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ON SEMICYCLES OF SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS WITH SEVERAL DELAYS

Abstract. In this paper, we study the semicycles of oscillatory solutions of nonlinear difference equation with several delays

$$(ax_{n+1}+bx_n)^k-(cx_n)^k+\sum_{i=1}^{m}p_i(n)x^k_{n-\sigma_i}=0,$$

where $a,b,c\in(0,\infty)$, $c-b>0$, $k=q/r$, q, r are positive odd integers, m,σ_i are positive integers, $\{p_i(n)\}$ is a real sequence with $p_i(n)\geq 0$ for all large n. Upper bound of numbers of terms of semicycles are determined.

1. Introduction

Consider the nonlinear delay difference equation

$$(ax_{n+1}+bx_n)^k-(cx_n)^k+\sum_{i=1}^{m}p_i(n)x^k_{n-\sigma_i}=0, \tag{1}$$

where $a,b,c\in(0,\infty)$, $c-b>0$, $k=q/r$, q, r, are positive odd integers, m,σ_i are positive integers, $\{p_i(n)\}$ is a real sequence with $p_i(n)\geq 0$ for all large n. In the following, we assume that

$$p_i(n)\geq p_i\geq 0,\qquad n\geq n_0,\quad i=1,2,\ldots,m. \tag{2}$$

Recently, the semicycles of solutions of the delay difference equations have been investigated by Zhou and Zhang [10-13]. Our purpose in this paper is to study the semicycles of solutions of Eq.(1). In Section 2, we determine upper bound of numbers of terms of semicycles of (1) in following two cases:

(i) when the characteristic equation of (1) has no positive real roots;
(ii) when the characteristic equation of (1) has positive real roots.

Key words and phrases: semicycles, oscillation, difference equations.

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For the sake of convenience, we set

$$N[u,v] = \{u, u+1, \dots, v-1, v\},$$

where u, v are positive integers, and

$$\sigma = \max_{1 \le i \le m} \sigma_i.$$

The following definitions can be found in [3], [4].

DEFINITION 1. [3] A nontrivial solution $\{y_n\}$ of Eq.(1) is said to be oscillating about zero or simply oscillates if the terms y_n of the sequence $\{y_n\}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

DEFINITION 2. [4] A positive semicycle of a solution $\{y_n\}$ of Eq.(1) consists of a "string" of terms $\{y_s, y_{s+1}, \dots, y_t\}$, all greater than or equal to zero, with $s \ge -\sigma$ and $t \le \infty$ and such that

$$\text{either } s = -\sigma \text{ or } s > -\sigma \text{ and } y_{s-1} < 0,$$

and

$$\text{either } t = \infty \text{ or } t < \infty \text{ and } y_{t+1} < 0.$$

A negative semicycle of a solution $\{y_n\}$ consists of a "string" of terms $\{y_s, y_{s+1}, \dots, y_t\}$, all less than zero, with $s \ge -\sigma$ and $t \le \infty$ and such that

$$\text{either } s = -\sigma \text{ or } s > -\sigma \text{ and } y_{s-1} \ge 0,$$

and

$$\text{either } t = \infty \text{ or } t < \infty \text{ and } y_{t+1} \ge 0.$$

2. Main results

First we define a sequence $\{A_r\}$ by

$$(3) \quad A_1 = \frac{a}{c-b}, \quad A_{r+1} = \frac{1}{c}\left[(a+bA_r)^k + \sum_{i=1}^{m} p_i A_r^{k(1+\sigma_i)}\right]^{\frac{1}{k}}, \quad r = 1, 2, \dots$$

One can check that $\{A_r\}$ is increasing.

Next, we also define a sequence $\{B_r\}$ by

$$(4) \quad B_1 = \frac{c-b}{a}, \; B_{r+1} = \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i B_r^{-k\sigma_i})^{\frac{1}{k}} - b\right], r = 1, 2, \dots .$$

LEMMA 1. *Assume that the sequence $\{A_r\}$ is defined by (3) and that*

$$(5) \qquad (a\lambda + b)^k - c^k + \sum_{i=1}^{m} p_i \lambda^{-k\sigma_i} = 0,$$

has no positive real roots. Then $\lim_{r\to\infty} A_r = \infty$.

