mean a sequence $\{y_{m,n}\}$ which satisfies Eq. (1.1) for $(m,n) \in \Omega \times \mathbb{N}_{n_0}$, satisfies (B) for $(m,n) \in \partial\Omega \times \mathbb{N}_{n_0}$ and satisfies IC (1.2) for $(m,n) \in \Omega \times \{n_0 - M, \dots, n_0\}$.

For the oscillation of the hyperbolic delay differential equation

$$\begin{aligned} \frac{\partial}{\partial t}\left(p(t)\frac{\partial}{\partial t}u(x,t)\right) &= a(t)\nabla^2 u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,t-\tau_i) - q(x,t)u(x,t) \\ &\quad - \sum_{j=1}^{n} q_j(x,t) f(u(x,t-\sigma_j)), \end{aligned} \tag{1.3}$$

together with the boundary condition

$$\frac{\partial u(x,t)}{\partial N} + \gamma(x,t)u(x,t) = 0, \quad \text{on } (x,t) \in \partial\Omega \times \mathbb{R}^+, \tag{B1}$$

where $\nabla^2 u$ is the Laplacian in $\mathbb{R}^n$, $(x,t) \in \Omega \times \mathbb{R}^+ \equiv G$, $\mathbb{R}^+ = [0,\infty)$, Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, $\gamma(x,t)$ is nonnegative continuous function on $\partial\Omega \times \mathbb{R}^+$, and N is the unit exterior normal vector to $\partial\Omega$, we refer to the papers [8]–[17],

Our aim in this paper is to establish some sufficient conditions for oscillation of the IBVP (1.1), (B) in $\Omega \times \mathbb{N}_{n_0}$, in the sense that there does not exist $n_1 \in \mathbb{N}_{n_0}$ such that $y_{m,n} > 0$ or $y_{m,n} < 0$ for $n \in \mathbb{N}_{n_1}$. Our results can be considered as the discrete analogues of some results in [8], [14]. To the best of our knowledge nothing is known up to now regarding the qualitative behavior of the IBVP (1.1), (B).

2. Main results

In this section we will establish some oscillation criteria for IBVP (1.1), (B). Before stating our main results we need the following:

LEMMA 2.1 ([7] (Discrete Gaussian formula)). *Let Ω be a convex connected solid net. Then we have*

$$\sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} = \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n}.$$

THEOREM 2.1. *Assume that* (h1)–(h4) *hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=1}^{\infty}$ *such that,*

$$\limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[k\rho_l q_l - \frac{(p_{l-\sigma})(\Delta\rho_l)^2}{4\rho_l} \right] = \infty. \tag{2.1}$$

Then every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

Proof. Suppose, to a contrary, that $\{y_{m,n}\}$ is a nonoscillatory solution of the IBVP (1.1), (B). Without loss of generality, we may assume that there exists $n_1 \in \mathbb{N}_{n_0}$ such that $y_{m,n-M} > 0$ for all $n \in \mathbb{N}_{n_1}$. Summing Eq. (1.1) over Ω, we have

$$\begin{aligned} &\Delta_2(p_n \sum_{m\in\Omega} \Delta_2 y_{m,n}) + \sum_{m\in\Omega} q_{m,n} f(y_{m,n-\sigma}) \\ &= r_n \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} + \sum_{j\in J} R_{j,n} \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1-\gamma_j}, \\ &\qquad \text{for } (m,n) \in \Omega \times \mathbb{N}_{n_1}. \end{aligned} \tag{2.2}$$

