DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

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ON BAIRE ONE POINT OF FUNCTIONS

Abstract. We show that the subspace of the space of almost continuous first recoverable with respect to some trajectory $\{x_n\}$ functions, consisting of first return continuous functions with respect to $\{x_n\}$, is porous at each point of whole space. Next we define a class of strongly $\mathcal{F}$-almost everywhere first return recoverable functions and we describe some properties of these functions. We also prove that the subspace of the space of strongly $\mathcal{F}$-almost everywhere first return recoverable functions consisting of measurable functions is superporous at each point of whole space.

Several standard subcollections of the class of real-valued Baire 1 functions defined on $[0, 1]$ have been characterized utilizing first return limiting notions. For example, for a function $f : [0, 1] \to \mathbb{R}$ it is known that f is Baire 1 function if and only if f is first return recoverable with respect to some trajectory ([1]) and f is almost continuous Baire 1 function if and only if f is first return continuous with respect to some trajectory ([3]). In this paper we shall show that for each trajectory the subset of the space of almost continuous first recoverable with respect to some trajectory functions, consisting of first return continuous functions with respect to this trajectory, is "small" in the sense of category. It will be formulated more precisely in Theorem 1. Next we define a class of functions which are different from first return recoverable functions on a "small" set. It turns out that in this class of functions the set of measurable functions is "small" in sense of category (Theorem 2).

We apply the classical symbols and notions. By $\mathbb{R}$ ($\mathbb{N}$) we denote the set of real (positive integers) numbers. The symbol m_1 stands for the Lebesque measure on the real line. Let $\overline{A}$ ($\operatorname{int}(A)$) denote a closure (an interior) of A, where $A \subset [0, 1]$.

By D_f we denote the set of all points of discontinuity of a function $f : [0, 1] \to \mathbb{R}$. A function χ_A, where $A \subset [0, 1]$, is characteristic function defined as follows: $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in [0, 1] \setminus A$.

By $\mathcal{B}_1$ ($\mathcal{A}$, $\mathcal{D}$) we denote a set of Baire 1 (almost continuous, Darboux) functions $f : [0, 1] \to \mathbb{R}$. By ρ we denote the metric of uniform convergence.

We say that a function $f : [0,1] \to \mathbb{R}$ satisfies the Young condition if for every $x \in [0,1]$ there exist sequences $x_n \nearrow x$ and $y_n \searrow x$ such that both $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to $f(x)$.

If (X,d) is a metric space, then the open ball with center at x and radius $R > 0$ we denote by $B(x,R)$. Let $M \subset X$, $x \in X$, $R > 0$. Then $\gamma(x,R,M)$ denotes the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$. The set M is porous at x if $p(M,x) = \limsup_{R \to 0^+} \frac{\gamma(x,R,M)}{R} > 0$.

By a trajectory we mean any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $[0,1]$, which is dense in $[0,1]$.

Let $\{x_n\}$ be a fixed trajectory. For a given interval, or finite union of intervals, $H \subseteq [0,1]$, $r(H)$ will be the first element of the trajectory $\{x_n\}$ in H.

For $0 < x \leq 1$, the left return path to x based on $\{x_n\}$, $P_x^l = \{t_k\}$, is defined recursively via

$$t_1 = r(0,x) \text{ and } t_{k+1} = r(t_k, x).$$

For $0 \leq x < 1$, the right first return path to x based on $\{x_n\}$, $P_x^r = \{s_k\}$, is defined analogously.

A function $f : [0,1] \to \mathbb{R}$ is first return continuous from the left [right] at x with respect to the trajectory $\{x_n\}$ provided

$$\lim_{t \to x, t \in P_x^l} f(t) = f(x) \quad [\lim_{s \to x, s \in P_x^r} f(s) = f(x)].$$

We say that for any $x \in (0,1)$, $f : [0,1] \to \mathbb{R}$ is first return continuous at x with respect to the trajectory $\{x_n\}$ provided it is both left and right first return continuous at x with respect to the trajectory $\{x_n\}$.