Proof. If not, since $\{A_r\}$ is increasing, then $\lim_{r\to\infty} A_r = d > 0$ exists. Let $r \to \infty$ in (3), we get

$$d = \frac{1}{c}\left[(a+bd)^k + \sum_{i=1}^{m} p_i d^{k(1+\sigma_i)}\right]^{\frac{1}{k}},$$

which implies

$$(a+bd)^k - (cd)^k + \sum_{i=1}^{m} p_i d^{k(1+\sigma_i)} = 0,$$

i.e.,

$$\left(a\frac{1}{d} + b\right)^k - c^k + \sum_{i=1}^{m} p_i (\frac{1}{d})^{-k\sigma_i} = 0.$$

This implies that $\lambda = \frac{1}{d}$ is a positive root of (5). This is a contradiction. The proof is completed.

LEMMA 2. *Assume that the sequence* $\{B_r\}$ *is defined by* (4) *and that* (5) *has positive real roots. Then* $\{B_r\}$ *is nonincreasing and* $\lim_{r\to\infty} B_r = \lambda_*$, *where* λ_* *is the largest root of equation* (5) *on* $(0, \frac{c-b}{a}]$.

Proof. First, we prove the sequence $\{B_r\}$ is nonincreasing. Since

$$B_2 = \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i B_1^{-k\sigma_i})^{\frac{1}{k}} - b\right] \le \frac{c-b}{a} = B_1.$$

Hence, by induction, suppose that $B_r \le B_{r-1}$. Then we have

$$B_{r+1} = \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i B_r^{-k\sigma_i})^{\frac{1}{k}} - b\right] \le \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i B_{r-1}^{-k\sigma_i})^{\frac{1}{k}} - b\right] = B_r.$$

Thus, the sequence $\{B_r\}$ is nonincreasing. From (5), it is obvious that $B_1 \ge \lambda_*$, by induction

$$B_{r+1} = \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i B_r^{-k\sigma_i})^{\frac{1}{k}} - b\right] \ge \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i \lambda_*^{-k\sigma_i})^{\frac{1}{k}} - b\right] = \lambda_*.$$

Hence, $\{B_r\}$ is nonincreasing and bounded. Therefore, $\lim_{r\to\infty} B_r$ exists. Letting $r \to \infty$, we have $\lim_{r\to\infty} B_r = \lambda_*$. The proof is completed.

THEOREM 1. *Assume that* (2) *holds and that* (5) *has no positive real roots. Then every nontrivial solution of* (1) *oscillates and every negative semicycle of every oscillatory solution of* (1) *has at most* $2 + R\sigma$ *terms, and every positive semicycle of every oscillatory solution of* (1) *has at most* $2 + R\sigma$

consecutive terms greater than zero, where

$$R = \min_{r \geq 2} \left\{ r \;\middle|\; \left(\frac{1}{c^k - b^k} \sum_{i=1}^{m} p_i A_{r-1}^{k(\sigma_i - 1)} \right)^{\frac{1}{k}} A_r \geq 1 \right\}. \tag{6}$$

Proof. Otherwise, without loss of generality, we assume that $\{x_n\}$ is a solution of (1) satisfying $x_n > 0$ on $N[n_1, n_1 + R\sigma + 2]$ for any $n_1 \geq n_0$. Then, $x_{n-\sigma_i} > 0$, $(i = 1, 2, \ldots, m)$ on $N[n_1 + \sigma, n_1 + R\sigma + 2]$. Therefore from (1) we have

$$x_n \geq \frac{a}{c-b} x_{n+1} = A_1 x_{n+1}, \quad n \in N[n_1 + \sigma, n_1 + R\sigma + 2]. \tag{7}$$

