DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

Stevo Stević

BOUNDEDNESS AND PERSISTENCE OF SOLUTIONS OF A NONLINEAR DIFFERENCE EQUATION

Abstract. In this paper we obtain sufficient conditions for the boundedness as well as for the unboundedness of the positive solutions of the difference equation

$$x_{n+1} = f(x_n, \dots, x_{n-k+1}), \quad n = 0, 1, 2, \dots,$$

where k is a positive integer and the initial conditions $x_{-k+1}, x_{-k+2}, \dots, x_0$ are arbitrary positive numbers.

1. Introduction

In [1] and [2] the authors investigated the boundedness, the persistence and the asymptotic behaviour of the positive solutions of the difference equation

$$x_{n+1} = \frac{A}{x_n^p} + \frac{B}{x_{n-1}^q}, \quad n = 0, 1, \dots, \tag{1}$$

where $A, B, p, q \in (0, \infty)$ and the initial conditions x_{-1} and x_0 are arbitrary positive numbers.

We say that a solution (x_n) of a difference equation is *bounded* and *presists* if there exist positive constants P and Q such that

$$P \le x_n \le Q \quad \text{for} \quad n = -1, 0, \dots$$

In [2] the authors established the following results:

THEOREM A. (a) *Assume that $pq \le 1$. Then every solution of Eq.(1) is bounded and persists.*

(b) *Assume that $pq > 1$. Then there exist solutions of Eq.(1) that are unbounded and do not persist.*

1991 *Mathematics Subject Classification*: Primary 39A10.
Key words and phrases: positive solution, difference equation, boundedness, persistence.

The special case $p = q = 1$ was investigated in [5] where it was shown that the unique positive equilibrium of Eq.(1) is globally asymptotically stable.

In [3] (see also [4]), the authors investigate the boundedness of the solutions of the difference equation

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{A_i}{x_{n-i}^{p_i}} + \frac{1}{x_{n-k}^{p_k}}, \quad n = 0, 1, 2, \ldots,$$

where $x_{-k}, \ldots, x_0 \in (0, \infty), k \in \mathbf{N}, p_i \in (0, \infty), i \in \{0, 1, \ldots, k\}$ and $A_i \in (0, \infty)$, $i \in \{0, 1, \ldots, k-1\}$.

Motivated by [1]-[5], in this note we investigate the boundedness of the solutions of the difference equation

$$x_{n+1} = f(x_n, \ldots, x_{n-k+1}), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $k \in \mathbf{N}$, the initial conditions $x_{-k+1}, \ldots, x_0$ are arbitrary positive numbers and where the function f satisfies the following conditions:

(a) $f \in C[(0, \infty]^k, [0, \infty))$;

(b) f is decreasing in each variable;

(c) there is $j \in \{1, \ldots, k\}$ such that $\lim_{x_j \to +0} f(\infty, \ldots, \infty, x_j, \infty, \ldots, \infty) = \infty$;

(d) $f(1, \ldots, 1) > 1$.

EXAMPLE 1. It is easy to see that the following functions satisfy the conditions (a)–(d).

1. $f(x_1, \ldots, x_k) = \sum_{i=1}^k \frac{A_i}{x_i^{p_i}}, \quad \sum_{i=1}^k A_i > 1, \ A_i, p_i > 0, \ i = 1, \ldots, k.$

2. $f(x, y) = \frac{a}{\sqrt[4]{x}} + \frac{b}{\sqrt[4]{y} + \sqrt{y}}, \quad a, b > 0, \ a + b > 1.$

REMARK 1. From (a) and (b) it follows that the equation $x - f(x, \ldots, x) = 0$ has a unique positive equilibrium $x = \bar{x}$ such that $(x - f(x, \ldots, x))(x - \bar{x}) > 0$ for $x \neq \bar{x}$.

From this and the condition (d) it follows $1 < \bar{x} < f(1, \ldots, 1)$.

REMARK 2. Note that if $\lim_{x \to +0} f(x, \ldots, x) < \infty$, then $x_n < \lim_{x \to +0} f(x, \ldots, x)$ for all $n \in \mathbf{N}$. Hence, in this case, every solution of Eq.(2) is bounded, moreover Eq.(2) is permanent.