From Lemma 2.1 and (B) we find that

$$\begin{aligned} \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1} &= \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1} \\ &= - \sum_{m\in\partial\Omega} g_{m,n+1} y_{m,n+1} \le 0, \quad \text{for } n \in \mathbb{N}_{n_1}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \sum_{m\in\Omega} \nabla^2 y_{m-1,n+1-\gamma_j} &= \sum_{m\in\partial\Omega} \Delta_N y_{m-1,n+1-\gamma_j} \\ &= - \sum_{m\in\partial\Omega} g_{m,n+1-\gamma_j} y_{m,n+1-\gamma_j} \le 0, \end{aligned} \tag{2.4}$$

for $j \in J$ and $n \in \mathbb{N}_{n_1}$. From (h2) and by using the discrete Jensens's inequality, we have

$$\begin{aligned} \sum_{m\in\Omega} q_{m,n} f(y_{m,n-\sigma}) &\ge q_n \sum_{m\in\Omega} f(y_{m,n-\sigma}) \\ &\ge q_n \sum_{m\in\Omega} f\left(\frac{1}{|\Omega|} \sum_{m\in\Omega} y_{m,n-\sigma} \right) |\Omega|, \text{ for } n \in \mathbb{N}_{n_1}. \end{aligned} \tag{2.5}$$

Set

$$x_n = \frac{1}{|\Omega|} \sum_{m\in\Omega} y_{m,n}. \tag{2.6}$$

Then, by (h4) and (2.2)-(2.6) we obtain

$$\Delta(p_n \Delta x_n) + k q_n x_{n-\sigma} \le 0, \tag{2.7}$$

where Δ is the ordinary difference operator. In view of (h2), we have

$$\Delta(p_n \Delta x_n) \le -k q_n x_{n-\sigma} \le 0, \qquad n \in \mathbb{N}_{n_1},$$

and so $\{p_n \Delta x_n\}$ is nonincreasing sequence. We first show that $p_n \Delta x_n$ is positive. Indeed, since $\{q_n\}$ has a positive subsequence, then $p_n \Delta x_n$ is of constant sign. Suppose there exists an integer $n_2 \geq n_1$ such that $p_{n_1} \Delta x_{n_1} = c < 0$ for $n \geq n_2$. Then $p_n \Delta x_n \leq c$ which implies that

$$\Delta x_n \leq \frac{c}{p_n},$$

and this yields

$$x_n \leq x_{n_2} + c \sum_{i=n_2}^{n-1} \frac{1}{p_i} \to -\infty \text{ as } n \to \infty,$$

which contradicts the fact that $x_n > 0$ for all sufficiently large n. Hence $p_n \Delta x_n$ is positive. Therefore, we see that

$$x_n > 0, \ \Delta x_n \geq 0, \ \Delta(p_n \Delta x_n) \leq 0, \quad n \in \mathbb{N}_{n_1}. \tag{2.8}$$

Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{p_n \Delta x_n}{x_{n-\sigma}}, \tag{2.9}$$

and observe that $w_n > 0$, and

$$\Delta w_n = p_{n+1} \Delta x_{n+1} \Delta \left[\frac{\rho_n}{x_{n-\sigma}} \right] + \frac{\rho_n \Delta(p_n \Delta x_n)}{x_{n-\sigma}}. \tag{2.10}$$

Then (2.7) and (2.10) imply that

$$\Delta w_n \leq -k \rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1} \Delta x_{n+1} \Delta x_{n-\sigma}}{x_{n+1-\sigma} x_{n-\sigma}}. \tag{2.11}$$

But from (2.8), we have

$$p_{n-\sigma} \Delta x_{n-\sigma} \geq p_{n+1} \Delta x_{n+1}, \quad \text{and} \quad x_{n+1-\sigma} \geq x_{n-\sigma}, \tag{2.12}$$

and thus from (2.11) and (2.12), we obtain

$$\Delta w_n \leq -k \rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{(\rho_{n+1})^2} \frac{1}{(p_{n-\sigma})} w_{n+1}^2. \tag{2.13}$$