By iteration, we get

$$x_{n-\sigma_i} \geq A_1^{\sigma_i + 1} x_{n+1}, \quad n \in N[n_1 + 2\sigma, n_1 + R\sigma + 2]. \tag{8}$$

Using (1), (2) and (8) we have

$$\begin{aligned} (cx_n)^k &= (ax_{n+1} + bx_n)^k + \sum_{i=1}^{m} p_i(n) x_{n-\sigma_i}^k \\ &\geq (ax_{n+1} + bA_1 x_{n+1})^k + \sum_{i=1}^{m} p_i A_1^{k(1+\sigma_i)} x_{n+1}^k, \end{aligned}$$

i.e.

$$\begin{aligned} x_n &\geq \frac{1}{c} \left[(a + bA_1)^k + \sum_{i=1}^{m} p_i A_1^{k(1+\sigma_i)} \right]^{\frac{1}{k}} x_{n+1} \\ &= A_2 x_{n+1}, \qquad n \in N[n_1 + 2\sigma, n_1 + R\sigma + 2], \end{aligned}$$

which gives by iteration

$$x_{n-\sigma_i} \geq A_2^{1+\sigma_i} x_{n+1}, \quad n \in N[n_1 + 3\sigma, n_1 + R\sigma + 2].$$

Repeating the above procedure, we get

$$x_n \geq A_{R-1} x_{n+1}, \qquad n \in N[n_1 + (R-1)\sigma, n_1 + R\sigma + 2], \tag{9}$$

which gives

$$x_{n_1 + R\sigma - \sigma_i} \geq A_{R-1}^{\sigma_i + 1} x_{n_1 + 1 + R\sigma}. \tag{10}$$

Hence, from (1), (9) and (10), we have

$$\begin{aligned} (cx_{n_1 + R\sigma})^k &= (ax_{n_1 + R\sigma + 1} + bx_{n_1 + R\sigma})^k + \sum_{i=1}^{m} p_i(n_1 + R\sigma) x_{n_1 + R\sigma - \sigma_i}^k \\ &\geq (a + bA_{R-1})^k x_{n_1 + R\sigma + 1}^k + \sum_{i=1}^{m} p_i A_{R-1}^{k(1+\sigma_i)} x_{n_1 + R\sigma + 1}^k \end{aligned}$$

$$\geq \Big[(a+bA_{R-1})^k + \sum_{i=1}^m p_i A_{R-1}^{k(1+\sigma_i)}\Big] x^k_{n_1+R\sigma+1}$$

i.e.,

$$x_{n_1+R\sigma} \geq \frac{1}{c}\Big[(a+bA_{R-1})^k + \sum_{i=1}^m p_i A_{R-1}^{k(1+\sigma_i)}\Big]^{\frac{1}{k}} x_{n_1+R\sigma+1} \tag{11}$$
$$= A_R x_{n_1+R\sigma+1}.$$

On the other hand, from (9) we have

$$x_{n_1+R\sigma+1-\sigma_i} \geq A_{R-1}^{\sigma_i-1} x_{n_1+R\sigma}. \tag{12}$$

Therefore, by (1) and (12)

$$(cx_{n_1+R\sigma+1})^k = (ax_{n_1+R\sigma+2} + bx_{n_1+R\sigma+1})^k$$
$$+ \sum_{i=1}^m p_i(n_1+R\sigma+1) x^k_{n_1+R\sigma+1-\sigma_i}$$
$$> (bx_{n_1+R\sigma+1})^k + \sum_{i=1}^m p_i A_{R-1}^{k(\sigma_i-1)} x^k_{n_1+R\sigma},$$

i.e.,

$$(c^k-b^k) x^k_{n_1+R\sigma+1} > \Big(\sum_{i=1}^m p_i A_{R-1}^{k(\sigma_i-1)}\Big) x^k_{n_1+R\sigma},$$

which implies

$$x_{n_1+R\sigma+1} > \Big(\frac{1}{c^k-b^k}\sum_{i=1}^m p_i A_{R-1}^{k(\sigma_i-1)}\Big)^{\frac{1}{k}} x_{n_1+R\sigma}. \tag{13}$$

