DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

D. Blezu, N. N.Pascu, R. N. Pascu

# INTEGRAL OPERATOR WHICH PRESERVES THE UNIVALENCE IN THE UPPER HALF-PLANE

**Abstract.** In this paper, using the Pfaltzgraff integral operator and the so called "parametric circles method" introduced by N.N.Pascu (1999), we can obtain an univalence criterion for the analytic function defined in the upper half-plane and also for comparison two univalence criteria obtained by a simple composition of functions.

## Introduction

We will denote by $D$ the upper half-plane and by $S(D)$ the class of analytic univalent functions in $D$ which are not necessarily hydrodinamic normalized.

$S(U)$ is the class of analytic, and univalent functions in $U$ $f(0) = 0$, $f'(0) = 1$.

The function $\varphi : U \to D$, $\varphi(u) = i\frac{1-u}{1+u}$ maps the unit disk $U$ in $D$.

For $0 < r < 1$, the image of the disk $U_r = \{z \in C, |z| = r\}$ under $\varphi$ is the disk $D_r = \{z \in C : |z - z_r| < R_r\}$ where

$$z_r = i\frac{1+r^2}{1-r^2}, \quad R_r = \frac{2r}{1-r^2}.$$

## 1. Preliminary results

It is known that an important problem is the preservation of the univalence of function through integral operators. Therefore we mention the classical Pfaltzgraff integral operator which will be used in this paper.

Theorem A [Pf]. *If $f \in S(U)$ then for $\alpha \in C$, $|\alpha| \leq \frac{1}{4}$ the function defined*

1991 *Mathematics Subject Classification*: 30C45.

Presented at the International Conference on Complex Analysis and Related Topics and the IX-th Romanian-Finnish Seminar August 27–31, 2001, Braşov, Romania *by*

$$G_\alpha(z) = \int_0^z [g'(u)]^\alpha du \tag{1}$$

*belong also to the class $S(U)$.*

LEMMA B (N. N. Pascu). *The family of domain $\{D_r\}_{r\in(0,1)}$ has the following properties*:

i) *for any positive real numbers $r < s < 1$ we have $D_r \subset D_s \subset D$;*

ii) *for any complex number $z \in D$ then exists $r_z \in (0,1)$ such that $z \in D_r$ for any $r \in (r_z, 1)$;*

iii) *for any $z \in D$ and $r \in (r_z, 1)$ arbitrarily fixed, there exists $u_r \in U$ such that $z = z_r + R_r \cdot u_r$.*

*Moreover, we have the following equalities*:

$$\lim_{r\to 1} u_r = -i,$$
$$\lim_{r\to 1} R_r(1 - |u_r|) = Im\ z.$$

## 2. Main results

In the following we can obtain an univalence criterion for the analytic functions in upper half-plane by using the integral operator of Pfaltzgraff type (1) and the "parametric circles method".

THEOREM 1. *If $f$ is analytic and univalent in $D$, $f \in S(D)$, $\alpha \in C$, $|\alpha| \le \frac{1}{4}$ then $F_\alpha$ also belong to $S(D)$ where*

$$F_\alpha(z) = \int_i^z (f'(u))^\alpha du. \tag{2}$$

Proof. a) Let $g(u) = [f(z_r + R_r \cdot u) - f(z_r)]/(R_r \cdot f'(z_r))$ and $r \in (0,1)$ be fixed. We have

$$g'(u) = f'(z_r + R_r \cdot u)/f'(z_r).$$

We have $g(0) = 0$, $g'(0) = 1$. By using (1) we obtain

$$G_\alpha(u) = \frac{1}{[f'(z_r)]^\alpha} \int_0^u [f'(z_r + R_r\tau)]^\alpha d\tau.$$

