DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

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THE CONJECTURE PARALLEL TO THE KRZYŻ CONJECTURE

Abstract. For any fixed integer k let us denote

$$F_k(t;z) = \frac{1}{(1-z)^{1+k}} \exp\left\{-t\frac{1+z}{1-z}\right\}$$
$$= e^{-t} + \sum_{n=1}^{\infty} A_n^{(k)}(t) z^n, \quad t > 0,\ z \in D = \{z : |z| < 1\}.$$

We say that a holomorphic function f in D of the form:

$$f(z) = e^{-t} + a_1 z + a_2 z^2 + \dots,\ z \in D,\ t > 0$$

belongs to the class $\mathcal{B}_0^k$ if and only if $f(z) \prec F_k(t;z)$, $z \in D$, where the sign $\prec$ denotes the subordination.

The class $\mathcal{B}_0^{-1} \equiv \mathcal{B}_0$, where $\mathcal{B}_0$ denotes the class of holomorphic, bounded and nonvanishing functions in D. We find the sharp bounds for $|a_1|$, $|a_2|$, $|a_3|$ and observe the different behaviour of these estimates depending on k. We conjecture that if $f(z) = e^{-t} + a_1 z + \dots \in \mathcal{B}_0^{-2}$, then

$$\max_{f \in \mathcal{B}_0^{-2}} |a_n| = \frac{2}{\sqrt{e}} = 1.21306\dots \quad n = 1,\ 2, \dots \tag{1}$$

and the sign of the equality holds (up to the rotation) only for the function

$$F_{-2}\left(\frac{1}{2}; z^n\right) = (1 - z^n)\exp\left\{-\frac{1}{2}\frac{1+z^n}{1-z^n}\right\} = \frac{1}{\sqrt{e}} - \frac{2}{\sqrt{e}} z^n + \dots$$

1. Introduction

Let $H(D)$ denote the set of holomorphic functions in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. In the sequel we consider the following families of functions:

$$\mathcal{B}_0 := \{f \in H(D) : f(z) = a_0 + a_1 z + \dots,\ 0 < |f(z)| < 1,\ z \in D\} \tag{2}$$

$$\Omega := \{\omega \in H(D) : \omega(z) = c_1 z + c_2 z^2 \dots,\ |\omega(z)| < 1,\ z \in D\}. \tag{3}$$

1991 *Mathematics Subject Classification*: 30C50.

With no loss of generality we may assume for $f \in \mathcal{B}_0$ the normalization

$$a_0 = e^{-t}, \quad t > 0.$$

The Krzyż conjecture [5] asserts that for $f(z) = e^{-t} + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{B}_0$:

$$\max_{f \in \mathcal{B}_0} |a_n| = \frac{2}{e} = 0.73575\ldots, \quad n = 1, 2, \ldots . \tag{4}$$

with the equality (up to the rotation) for the function $F_n(z) = F(z^n)$, $n = 1, 2, \ldots$, where

$$F(z) = \exp\left(-\frac{1+z}{1-z}\right) = \frac{1}{e} - \frac{2}{e}z + \ldots, \; z \in D. \tag{5}$$

So far the conjecture has been proved only for $n = 1, 2, 3, 4$. In general, it is known that $|a_n| < 0.99918$ ([3] and [2]). For more information concerning the problem as well as some of its generalizations or special versions we refer to [4], [10], [1], [7], [11], and [6].

The connection of the Krzyż conjecture with the Laguerre polynomials $L_n^{(-1)}$ is well-known. Namely we have (e.g.[6]):

$$f(z) = e^{-t} + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{B}_0 \iff f(z) = \exp\left\{-t\frac{1+\omega(z)}{1-\omega(z)}\right\} \tag{6}$$

where $\omega \in \Omega$. Denoting

$$F(t; z) = \exp\left\{-t\frac{1+z}{1-z}\right\} = e^{-t} + \sum_{n=1}^{\infty} A_n(t) z^n, \; z \in D \tag{7}$$

we have

$$A_n(t) = e^{-t} L_n^{(-1)}(2t), \; t > 0, \quad n = 1, 2, \ldots, \quad A_0 = e^{-t}, \tag{8}$$

because the generating function for the Laguerre polynomials $L_n^{(\alpha)}(x)$, $\alpha \in R$, $x > 0$ has the form:

$$\frac{1}{(1-z)^{1+\alpha}} \exp\left\{-\frac{xz}{1-z}\right\} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n, \; z \in D. \tag{9}$$