So

$$\Delta w_n \leq -k \rho_n q_n + \frac{(p_{n-\sigma})(\Delta \rho_n)^2}{4 \rho_n} \left[\frac{\sqrt{\rho_n}}{\rho_{n+1} \sqrt{(p_{n-\sigma})}} w_{n+1} - \frac{\sqrt{(p_{n-\sigma})} \Delta \rho_n}{2 \sqrt{\rho_n}} \right]^2$$
$$< - \left[\rho_n q_n - \frac{(p_{n-\sigma})(\Delta \rho_n)^2}{4 \rho_n} \right],$$

and hence

$$\Delta w_n < - \left[\rho_n q_n - \frac{(p_{n-\sigma})(\Delta \rho_n)^2}{4 \rho_n} \right]. \tag{2.14}$$

Summing (2.14) up from n_1 to n, we obtain

$$-w_{n_1} < w_{n+1} - w_{n_1} < -\sum_{l=n_1}^{n} \left[k\rho_l q_l - \frac{(p_{l-\sigma})(\Delta\rho_l)^2}{4\rho_l} \right],$$

and this implies that

$$\sum_{l=n_1}^{n} \left[k\rho_l q_l - \frac{(p_{l-\sigma})\,(\Delta\rho_l)^2}{4\rho_l} \right] < c_1 \tag{2.15}$$

for all sufficiently large n. This is a contrary to (2.1). The proof is complete.

For $\rho_n = n$ and $\mathrm{p}_n = 1$ in Theorem 2.1 then the condition (2.1) reduces to

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[klq_l - \frac{1}{4l} \right] = \infty. \tag{2.16}$$

From Theorem 2.1, by different choices of $\{\rho_n\}$ we can obtain different conditions for oscillation of all solutions of the IBVP (1.1), (B) in $\Omega \times \mathbb{N}_{n_0}$. Let $\rho_n = n^\lambda$, n$\geq n_0$ and $\lambda > 1$ is a constant.

COROLLARY 2.1. *Assume that all the assumption of Theorem* 2.1 *hold, except that the condition* (2.1) *is replaced by*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[ks^\lambda q_s - \frac{(p_{s-\sigma})\left((s+1)^\lambda - s^\lambda\right)^2}{4s^\lambda} \right] = \infty. \tag{2.17}$$

Then, every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

In the following theorem, we give the discrete analogue of Theorem 2 in [8], [14].

THEOREM 2.2. *Assume that* (h1)–(h4) *hold. Let* $\{\rho_n\}_{n=1}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *such that*

(i) $H_{m,m} = 0$ *for* $m \geq 0$,
(ii) $H_{m,n} > 0$ *for* $m > n \geq 0$,
(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$.

Assume that

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \left[kH_{m,n}\rho_n q_n - \frac{(p_{n-\sigma})\rho_{n+1}^2}{\rho_n} \left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] = \infty, \tag{2.18}$$

where

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\ m > n \geq 0. \tag{2.19}$$

Then every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

Proof. We proceed as in the proof of Theorem 2.1. Assume that IBVP (B) has a nonoscillatory solution. Without loss of generality, we may assume that it has a positive solution in $\Omega \times \mathbb{N}_{n_0}$. Following the proof of Theorem 2.1 we have $x_n > 0$, $\Delta x_n > 0$, $\Delta(p_n \Delta x_n) \leq 0$ for $n \geq n_1$. Defining again $\{w_n\}$ by (2.9) we have that $w_n > 0$ and (2.13) holds. Then from (2.13) we have

$$k\rho_n q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2, \tag{2.20}$$

where $\overline{\rho}_n = \frac{\rho_n}{(p_{n-\sigma})}$. Therefore we have

$$\begin{aligned} &\sum_{n=n_1}^{m-1} kH_{m,n}\rho_n q_n \\ &\leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 \end{aligned} \tag{2.21}$$

which yields, after summing up by part,

$$\begin{aligned} &\sum_{n=n_1}^{m-1} kH_{m,n}\rho_n q_n \\ &\leq H_{m,n_1} w_{n_1} + \sum_{n=n_1}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ &\quad - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 \\ &= H_{m,n_1} w_{n_1} - \sum_{n=n_1}^{m-1} h_{m,n}\sqrt{H_{m,n}} w_{n+1} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ &\quad - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\overline{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 = H_{m,n_1} w_{n_1} \\ &\quad - \sum_{n=n_1}^{m-1}\left[\frac{\sqrt{H_{m,n}\overline{\rho}_n}}{\rho_{n+1}} w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\overline{\rho}_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n}\right)\right]^2 \\ &\quad + \frac{1}{4}\sum_{n=n_1}^{m-1}\frac{(\rho_{n+1})^2}{\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2. \end{aligned}$$