With the notation $\zeta = z_r + R_r\tau$, $d_\zeta = R_r \cdot d\tau$ it follows that

$$G_\alpha(z) = \frac{1}{[f'(z_r)]^\alpha \cdot R_r} \int_{z_r}^z [f'(\zeta)]^\alpha d\zeta$$

or

$$R_r[f'(z_r)]^\alpha \cdot G_\alpha(z) = \int_{z_r}^z [f'(\zeta)]^\alpha d\zeta, \qquad \forall z \in D_r.$$

By adding a suitable constant we obtain

$$F_\alpha(z) = R_r[f'(z_r)]^\alpha \cdot G_\alpha(z) + \int_i^{z_r} [f'(\zeta)]^\alpha d\zeta = \int_i^{z} [f'(\zeta)]^\alpha d\zeta$$

$\forall z \in D_r$, $\forall r \in (0,1)$.

b) Let $\widetilde{F}_\alpha(z)$ be the analytic extension of the $F_\alpha(z)$ from $D_r$ until $D$. This exists because $f$ if it is univalent in $D$ and $f'(0) \neq 0$ in $D$, that is $[f'(z)]^\alpha$ has an analytic branch in $D$. It follows that

$$F_\alpha(z) = \int_i^{z} [f'(\zeta)]^\alpha d\zeta \text{ is analytic in } D.$$

c) Now we can prove that $\widetilde{F}_\alpha(z)$ is univalent in $D$. Let be $z_1 \neq z_2$, $z_1, z_2 \in D$. According to the Lemma B if $z_1 \in D$, $\exists\ r_{z_1}$ so that $z_1 \in D_r$ $\forall r \in (r_{z_1}, 1)$ and if $z_2 \in D$, $\exists r_{z_2}$ so that $z_2 \in D_r$, $\forall r \in (r_{z_2}, 1)$.

Let $\rho = \max(r_{z_1}, r_{z_2}) \Rightarrow z_1$ and $z_2 \in D_r$, $\forall r \in (\rho, 1)$.

If $r \in (\rho, 1)$ is fixed, it follows that $\widetilde{F}_\alpha \equiv F_\alpha$ is univalent in $D_r$, $\widetilde{F}_\alpha(z_1) \neq \widetilde{F}_\alpha(z_2)$ that is $\widetilde{F}_\alpha(z)$ – is univalent in $D$. ∎

In order to show the part of the above method for the study of the analytic functions in the half-plane we can obtain another univalence criterion by a simple composition of functions.

THEOREM 2. *Let $f \in S(D)$, $\alpha \in C$, $|\alpha| \leq \frac{1}{4}$, then $F_\alpha(w)$ is analytic and univalent in $D$ where*

$$F_\alpha(w) = \int_i^{w} [f'(\zeta)]^\alpha (i+\zeta)^{2(\alpha-1)} d\zeta.$$

Proof. Let $\varphi$ be the function $\varphi(u) = i\dfrac{1-u}{1+u}$, $\varphi(0) = i$ which maps $U$ in $D$, $\varphi(U) = D$.

Let $g(u) = [f(\varphi(u)) - f(\varphi(0))]/(-2if'(i))$, $g(0) = 0$

$$g'(u) = f'(\varphi(u)) \cdot \varphi'(u)/(-2f'(i))$$

$$= f'(\varphi(u)) \cdot \frac{-2i}{(1+u)^2}/(-2if'(i)) = \frac{f'(\varphi(u))}{(1+u)^2 f'(i)} \qquad g'(0) = 1,$$

$$G_\alpha(z) = \frac{1}{[f'(i)]^\alpha} \int_0^{z} [f'(\varphi(u))]^\alpha \cdot \frac{1}{(1+u)^{2\alpha}} du.$$

By notation

$$\zeta = \varphi(u) = i\frac{1-u}{1+u} \qquad d_\zeta = \frac{-2i}{(1+u)^2} du$$

$$u = \frac{i-\zeta}{i+\zeta} \qquad du = \frac{-2i}{(i+\zeta)^2} d\zeta$$

we obtain

$$F_\alpha(w) = [f'(i)]^\alpha \cdot \frac{(-4)^\alpha}{-2i} G_\alpha(w) = \int_i^w [f'(\zeta)]^\alpha \cdot (i+\zeta)^{2(\alpha-1)} d\zeta. \ \blacksquare$$

By reverse we obtain a new integral operator which preserves the univalence in unit disk by using Theorem 1 and again the composition of functions.