The relation (6) can be written equivalently in terms of subordination, namely

$$f(z) = e^{-t} + a_1 z + a_2 z^2 + \ldots \in \mathcal{B}_0 \iff f(z) \prec F(t; z), \quad z \in D.$$

One can extend the above coefficient problem (4) as follows. For any fixed integer k (positive, negative or zero) let us set

$$\begin{aligned} F_k(t; z) &= \frac{1}{(1-z)^{1+k}} \exp\left\{-t\frac{1+z}{1-z}\right\} \\ &= e^{-t} + \sum_{n=1}^{\infty} A_n^{(k)}(t) z^n, \quad t > 0, \; z \in D. \end{aligned} \tag{10}$$

By the relation (9) we have the following equality

$$A_n^{(k)}(t) = e^{-t} L_n^{(k)}(2t). \tag{11}$$

We propose the study of the coefficient problems in the classes $\mathcal{B}_0^k$ defined as follows:

$$f(z) = e^{-t} + a_1 z + a_2 z^2 + \dots \in \mathcal{B}_0^k \iff f(z) \prec F_k(t; z), \; z \in D. \tag{12}$$

Naturally, we have $\mathcal{B}_0^{-1} \equiv \mathcal{B}_0$. In this note we obtain the sharp bounds for $|a_n|$, $n = 1, 2, 3$; $f \in \mathcal{B}_0^k$ and observe their different behaviour depending on whether k is a positive or negative integer. At the end of this paper we propose the parallel conjecture to the Krzyż conjecture for the class $\mathcal{B}_0^{-2}$.

2. The coefficient problem in the classes $\mathcal{B}_0^k$

From the definition (12) of the class $\mathcal{B}_0^k$ we have:

$$f(z) = e^{-t} + a_1 z + a_2 z^2 + \dots \in \mathcal{B}_0^k \iff f(z) = F_k(t; \omega(z)) \tag{13}$$

where $\omega \in \Omega$.

Therefore, by (3), (10) and (13) we easily find that

$$\begin{cases} a_1 = A_1^{(k)}(t) c_1 \\ a_2 = A_1^{(k)}(t) c_2 + A_2^{(k)}(t) c_1^2 \\ a_3 = A_1^{(k)}(t) c_3 + 2A_2^{(k)}(t) c_1 c_2 + A_3^{(k)}(t) c_1^3 \end{cases} \tag{14}$$

where $A_n^{(k)}(t) = e^{-t} L_n^{(k)}(2t)$ are given by the following formulae:

$$\begin{aligned} A_1^{(k)}(t) &= e^{-t}(k+1-2t) \\ A_2^{(k)}(t) &= e^{-t}\left[\frac{1}{2}(k+1)(k+2) - 2(k+2)t + 2t^2\right] \\ A_3^{(k)}(t) &= e^{-t}\left[\frac{1}{6}\,(k+1)(k+2)(k+3)\right. \\ &\qquad \left. - (k+2)(k+3)t + 2(k+3)t^2 - \frac{4}{3}t^3\right]. \end{aligned} \tag{15}$$

In what follows we assume $k \neq -1$. The cases $k \geq 0$, $k = -2$ and $k \leq -3$ will be treated separately.

THEOREM 1. *If $f(z) \in \mathcal{B}_0^k$ then we have the following sharp estimates:*

$$|a_1| \leq e^{-t}|k+1-2t| \leq |k+1|, \qquad k \neq -1, -2; \tag{16}$$

$$\begin{aligned} |a_2| &\leq \max\left\{e^{-t}|k+1-2t|, e^{-t}\left|\frac{1}{2}(k+1)(k+2) - 2(k+2)t + 2t^2\right|\right\} \\ &\leq \frac{1}{2}(k+1)(k+2), \qquad k \geq 0, \; k \leq -5. \end{aligned}$$

Proof. From (14) and (15) we easily get (16) by using the inequalities $|c_1| \le 1$ and $|c_2| \le 1 - |c_1|^2$ valid for $\omega \in \Omega$.

The maximal value in both cases for the right-hand side is attained at $t = 0$ and $\omega(z) = z$ respectively. Therefore, the extremal function for $|a_1|$ and $|a_2|$ has the form (up to the rotation)

$$\begin{aligned} (17) \qquad f_0(z) &= \frac{1}{(1-z)^{k+1}} = (1-z)^{-(k+1)} = 1 - \binom{-(k+1)}{1} z + \dots \\ &= 1 + (k+1)z + \frac{1}{2}(k+1)(k+2)z^2 + \dots \ \blacksquare \end{aligned}$$

In the case $k = -2, -3$ and -4 the situation is different.