Then (11) and (13) can be reduced to

$$\frac{1}{A_R} \geq \frac{x_{n_1+R\sigma+1}}{x_{n_1+R\sigma}} > \Big(\frac{1}{c^k-b^k}\sum_{i=1}^m p_i A_{R-1}^{k(\sigma_i-1)}\Big)^{\frac{1}{k}}.$$

The latter implies

$$\Big(\frac{1}{c^k-b^k}\sum_{i=1}^m p_i A_{R-1}^{k(\sigma_i-1)}\Big)^{\frac{1}{k}} A_R < 1,$$

what contradicts with (6) and completes the proof.

THEOREM 2. *Assume that* (2) *holds and that* (5) *has positive real roots. Further assume that there exists a subsequence* $\{n_s\}_{s=1}^\infty \subset \{1,2,\ldots\}$ *such* *that* $|n_{s+1} - n_s| \leq N$ *and* $p_i(n_s + 1) \geq L_i$ *and that*

$$\left(\frac{1}{c^k - b^k}\sum_{i=1}^{m} L_i \lambda_*^{k(1-\sigma_i)}\right)^{\frac{1}{k}} > \frac{c-b}{a},$$

where λ_* *is the largest root of* (5) *on* $(0, \frac{c-b}{a}]$. *Then every nontrivial solution of* (1) *oscillates and every negative semicycle of every oscillatory solution of* (1) *has at most* $N + R\sigma$ *terms, and every positive semicycle of every oscillatory solution of* (1) *has at most* $N + R\sigma$ *consecutive terms greater than zero, where*

$$R = \min_{r \geq 2}\left\{ r \;\middle|\; \left(\frac{1}{c^k - b^k}\sum_{i=1}^{m} L_i B_{r-1}^{k(1-\sigma_i)}\right)^{\frac{1}{k}} \geq \frac{c-b}{a}\right\}. \tag{14}$$

Proof. Otherwise, without loss of generality, we assume that $\{x_n\}$ is a solution of (1) satisfying $x_n > 0$ on $N[n_1, n_1 + N + R\sigma]$ for any $n_1 \geq n_0$. Then there exists an $\overline{n}_s$ such that $N[\overline{n}_s - R\sigma, \overline{n}_s + 1] \subset N[n_1, n_1 + N + R\sigma]$. Hence, $x_{n-\sigma_i} > 0$, $i = 1, 2, \ldots, m$ on $N[\overline{n}_s - R\sigma + \sigma, \overline{n}_s + 1]$. Therefore, from (1) we have

$$x_n \geq \frac{a}{c-b} x_{n+1} = B_1^{-1} x_{n+1}, n \in N[\overline{n}_s - R\sigma + \sigma, \overline{n}_s + 1]. \tag{15}$$

By iteration, we get

$$x_{n-\sigma_i} \geq B_1^{-\sigma_i} x_n, n \in N[\overline{n}_s - R\sigma + 2\sigma, \overline{n}_s + 1]. \tag{16}$$

Using (1), (2) and (16) we have

$$(ax_{n+1} + bx_n)^k \leq (cx_n)^k - \sum_{i=1}^{m} p_i B_1^{-k\sigma_i} x_n^k, n \in N[\overline{n}_s - R\sigma + 2\sigma, \overline{n}_s + 1],$$

i.e.,

$$x_{n+1} \leq \frac{1}{a}\left[(c^k - \sum_{i=1}^{m} p_i B_1^{-k\sigma_i})^{\frac{1}{k}} - b\right] x_n = B_2 x_n, n \in N[\overline{n}_s - R\sigma + 2\sigma, \overline{n}_s + 1],$$

which gives by iteration

$$x_{n-\sigma_i} \geq B_2^{-\sigma_i} x_n, n \in N[\overline{n}_s - R\sigma + 3\sigma, \overline{n}_s + 1].$$

