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A PROOF OF THE EHRENPREIS-MARTINEAU THEOREM USING THE BOCHNER-MARTINELLI KERNEL

Abstract. We give an elementary proof of a version of the Ehrenpreis-Martineau theorem, which describes the entire holomorphic functions of exponential type as combinations of exponential functions. The proof uses the Bochner-Martinelli kernel and appropriate derivatives of it.

1. Introduction

To each measure μ with compact support in $\mathbb{C}^n$, we may associate an entire holomorphic function F_μ which is the combination of the functions $e^{\langle z,\zeta\rangle}$, $z \in \mathbb{C}^n$, with respect to this measure, i.e.,

$$F_\mu(\zeta) = \int_{z\in\mathbb{C}^n} e^{\langle z,\zeta\rangle} d\mu(z), \quad \zeta \in \mathbb{C}^n,$$

where $\langle z,\zeta\rangle = \sum z_j\zeta_j$. If $K = \operatorname{supp}(\mu)$ then

$$|F_\mu(\zeta)| \le \int_{z\in K} |e^{\langle z,\zeta\rangle}| d|\mu|(z) \le A \cdot e^{B|\zeta|}, \quad \text{for} \quad \zeta \in \mathbb{C}^n,$$

where

$$B = \sup\{|z| : z \in K\} \quad \text{and} \quad A = |\mu|(K).$$

Thus the function F_μ is an entire function of exponential type. As usual, we say that an entire function F is of exponential type if there exist positive constants A and B, depending only on F, so that

$$(*) \qquad |F(\zeta)| \le A \cdot e^{B|\zeta|}, \quad \text{for every} \quad \zeta \in \mathbb{C}^n.$$

A version of the Ehrenpreis-Martineau theorem asserts that, conversely, any entire function of exponential type is a combination of the exponential functions $e^{\langle z,\zeta\rangle}$, $z \in \mathbb{C}^n$. (For variations of this theorem see for example [2] and [3]). We will give a proof of this version of the theorem using only the

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Bochner-Martinelli formula for holomorphic functions. Thus we will prove the following theorem.

THEOREM. *Let F be an entire holomorphic function in $\mathbb{C}^n$, of exponential type. Then there exists a measure μ with compact support in $\mathbb{C}^n$, so that*

$$F(\zeta) = \int_{z\in\mathbb{C}^n} e^{\langle z,\zeta\rangle} d\mu(z), \quad \text{for} \quad \zeta \in \mathbb{C}^n.$$

Before we prove this theorem we will recall some facts about the Bochner-Martinelli kernel.

2. The Bochner-Martinelli kernel

For $z \neq w$, set

$$M(z,w) = \frac{\beta_n}{|z-w|^{2n}} \sum_{j=1}^{n} (-1)^{j-1} (\bar{z}_j - \bar{w}_j) d\bar{z}_1 \wedge \ldots (j) \ldots \wedge d\bar{z}_n,$$

where $\beta_n = (n-1)!/(2\pi i)^n$, and, as in [1], for each $k = (k_1, \ldots, k_n)$ (where k_j are non-negative integers), define

$$\begin{aligned}\eta_k(z) &= \left.\frac{\partial^{k_1+\cdots+k_n} M(z,w)}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}}\right|_{w=0} \\ &= \beta_n n(n+1)\cdots(n+k_1+\cdots+k_n-1)\frac{\bar{z}_1^{k_1}\cdots \bar{z}_n^{k_n}}{|z|^{2(n+k_1+\cdots+k_n)}} \times \\ &\quad \times \sum_{j=1}^{n} (-1)^{j-1} \bar{z}_j d\bar{z}_1 \wedge \ldots (j) \ldots \wedge d\bar{z}_n.\end{aligned}$$

Then $\eta_k \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$, i.e., η_k is a $(0,n-1) - \bar{\partial}$-closed form with C^∞ coefficients in $\mathbb{C}^n - \{0\}$.

Also, by the Bochner-Martinelli formula (see [4]), for $f \in \mathcal{O}(\mathbb{C}^n)$, (i.e., a function f, holomorphic in $\mathbb{C}^n$),

$$\int_{z\in S(0,r)} f(z) M(z,w) \wedge \omega(z) = f(w) \quad (\text{for } w \text{ close to } 0 \in \mathbb{C}^n),$$

where $\omega(z) = dz_1 \wedge \ldots \wedge dz_n$, $S(0,r) = \{z \in \mathbb{C}^n : |z| = r\}$ and $r > 0$.