THEOREM 2. *If $f \in \mathcal{B}_0^{-2}$, then we have the following sharp estimates:*

$$(18) \qquad |a_1| \le e^{-t}(1+2t) \le \frac{2}{\sqrt{e}}$$

$$(19) \qquad |a_2| \le \max\{e^{-t}(1+2t), e^{-t}2t^2\} \le \frac{2}{\sqrt{e}}.$$

The extremal functions have the form:

$$(20) \qquad f_0(z) = (1-z)\exp\left\{-\frac{1}{2}\frac{1+z}{1-z}\right\}$$

in (18) *and*

$$(21) \qquad f_{00}(z) = (1-z^2)\exp\left\{-\frac{1}{2}\frac{1+z^2}{1-z^2}\right\}$$

in (19).

Proof. From the formulae (14) and (15) we obtain

$$(22) \qquad |a_1| \le \left|A_1^{(-2)}(t)\right| = e^{-t}(1+2t) \le \frac{2}{\sqrt{e}}$$

and

$$|a_2| \le \max\{e^{-t}(1+2t), e^{-t}2t^2\} = e^{-t}\begin{cases} 2t+1 & \text{if } 0 \le t \le \frac{1+\sqrt{3}}{2} \\ 2t^2 & \text{if } t \ge \frac{1+\sqrt{3}}{2}. \end{cases}$$

The maximal values are $\frac{2}{\sqrt{e}}$ for $t = \frac{1}{2}$ and $\frac{8}{e^2}$ for $t = 2$ respectively, which implies (19).

The form of the extremal function follows from the fact that we have $|c_1| = 1$, i.e. $\omega(z) = z$ in the case of $\max|a_1|$, and $|c_2| = 1$, i.e. $\omega(z) = z^2$ in the case of $\max|a_2|$. $\blacksquare$

THEOREM 3. *If $f \in \mathcal{B}_0^{-3}$, then we have the following sharp estimates:*

$$|a_1| \le 2(1+t)e^{-t} \le 2 \tag{23}$$

$$|a_2| \le \max\{e^{-t}(2t+2), e^{-t}(2t^2+2t+1)\} \le 2. \tag{24}$$

Proof. The estimates (23) and (24) again follow from the formulae (14) and (15) and the extremal functions have the form

$$f_1(z) = (1-z)^2$$

and

$$f_2(z) = (1-z^2)^2$$

respectively. ■

THEOREM 4. *If $f \in \mathcal{B}_0^{-4}$, then we have the following sharp estimates:*

$$|a_1| \le e^{-t}(2t+3) \le 3, \tag{25}$$

$$\begin{aligned} |a_2| &\le \max\{e^{-t}(2t+3), e^{-t}(2t^2+4t+3)\} \\ &\le 2(\sqrt{2}+2)e^{-\frac{\sqrt{2}}{2}} \approx 3.34\ldots \end{aligned} \tag{26}$$

The extremal functions have the form:

$$\widehat{f}_1(z) = (1-z)^3,$$

and

$$\widehat{f}_2(z) = (1-z)^3 \exp\left\{-\frac{\sqrt{2}}{2}\frac{1+z}{1-z}\right\}$$

respectively.

REMARK 1. [4] If $f \in \mathcal{B}_0^{-1}$, then the following sharp estimates hold:

$$|a_1| \le e^{-t}2t \le \frac{2}{e}, \tag{27}$$

$$|a_2| \le \max\{e^{-t}2t, e^{-t}|2t^2-2t|\} \le \frac{2}{e}. \tag{28}$$

The extremal functions have the form:

$$f^0(z) = \exp\left(-\frac{1+z}{1-z}\right),$$

and

$$f^{00}(z) = \exp\left(-\frac{1+z^2}{1-z^2}\right).$$

respectively.

The problem of estimating $|a_3|$ is much more complicated. However, it could be resolved by the following Lemma from [8].