Repeating the above procedure, we get

$$x_{n+1} \leq B_{R-1} x_n, n \in N[\overline{n}_s - \sigma, \overline{n}_s + 1], \tag{17}$$

which gives

$$x_{\overline{n}_s + 1 - \sigma_i} \geq B_{R-1}^{1-\sigma_i} x_{\overline{n}_s}. \tag{18}$$

Hence, from (1), (17) and (18)

$$\frac{c-b}{a} = B_1 \ge B_{R-1} \ge \frac{x_{\overline{n}_s+1}}{x_{\overline{n}_s}} > \left[\frac{1}{c^k-b^k}\sum_{i=1}^{m} p_i(\overline{n}_s+1)\left(\frac{x_{\overline{n}_s+1-\sigma_i}}{x_{\overline{n}_s}}\right)^k\right]^{\frac{1}{k}}$$

$$> \left(\frac{1}{c^k-b^k}\sum_{i=1}^{m} L_i B_{R-1}^{k(1-\sigma_i)}\right)^{\frac{1}{k}},$$

i.e.,

$$\left(\frac{1}{c^k-b^k}\sum_{i=1}^{m} L_i B_{R-1}^{k(1-\sigma_i)}\right)^{\frac{1}{k}} < \frac{c-b}{a}.$$

This contradicts with (14) and completes the proof.

EXAMPLE 1. Consider the nonlinear difference equation with several delays

$$(ax_{n+1}+bx_n)^k - (cx_n)^k + \sum_{i=1}^{m} p_i(n)x_{n-\sigma_i}^k = 0. \tag{19}$$

Let

$$p_1(0) = \frac{4}{3},\ p_1(1) = \frac{3}{4},\ p_1(n+2) = p_1(n),$$

$$p_2(0) = \frac{3}{4},\ p_2(1) = \frac{1}{2},\ p_2(n+2) = p_2(n), \quad n = 0,1,2,\dots,$$

$$k=3, \quad a=1, \quad b=1, \quad c=2, \quad m=2, \quad \sigma_1=1, \quad \sigma_2=2.$$

By Corollary 2 in [14], we know that (5) has no positive roots. Hence, by Theorem 1, every solution of equation (19) oscillates. By simple calculation, we have

$$A_1 = 1,\ A_{r+1} = \frac{1}{2}\left((1+A_r)^3 + \frac{3}{4}A_r^6 + \frac{1}{2}A_r^9\right)^{\frac{1}{3}}, \quad r = 1,2,\dots$$

$$D_r = \left(\frac{1}{c^k-b^k}\sum_{i=1}^{m} p_i A_{r-1}^{k(\sigma_i-1)}\right)^{\frac{1}{k}} A_r$$

$$= \left(\frac{1}{7}\left(\frac{3}{4}+\frac{1}{2}A_r^9\right)\right)^{\frac{1}{3}} A_{r+1}, \quad r = 2,3,\dots,$$

$$A_2 = 1.04958411345210, \qquad D_2 = 0.591045941561356$$
$$A_3 = 1.09088444064887, \qquad D_3 = 0.626843391013837$$
$$A_4 = 1.12854970603156, \qquad D_4 = 0.659837157231824$$
$$A_5 = 1.16588262341269, \qquad D_5 = 0.692783138623273$$
$$A_6 = 1.20604337743297, \qquad D_6 = 0.728440715710422$$
$$A_7 = 1.25314448952005, \qquad D_7 = 0.770504103651539$$
$$A_8 = 1.31408290707526, \qquad D_8 = 0.825281967414030$$

$$A_9 = 1.40305282844783, \quad D_9 = 0.905969354928884$$
$$A_{10} = 1.55591395623001, \quad D_{10} = 1.04667945569338$$

Hence, $R = 10$. Therefore, every negative semicycle of every eventually nontrivial solution of (19) has at most $2 + 2 \times 10 = 22$ terms, and every positive semicycle of every eventually nontrivial solution of (19) has at most 22 consecutive terms greater than zero.

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