Let

$$g_i(x) = f(\underbrace{\infty, \ldots, \infty}_{i-1}, x, \underbrace{\infty, \ldots, \infty}_{k-i}), \qquad i = 1, \ldots, k, \qquad x \in (0, \infty]$$

and

$$h_i(x) = f(\underbrace{\bar{x}, \ldots, \bar{x}}_{i-1}, x, \underbrace{\bar{x}, \ldots, \bar{x}}_{k-i}), \qquad i = 1, \ldots, k, \qquad x \in (0, \infty].$$

Note that $g_i(x), i = 1, \ldots, k$ are decreasing continuous functions on $(0, \infty]$ and consequently there are their continuous inverses $g_i^{-1}(x), i = 1, \ldots, k$, which are decreasing functions, too.

We prove the following results:

THEOREM 1. *Consider Eq.* (2), *where f satisfies the conditions* (a), (b) *and* (d). *If the function*

$$\phi(x) = \frac{f(h_k(x), \ldots, h_1(x))}{x} \tag{3}$$

is decreasing on $(0, \infty)$, *then every positive solution of Eq.*(2) *is bounded and persists.*

THEOREM 2. *Consider Eq.*(2), *where f satisfies the conditions* $(a - d)$. *If there is a* $\delta > 0$ *such that*

$$f(g_k(x), \ldots, g_1(x)) < x \quad \text{for} \quad x \in (0, \delta), \tag{4}$$

then there exist solutions of Eq.(1) *that are unbounded and do not persist.*

2. Auxiliary results

The following lemmas summarize some of the important properties of the semicycles of Eq.(2).

LEMMA 1. *Consider Eq.* (2), *where f satisfies the condition* (b). *Then every semicycle of a nontrivial solution of Eq.* (2) *contains at most k terms.*

Proof. Assume that for a nontrivial solution (x_n) and for some $N \geq 0$, $x_{N-k+1}, \ldots, x_N \geq \bar{x}$. Then

$$x_{N+1} = f(x_N, \ldots, x_{N-k+1}) < f(\bar{x}, \ldots, \bar{x}) = \bar{x}.$$

The case where $x_{N-k+1}, \ldots, x_N \leq \bar{x}$ is similar.

LEMMA 2. *Consider Eq.* (2), *where f satisfies the conditions* (a), (b) *and* (d). *Then the following statements are true.*

(a) *If for some $N \geq 0$ and some $j \in \{1, \ldots, k\}$,*

$$x_N \leq g_j^{-1}(f(1, \ldots, 1)), \tag{5}$$

then $x_{N+j} > \bar{x}$.

(b) *If for some $N \geq 1$ and some $j \in \{1, \ldots, k\}$,*

$$x_N \leq g_j(f(1, \ldots, 1)), \tag{6}$$

then $x_{N-j} > \bar{x}$.

(c) *If for some $N \geq 1$,*

$$x_N \leq m_k = \min\{g_j^{-1}(f(1, \ldots, 1)),\ g_j(f(1, \ldots, 1)),\ j = 1, \ldots, k\},$$

then $x_{N \pm j} > \bar{x}$ for $j = 1, \ldots, k$.

P r o o f. (a) If for some $N \geq 0$ and some $j \in \{1, \ldots, k\}$, (5) holds, then

$$x_{N+j} = f(x_{N+j-1}, \ldots, x_{N+j-k}) > g_j(x_N) \geq f(1, \ldots, 1) > \bar{x}.$$

(b) Suppose the contrary $x_{N-j} \leq \bar{x}$, if for some $N \geq 0$ and some $j \in \{1, \ldots, k\}$, (6) holds, then

$$x_N = f(x_{N-1}, \ldots, x_{N-k}) > g_j(x_{N-j}) \geq g_j(\bar{x}) > g_j(f(1, \ldots, 1)),$$

which is a contradiction.

(c) The proof is a direct consequence of (a) and (b).

REMARK 3. If $\phi(x)$ is decreasing on $(0, \infty)$, then $\bar{x}$ is the unique solution of the equation $\phi(x) = 1$ and consequently $\phi(x) > 1$ for $x \in (0, \bar{x})$.