LEMMA 1. *If $\omega \in \Omega$, then for any real numbers p and q the following sharp estimate holds:*

$$\Psi(\omega) := \left|c_3 + pc_1c_2 + qc_1^3\right| \le H(p,q) \tag{29}$$

where

$$H(p,q) = \begin{cases} 1 & \text{for } (p,q) \in D_1 \cup D_2 \\ |q| & \text{for } (p,q) \in \bigcup_{n=3}^{7} D_n \\ \frac{2}{3}(|p|+1)\left(\frac{|p|+1}{3(|p|+1+q)}\right)^{\frac{1}{2}} & \text{for } (p,q) \in D_8 \cup D_9 \\ \frac{1}{3}q\left(\frac{p^2-4}{p^2-4q}\right)\left(\frac{p^2-4}{3(q-1)}\right)^{\frac{1}{2}} & \text{for } (p,q) \in D_{10} \cup D_{11} \backslash (\pm 2, 1) \\ \frac{2}{3}(|p|-1)\left(\frac{|p|-1}{3(|p|-1-q)}\right)^{\frac{1}{2}} & \text{for } (p,q) \in D_{12}. \end{cases} \tag{30}$$

The sets $D_1, \ldots, D_{12}$ are defined as follows:

$$\begin{aligned} D_1 &:= \left\{(p,q) : \ |p| \le \frac{1}{2}, \quad |q| \le 1\right\}, \\ D_2 &:= \left\{(p,q) : \ \frac{1}{2} \le |p| \le 2, \quad \frac{4}{27}(|p|+1)^3 - (|p|+1) \le q \le 1\right\}, \\ D_3 &:= \left\{(p,q) : \ |p| \le \frac{1}{2}, \quad q \le -1\right\}, \\ D_4 &:= \left\{(p,q) : \ |p| \ge \frac{1}{2}, \quad q \le -\frac{2}{3}(|p|+1)\right\}, \\ D_5 &:= \{(p,q) : \ |p| \le 2, \quad q \ge 1\}, \\ D_6 &:= \left\{(p,q) : \ 2 \le |p| \le 4, \quad q \ge \frac{1}{12}(p^2+8)\right\}, \\ D_7 &:= \left\{(p,q) : \ |p| \ge 4, \quad q \ge \frac{2}{3}(|p|-1)\right\}, \\ D_8 &:= \left\{(p,q) : \ \frac{1}{2} \le |p| \le 2, \right. \\ &\qquad \left. -\frac{2}{3}(|p|+1) \le q \le \frac{4}{27}(|p|+1)^3 - (|p|+1)\right\}, \\ D_9 &:= \left\{(p,q) : \ |p| \ge 2, \quad -\frac{2}{3}(|p|+1) \le q \le \frac{2|p|(|p|+1)}{p^2+2|p|+4}\right\}, \\ D_{10} &:= \left\{(p,q) : \ 2 \le |p| \le 4, \quad \frac{2|p|(|p|+1)}{p^2+2|p|+4} \le q \le \frac{1}{12}(p^2+8)\right\}, \\ D_{11} &:= \left\{(p,q) : \ |p| \ge 4, \quad \frac{2|p|(|p|+1)}{p^2+2|p|+4} \le q \le \frac{2|p|(|p|-1)}{p^2-2|p|+4}\right\}, \\ D_{12} &:= \left\{(p,q) : \ |p| \ge 4, \quad \frac{2|p|(|p|-1)}{p^2-2|p|+4} \le q \le \frac{2}{3}(|p|-1)\right\}. \end{aligned} \tag{31}$$

THEOREM 5. *If* $f(z) = e^{-t} + a_1 z + \dots \in \mathcal{B}_0^k$, *then we have the following sharp estimate* ($x = 2t > 0$)

$$|a_3| \leq e^{-\frac{x}{2}}|(k+1) - x|H(p,q) \quad \text{if} \quad x \neq k+1 \tag{32}$$

where $H(p,q)$ *is given by* (30) *with*

$$\begin{cases} p = \dfrac{(k+1)(k+2) - 2(k+2)x + x^2}{(k+1) - x} \\ q = \dfrac{1}{6}\dfrac{(k+1)(k+2)(k+3) - 3(k+2)(k+3)x + 3(k+3)x^2 - x^3}{(k+1) - x} \end{cases} \tag{33}$$

and

$$|a_3| \leq \frac{2}{3}(k+1)e^{-\frac{k+1}{2}} \quad \text{if} \quad x = k+1, \ k = 0, 1, 2, \dots \tag{34}$$

Proof. From the formulae (14) and (15) we obtain (32) with p and q given by (33).