Applying to the above equation the operator $\frac{\partial^{k_1+\cdots+k_n}}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}$ and evaluating at $w = 0$, we obtain

$$(1) \qquad \int_{z\in S(0,r)} f(z)\eta_{k_1,\ldots,k_n}(z) \wedge \omega(z) = \left.\frac{\partial^{k_1+\cdots+k_n} f(w)}{\partial w_1^{k_1}\cdots \partial w_n^{k_n}}\right|_{w=0}.$$

3. The proof of the Theorem

Let F be an entire function which satisfies $(*)$. In order to construct a measure μ with the required property, we expand $F(\zeta)$ in power series:

$$F(\zeta) = \sum_{k_1,\dots,k_n \geq 0} c_{k_1,\dots,k_n} \zeta_1^{k_1} \dots \zeta_n^{k_n}, \quad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n,$$

where

$$c_{k_1,\dots,k_n} = \frac{1}{k_1! \dots k_n!} \left. \frac{\partial^{k_1+\dots+k_n} F(\zeta)}{\partial \zeta_1^{k_1} \dots \partial \zeta_n^{k_n}} \right|_{\zeta=0}.$$

It follows from $(*)$ and Cauchy's inequalities that

$$|c_{k_1,\dots,k_n}| \leq A \frac{e^{B(R_1+\dots+R_n)}}{R_1^{k_1} \dots R_n^{k_n}}, \quad \text{for every} \quad R_1, \dots, R_n > 0.$$

Applying this inequality with $R_1 = k_1/B, \dots, R_n = k_n/B$ we obtain that

$$|c_{k_1,\dots,k_n}| \leq A \frac{(eB)^{k_1+\dots+k_n}}{k_1^{k_1} \dots k_n^{k_n}}, \quad \text{for all} \quad k_1, \dots, k_n.$$

We claim that this implies that

$$(2) \quad \sum_{k_1,\dots,k_n \geq 0} n(n+1) \cdots (n + k_1 + \dots + k_n - 1) |c_{k_1,\dots,k_n}| s_1^{k_1} \dots s_n^{k_n} < \infty$$

when

$$0 < s_j < \frac{1}{(n+1)eB}.$$

Indeed, it suffices to observe that

$$(3) \quad \sum_{k_1,\dots,k_n \geq 0} \frac{n(n+1) \cdots (n + k_1 + \dots + k_n - 1)}{k_1^{k_1} \dots k_n^{k_n}} \sigma_1^{k_1} \dots \sigma_n^{k_n} < \infty$$

when

$$0 < \sigma_j < 1/(n+1).$$

To justify (3), let us first consider the series

$$(4) \quad \sum_{n \leq k_1 \leq \dots \leq k_n} \frac{n(n+1) \cdots (n + k_1 + \dots + k_n - 1)}{k_1^{k_1} \dots k_n^{k_n}} \sigma_1^{k_1} \dots \sigma_n^{k_n}.$$

Writing

$$\frac{n(n+1) \cdots (n + k_1 + \dots + k_n - 1)}{k_1^{k_1} \cdots k_n^{k_n}} \sigma_1^{k_1} \cdots \sigma_n^{k_n} = \frac{n \cdots (n + k_1 - 1)}{(2k_1)^{k_1}} \cdots$$

$$\frac{(n + k_1 + \dots + k_{n-1}) \cdots (n + k_1 + \dots + k_n - 1)}{[(n+1)k_n]^{k_n}} (2\sigma_1)^{k_1} \dots [(n+1)\sigma_n]^{k_n}$$

and observing that $n \le k_1 \le \cdots \le k_n$ implies

$$\frac{n\cdots(n+k_1-1)}{(2k_1)^{k_1}} \le 1,\ \ldots,$$
$$\frac{(n+k_1+\cdots+k_{n-1})\cdots(n+k_1+\cdots+k_n-1)}{[(n+1)k_n]^{k_n}} \le 1,$$

we see that the general term of (4) is dominated by $(2\sigma_1)^{k_1}\cdots[(n+1)\sigma_n]^{k_n}$.