Thus,

$$\sum_{n=n_1}^{m-1}\left[kH_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_1}w_{n_1} \leq H_{m,n_0}w_{n_1}$$

which implies that

$$\sum_{n=n_0}^{m-1}\left[kH_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_0}\Big(w_{n_1} + \sum_{n=n_0}^{n_1-1} k\rho_n q_n\Big).$$

Hence

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\left[kH_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\overline{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < \Big(w_{n_1} + \sum_{n=n_0}^{n_1-1} k\rho_n q_n\Big) < \infty,$$

which is a contrary to (2.18). The proof is complete.

As an immediate consequence of Theorem 2.2, we get the following:

COROLLARY 2.2. *Assume that all the assumptions of Theorem 2.2 hold, except that the condition* (2.18) *is replaced by*

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1} H_{m,n}\rho_n q_n = \infty,$$

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\frac{(p_{n-\sigma})\,\rho_{n+1}^2}{\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2 < \infty.$$

Then every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

By choosing the sequence $\{H_{m,n}\}$ in an appropriate manner, we can derive several oscillation criteria for the IBVP (1.1), (B) in $\Omega \times \mathbb{N}_{n_0}$. For instance, let us consider the double sequence $\{H_{m,n}\}$! defined by

$$\begin{cases} H_{m,n} = (m-n)^\lambda, & \lambda \geq 1, m \geq n \geq 0, \\ H_{m,n} = \left(\log\frac{m+1}{n+1}\right)^\lambda, & \lambda \geq 1, m \geq n \geq 0, \\ H_{m,n} = (m-n)^{(\lambda)} & \lambda > 2,\ m \geq n \geq 0, \end{cases} \tag{2.22}$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)\dots(m-n+\lambda-1)$, and

$$\Delta_2(m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for m≥ 0 and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Hence we have the following results.

COROLLARY 2.3. *Assume that all the assumptions of Theorem* 2.2 *hold, except that the condition* (2.18) *is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^\lambda} \sum_{n=0}^{m-1} \Bigg[(m-n)^\lambda k\rho_n q_n - \frac{\rho_{n+1}^2}{4\overline{\rho}_n} \left(\lambda (m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{(m-n)^\lambda} \right)^2 \Bigg] = \infty.$$

Then every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

COROLLARY 2.4. *Assume that all the assumptions of Theorem* 2.2 *hold, except that the condition* (2.18) *is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{(\log(m+1))^\lambda} \sum_{n=0}^{m-1} \Bigg[\left(\log \frac{m+1}{n+1} \right)^\lambda k\rho_n q_n - \frac{\rho_{n+1}^2}{4\overline{\rho}_n} \left(\frac{\lambda}{n+1} \left(\log \frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{\left(\log \frac{m+1}{n+1} \right)^\lambda} \right)^2 \Bigg] = \infty.$$

Then every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

COROLLARY 2.5. *Assume that all the assumptions of Theorem* 2.2 *hold, except that the condition* (2.18) *is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^{(\lambda)}} \sum_{n=0}^{m-1} (m-n)^{(\lambda)} \Bigg[k\rho_n q_n - \frac{\rho_{n+1}^2}{4\overline{\rho}_n} \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta\rho_n}{\rho_{n+1}} \right)^2 \Bigg] = \infty.$$

Then every solution of IBVP (1.1), (B) *is oscillatory in* $\Omega \times \mathbb{N}_{n_0}$.

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Received February 14, 2002; revised version August 27, 2002.