In order to apply Lemma 1 we have to find the equation of the curve (33), call it Γ_k, in the form $q = q(p)$ and find the intersection of Γ_k and the boundary curves of the domains D_k given by (31).

The elimination of the parameter x (obtained by long calculations) from (33) gives the equation of Γ_k which consists of two parts Γ_k^+ and Γ_k^-:

$$\Gamma_k^+ \ : \ q = q(p) = \frac{1}{12}\left\{p^2 + 6p + p\sqrt{p^2 - 4p + 4(k+2)} - 4(k+2)\right\}, \tag{35}$$

if $k = 0, 1, 2 \dots$; and p is running from $(k+2)$ to $-\infty$, $(x \in (0, k+1))$;

and

$$\Gamma_k^- \ : \ q = q(p) = \frac{1}{12}\left\{p^2 + 6p - p\sqrt{p^2 - 4p + 4(k+2)} - 4(k+2)\right\}, \tag{36}$$

if $k = 0, 1, 2 \dots$; and p is running from $+\infty$ to $-\infty$, $(x \in (k+1, \infty))$;

and if $k = -2, -3, \dots$; and p is running from $(k+2)$ to $-\infty$, $(x \in (0, \infty))$;

and

$$\Gamma_{-1}^- \ : \ q = q(p) = \frac{1}{6}(p^2 + 2p - 2), \ p \in (-\infty, -2]. \ \blacksquare \tag{37}$$

Because of the complicated nature of the equation of Γ_k we explicitly consider only the cases $k = -2$ and $k = -3$.

THEOREM 6. *If $f \in \mathcal{B}_0^{-2}$ then we have the following sharp estimates*

$$
(38)\quad |a_3| \le
e^{-\frac{x}{2}}\begin{cases}
(x+1) & \text{if } x \in (0, x_1] \\
\dfrac{2\sqrt{2}}{3}\dfrac{(x^2+x+1)^{\frac{3}{2}}}{(x^3+3x^2+6x+6)^{\frac{1}{2}}} & \text{if } x \in [x_1, x_2] \\
\dfrac{\sqrt{2}}{6}\dfrac{(x-3)(x^4-4(x+1)^2)^{\frac{3}{2}}}{(x+1)^{\frac{1}{2}}(x^2+4x+6)(x^3-3x^2-6x-6)^{\frac{1}{2}}} & \text{if } x \in [x_2, x_3] \\
\dfrac{2\sqrt{2}}{3}\dfrac{(x^2-x-1)^{\frac{3}{2}}}{(-x^3+9x^2-6x-6)^{\frac{1}{2}}} & \text{if } x \in [x_3, x_4] \\
\dfrac{1}{6}x^2(x-3) & \text{if } x \in [x_4, +\infty).
\end{cases}
$$

The numbers $x_1 \approx 2.17\ldots, x_2 \approx 5.06\ldots, x_3 \approx 5.62\ldots, x_4 \approx 6.26\ldots$ *are the roots of the following equations*

$$(39)\qquad 9(x+1)^2(x^3+3x^2+6x+6) - 8(x^2+x+1)^3 = 0$$

$$(40)\qquad 12(x+1)(x^2+x+1) - (x-3)(x^2+x+1)^2 - 3(x+1)^2(x-3) = 0$$

$$(41)\qquad 12(x+1)(x^2-x-1) - (x-3)(x^2-x-1)^2 - 3(x+1)^2(x-3) = 0$$

$$(42)\qquad x^3 - 7x^2 + 4x + 4 = 0$$

respectively.

Proof. In this particular case the equation of $\Gamma_k = \Gamma_{-2}^{-}$ is of the form

$$(43)\qquad q = \frac{1}{12}p\left\{p + 6 - \sqrt{p^2 - 4p}\right\} = \frac{1}{6}x^2\frac{x-3}{x+1},\ x \ge 0,$$

because $p = -\frac{x^2}{x+1} \le 0$. ■

The results (38)–(42) follow from Lemma 1 by the investigation of the intersection of Γ_{-2}^{-} given by (43) with the boundary curves of the domains D_k given by (31).

COROLLARY 1. *If $f \in \mathcal{B}_0^{-2}$ then*

$$(44)\qquad |a_3| \le \frac{2}{\sqrt{e}} \approx 1.21\ldots$$

and the extremal function has the form

$$f(z) = (1 - z^3)\exp\left\{-\frac{1}{2}\frac{1+z^3}{1-z^3}\right\}.$$

Maximization of the corresponding functions in (38) with the use of Mathematica leads to (44).