Therefore the series (4) converges if $2\sigma_1 < 1, \ldots, (n+1)\sigma_n < 1$, i.e., if $\sigma_1 < 1/2, \ldots, \sigma_n < 1/(n+1)$.

Since the general term of the series in (3) is symmetric with respect to $k_1, \ldots, k_n$, we conclude that (3) holds.

Next writing the factor $\frac{\bar{z}_1^{k_1}\cdots\bar{z}_n^{k_n}}{|z|^{2(n+k_1+\cdots+k_n)}}$ of $\eta_{k_1,\ldots,k_n}(z)$ in the form

$$\frac{1}{|z|^{2n}}\left(\frac{\bar{z}_1}{|z|^2}\right)^{k_1}\cdots\left(\frac{\bar{z}_n}{|z|^2}\right)^{k_n},$$

we see that (2) implies that the series

$$\eta(z) = \sum_{k_1,\ldots,k_n \ge 0} c_{k_1,\ldots,k_n}\eta_{k_1,\ldots,k_n}(z)$$

converges for $|z| > (n+1)eB$ and defines a $(0,n-1)$-form with C^∞ coefficients in $\mathbb{C}^n - E_\rho$, where $E_\rho = \{z \in \mathbb{C}^n : |z| \le \rho\}$ and $\rho = (n+1)eB$.

Finally we claim that

$$\int_{z\in S(0,r)} e^{\langle z,\zeta\rangle}\eta(z)\wedge\omega(z) = F(\zeta), \quad \textit{for} \quad \zeta \in \mathbb{C}^n,$$

provided that $r > \rho$.

To prove it, we apply (1) with $f(w) = e^{\langle\zeta,w\rangle}$ (with ζ fixed) and we find that

$$\int_{z\in S(0,r)} e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z) = \zeta_1^{k_1}\cdots\zeta_n^{k_n}.$$

But by (2), the series $\sum_{k_1,\ldots,k_n\ge 0} c_{k_1,\ldots,k_n}e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z)$ converges uniformly for $z \in S(0,r)$, and therefore

$$\begin{aligned}\int_{z\in S(0,r)} e^{\langle\zeta,z\rangle}\eta(z)\wedge\omega(z) &= \sum_{k_1,\ldots,k_n\ge 0} c_{k_1,\ldots,k_n}\int_{z\in S(0,r)} e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z)\\ &= \sum_{k_1,\ldots,k_n\ge 0} c_{k_1,\ldots,k_n}\zeta_1^{k_1}\cdots\zeta_n^{k_n} = F(\zeta).\end{aligned}$$

Thus as a measure μ we may choose the restriction of the differential form $\eta(z)\wedge\omega(z)$ to the sphere $S(0,r)$:

$$d\mu(z) = \eta(z)\wedge\omega(z)|_{z\in S(0,r)}.$$

This completes the proof of the theorem.

REMARKS. It follows from (2) that

$$\begin{aligned}\bar{\partial}\eta &= \bar{\partial}\Big(\sum_{k_1,\dots,k_n\geq 0} c_{k_1,\dots,k_n}\eta_{k_1,\dots,k_n}\Big)\\ &= \sum_{k_1,\dots,k_n\geq 0} c_{k_1,\dots,k_n}\bar{\partial}\eta_{k_1,\dots,k_n} = 0, \quad \text{in } \mathbb{C}^n - E_\rho,\end{aligned}$$

i.e., $\eta \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - E_\rho)$.

In particular, any measure λ with compact support in $\mathbb{C}^n$, is equivalent to a $(n, n-1) - \bar{\partial}$-closed form $\eta(z) \wedge \omega(z)$ in $\mathbb{C}^n - E_\rho$ (when ρ is sufficiently large), in the sense that, for $r > \rho$,

$$\int_{z\in S(0,r)} f(z)\eta(z)\wedge\omega(z) = \int_{w\in\mathbb{C}^n} f(w)d\lambda(w), \quad \text{for every } f \in \mathcal{O}(\mathbb{C}^n).$$

And of course the sphere $S(0,r)$, in the first of the above integrals, may be replaced by any simple closed $(2n-1)$-dimensional surface in $\mathbb{C}^n - E_\rho$, which surrounds 0.

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