THEOREM 3. *Let $h$ be analytic and univalent in $U$, $h(0) = 0$, $h'(0) = 1$, $\alpha \in C$, $|\alpha| \le \frac{1}{4}$. Then $H_\alpha$ is also analytic and univalent, where*

$$H_\alpha(v) = \int_0^v [h'(\tau)]^\alpha \cdot (1+\tau)^{2(\alpha-1)} d\tau.$$

Proof. The function $\varphi^{-1} : D \to U$

$$\varphi^{-1}(u) = \frac{i-u}{i+u} = \tau$$

and

$$u = \varphi(\tau) = i\frac{1-\tau}{1+\tau}$$

$$du = \frac{-2i}{(1+\tau)^2} d\tau = \varphi'(\tau) d\tau.$$

Replacing the function $u = \varphi(\tau)$ in the integral operator (2) we can obtain

$$h(\tau) = f(\varphi(\tau)) \text{ and } h'(\tau) = f'(\varphi) \cdot \varphi'(\tau),$$

$$F_\alpha(z) = \int_i^z \{f'[\varphi(\tau)]\}^\alpha du; \qquad F_\alpha(v) = \int_0^v \left[f'\left(i\frac{1-\tau}{1+\tau}\right)\right]^\alpha \cdot \frac{-2i}{(1+\tau)^2} d\tau,$$

$$= \int_0^v \left[f'\left(i\frac{1-\tau}{1+\tau}\right) \cdot \frac{-2i}{(1+\tau)^2}\right]^\alpha \cdot \left[\frac{-2i}{(1+\tau)^2}\right]^{1-\alpha} d\tau,$$

$$F_\alpha(v) = (-2i)^{1-\alpha} \int_0^v [h'(\tau)]^\alpha \cdot (1+\tau)^{2(\alpha-1)} d\tau,$$

$$H_\alpha(v) = (-2i)^{\alpha-1} F_\alpha(v) = \int_0^v [h'(\tau)]^\alpha (1+\tau)^{2(\alpha-1)} d\tau.$$

We observe that if

$$h(\tau) = [f(\varphi(\tau)) - f(\varphi(0))]/(-2if'(i))$$

then $h(0) = 0$ and $h'(0) = 1$ that is $h$ verifies the usual conditions from $S(U)$

$$h'(\tau) = f'(\varphi(\tau)) \cdot \varphi'(\tau) = f'(\varphi(\tau)) \cdot \frac{-2i}{(1+\tau)^2}. \ \blacksquare$$

REMARK. This univalence criterion is distinct from the classical Pfaltzgraff criterion (see Theorem A).

## References


[Pf] I. Pfaltzgraff, *Univalence of the integral* $(f'(z))^c$, Bull. London Math. Soc. 7 (1975), No. 3, 254–256.

[Pas] N. N. Pascu, *Univalent functions in a half-plane*, International Conference on Complex Analysis and Related Topics, The VIII-th Romanian-Finnish Seminar, Iassy, Romania, 1999.



Blezu Dorin
DEPARTMENT OF MATHEMATICS
"LUCIAN BLAGA" UNIVERSITY
Str. Panait Cerna 10
2400 SIBIU, ROMANIA
E-mail: blezu@roger-univ.ro

Nicolae N.Pascu, Radu N.Pascu
DEPARTMENT OF MATHEMATICS
"TRANSILVANIA" UNIVERSITY OF BRAŞOV
Str. Iuliu Maniu 50
2200 BRAŞOV, ROMANIA
E-mail: n.pascu@unitbv.ro


*Received January 29, 2002.*