THEOREM 7. *If $f \in \mathcal{B}_0^{-3}$ then we have the following sharp estimates*

$$(45)\quad |a_3| \leq e^{-\frac{x}{2}} \begin{cases} (x+2) & \text{if } x \in (0, x_1^*] \\ \frac{2\sqrt{2}}{3} \frac{(x^2+3x+4)^{\frac{3}{2}}}{(x^3+6x^2+18x+24)^{\frac{1}{2}}} & \text{if } x \in [x_1^*, x_2^*] \\ \frac{\sqrt{2}}{6} \frac{x^3(x^4+4x^3+4x^2-8x-12)^{\frac{3}{2}}}{(x+1)^{\frac{1}{2}}(x^3-6x-12)^{\frac{1}{2}}(x^4+8x^3+24x^2+24x+12)} & \text{if } x \in [x_2^*, x_3^*] \\ \frac{2\sqrt{2}}{3} \frac{x(x+1)^{\frac{3}{2}}}{(-x^2+6x+6)^{\frac{1}{2}}} & \text{if } x \in [x_3^*, x_4^*] \\ \frac{1}{6}x^3 & \text{if } x \in [x_4^*, +\infty). \end{cases}$$

The numbers $x_1^* \approx 0.94\ldots, x_2^* \approx 3.64\ldots, x_3^* \approx 4.27\ldots, x_4^* = 2+2\sqrt{2}$ are the roots of the following equations

$$(46)\qquad 8x^6 + 63x^5 + 222x^4 + 378x^3 + 168x^2 - 360x - 352 = 0,$$

$$(47)\qquad x^7 + 6x^6 + 8x^5 - 48x^4 - 236x^3 - 456x^2 - 432x - 192 = 0,$$

$$(48)\qquad x^6 + 2x^5 - 8x^4 - 48x^3 - 108x^2 - 120x - 48 = 0,$$

$$(49)\qquad x^2 - 4x - 4 = 0,$$

respectively.

Proof. In this case the equation of $\Gamma_k = \Gamma_{-3}^-$ is of the form

$$(50)\qquad q = \frac{1}{12}p\left\{p + 6 - \sqrt{p^2 - 4p - 4}\right\} + 4 = \frac{1}{6}\frac{x^3}{x+2},\ x \geq 0,$$

because $p = -\frac{x^2+2x+2}{x+2} \leq -1$. ∎

The results (45)–(49) follow from Lemma 1 by the investigation of the intersection of Γ_{-3}^- given by (50) with the boundary curves of the domains D_k given by (31).

COROLLARY 2. *If $f \in \mathcal{B}_0^{-3}$ then*

$$(51)\qquad |a_3| \leq 2$$

and the extremal function has the form

$$f(z) = (1 - z^3)^2.$$

Maximization of the corresponding functions in (45) with the use of Mathematica leads to (51).

3. Concluding remarks

The behaviour of the few first coefficients of a function $f \in \mathcal{B}_0^{-2}$ as well as those of functions $f \in \mathcal{B}_0^k$, $k \leq -3$ leads us to the following conjecture, parallel to the Krzyż conjecture:

if $f(z) = e^{-t} + a_1 z + \ldots \in \mathcal{B}_0^{-2}$, then

$$\max_{f \in \mathcal{B}_0^{-2}} |a_n| = \frac{2}{\sqrt{e}} = 1.21306\ldots \qquad n = 1,\ 2, \ldots$$

and the sign of the equality holds (up to the rotation) only for the function

$$F_{-2}\left(\frac{1}{2}; z^n\right) = (1 - z^n)\exp\left\{-\frac{1}{2}\frac{1+z^n}{1-z^n}\right\} = \frac{1}{\sqrt{e}} - \frac{2}{\sqrt{e}} z^n + \ldots$$

REMARK 2. Some support for the above conjecture is given in particular by the function $F_{-2}(t, z)$. Indeed, for $n \geq 10$ from the estimate obtained by Rooney [9]:

$$|A_n^{(-2)}(t)| = e^{-t}|L_n^{(-2)}(2t)| \leq 4\frac{[(2n)!]^{1/2}}{2^{n+(1/2)}n!}, \qquad n = 1, 2, \ldots$$

and for $n = 4, 5, \ldots, 9$ by direct calculations with the use of Mathematica one can show that $|A_n^{(-2)}(t)| < 1$.

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Received January 28, 2002.