LEMMA 3. *Consider Eq.*(2), *where f satisfies the conditions* (*a*), (*b*) *and* (*d*), *and $\phi(x)$ is decreasing on* $(0, \infty)$. *Let m_k be as in Lemma 2. Suppose that* (x_n) *is a solution of Eq.*(2) *such that $x_N \leq m_k$ for some $N \geq 1$.*

Then

$$x_{N\pm 1}, \ldots, x_{N\pm k} \in (\bar{x}, \infty) \quad \text{and} \quad x_N < x_{N+k+1} < \bar{x}.$$

P r o o f. First note that $m_k < \bar{x}$, since $m_k \leq g_j(f(1, \ldots, 1)) < g_j(\bar{x}) < \bar{x}$. By Lemma 2 *(c)* we have $x_{N\pm j} \in (\bar{x}, \infty)$, $j = 1, \ldots, k$. Hence

$$x_{N+j} < h_j(x_N) \quad \text{for} \quad j = 1, \ldots, k$$

and consequently

$$x_{N+k+1} > f(h_k(x_N), \ldots, h_1(x_N)) = x_N \phi(x_N) \geq x_N,$$

where the last inequality holds because $x_N \leq m_k < \bar{x}$. By Lemma 1 we have $x_{N+k+1} < \bar{x}$, as desired.

3. Proof of the main results

We are now in a position to prove the main results.

P r o o f o f T h e o r e m 1. Let $m = m_k$ and set

$$M = f(m, \ldots, m) \quad \text{and} \quad c = \min\{m, f(M, \ldots, M)\}.$$

Let (x_n) be a solution of Eq.(2). Clearly if l is a lower bound of (x_n) then $L = f(l, \ldots, l)$ is an upper bound. Hence it suffices to show that (x_n) is bounded from below. Now either c is a lower bound of (x_n) or there is $N \geq k$ such that $x_N < c$. We show that x_N is a lower bound of (x_n) for $n \geq N$. In contrary there is the first natural number $K > N$ such that $x_K < x_N$. By Lemma 3 we have

$$x_{K-j} > \bar{x} > m \quad \text{for} \quad j = 1, \ldots, k.$$

We show that $x_{K-j} < M$ for $j = 1, \ldots, k$. If $x_{K-k-1} \leq m$, then by Lemma 3 we have

$$x_N \leq x_{K-k-1} \leq x_K < x_N,$$

which is a contradiction. Hence, $x_{K-k-1} > m$. If $x_{K-k-j-1} \leq m$, for some $j \in \{1, \dots, k\}$, then by Lemma 3, $x_{K-k-j-1} < x_{K-j} < \bar{x}$. Since $x_{K-j} > \bar{x}$, we arrive at a contradiction. Therefore $x_{K-k-j-1} > m$, for $j = 1, \dots, k$.

Hence

$$x_{K-j} = f(x_{K-j-1}, \dots, x_{K-j-k}) < f(m, \dots, m) = M, \tag{7}$$

$j = 1, \dots, k$.

From (2) and (7) we obtain

$$x_N > x_K > f(M, \dots, M) \geq c > x_N,$$

from which the result follows.

Proof of Theorem 2. Let $x_0 \in (0, \delta)$. Then

$$x_j = f(x_{j-1}, \dots, x_{j-k}) > g_j(x_0) \quad \text{for} \quad j = 1, \dots, k,$$

and consequently

$$x_{k+1} = f(x_k, \dots, x_1) < f(g_k(x_0), \dots, g_1(x_0)) < x_0.$$

By induction, we can obtain

$$x_{n(k+1)} < f(g_k(x_{(n-1)(k+1)}), \dots, g_1(x_{(n-1)(k+1)})) < x_{(n-1)(k+1)}, \tag{8}$$

for all $n \in \mathbf{N}$.

Hence, there is $\lim_{n\to\infty} x_{n(k+1)} = x^* \in [0, \delta)$. Suppose $x^* \in (0, \delta)$. Then from (4) we have $f(g_k(x^*), \dots, g_1(x^*)) < x^*$. On the other hand, letting $n \to \infty$ in (8) we obtain $f(g_k(x^*), \dots, g_1(x^*)) \geq x^*$, arriving at a contradiction. Hence $x^* = 0$.

From the condition (c) it follows that there is a $j \in \{1, .., k\}$ such that $g_j(x) \to \infty$ as $x \to 0$. Thus

$$x_{n(k+1)+j+1} = f(x_{n(k+1)+j}, \dots, x_{n(k+1)+j-k+1}) > g_j(x_{n(k+1)}) \to \infty$$

as $n \to \infty$, finishing the proof.

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MATEMATIČKI FAKULTET
Studentski Trg 16
11000 BEOGRAD, SERBIA, YUGOSLAVIA
E-mail: sstevo@matf.bg.ac.yu, sstevic@ptt.yu

Received October 31st, 2001; revised version September 4, 2002.