We say that $x \in [0,1]$ is a first return continuity (from the left, from the right) point of $f : [0,1] \to \mathbb{R}$ with respect to $\{x_n\}$ if f is first return continuous (from the left, from the right) with respect to $\{x_n\}$ at x. For a fixed function f let $\mathcal{C}(f,\{x_n\})$ denote the set of all first return continuity points of f with respect to $\{x_n\}$. Moreover let $\mathcal{C}^\wedge(f,\{x_n\}) = [0,1] \setminus \mathcal{C}(f,\{x_n\})$.

For $x \in [0,1]$ we define what we shall mean by the first return route to x based on the trajectory $\{x_n\}$. The first return route to x, $R_x = \{y_k\}_{k=1}^{\infty}$, is defined recursively via

$$y_1 = x_0,$$
$$y_{k+1} = \begin{cases} r(B(x, |x - y_k|)) & \text{if } x \neq y_k; \\ y_k & \text{if } x = y_k. \end{cases}$$

We say that $f : [0,1] \to \mathbb{R}$ is first return recoverable with respect to $\{x_n\}$ at x provided that

$$\lim_{k \to \infty} f(y_k) = f(x),$$

and if this happens for each $x \in [0,1]$, we say that $f : [0,1] \to \mathbb{R}$ is first return recoverable with respect to $\{x_n\}$.

We say that $x \in [0,1]$ is Baire one point of $f : [0,1] \to \mathbb{R}$ with respect to $\{x_n\}$ if f is first return recoverable with respect to $\{x_n\}$ at x. For a fixed function f let $\mathcal{B}_1(f,\{x_n\})$ denote the set of all Baire one points of f with respect to $\{x_n\}$. Moreover let $\mathcal{B}_1^{\wedge}(f,\{x_n\}) = [0,1] \setminus \mathcal{B}_1(f,\{x_n\})$.

Let $\{x_n\}_{|A}$ denote a subsequence of a sequence $\{x_n\}$ consisting of all x_n such that $x_n \in A$.

REMARK 1. There exists a function $f : [0,1] \to \mathbb{R}$ such that $[a,b] \subset [0,1] \subset \mathcal{B}_1(f,\{x_n\})$ and $[a,b] \setminus \mathcal{B}_1(f_{|[a,b]},\{x_n\}_{|[a,b]}) \neq \emptyset$ for some trajectory $\{x_n\}$.

Proof. It is enough to consider a function $f : [0,1] \to \mathbb{R}$ defined as follows: $f(x) = \chi_{|(\frac{1}{2},1]}(x)$. Then there exists a trajectory $\{x_n\}$ such that $[0,1] \subset \mathcal{B}_1(f,\{x_n\})$ and $\{\frac{1}{2}\} \in [\frac{1}{2},1] \setminus \mathcal{B}_1(f_{|[\frac{1}{2},1]},\{x_n\}_{|[\frac{1}{2},1]}) \neq \emptyset$. ∎

But it turns out that we can slightly modify a trajectory $\{x_n\}$ in such a way that above situation is impossible.

We say that a trajectory $\{z_n\}$ is a finite extension of a trajectory $\{x_n\}$ if $\{x_n\}$ is a subsequence of the sequence $\{z_n\}$ and $\mathrm{card}(\{z_n : n = 1,2,\ldots\} \setminus \{x_n : n = 1,2,\ldots\}) < \aleph_0$.

PROPOSITION 1. *Let $f : [0,1] \to \mathbb{R}$ be such that $(a,b) \subset [0,1] \cap \mathcal{B}_1(f,\{x_n\})$. Then there exists a finite extension $\{z_n\}$ of a trajectory $\{x_n\}_{|(a,b)}$ such that $[a,b] \subset \mathcal{B}_1(f_{|[a,b]},\{z_n\})$.*

Proof. Let $\{x_n\}$ be a fixed trajectory. Consider a trajectory $\{z_n\}$ defined in the following way:

$$z_0 = a, \quad z_1 = b,$$
$$z_t = r((a,b) \setminus \{z_i : i < t\}) \text{ for } t \geq 2.$$

Obviously $\{z_n\}$ is a finite extension of the trajectory $\{x_n\}$. In order to complete the proof, it is sufficient to show that

$$[a,b] \subset \mathcal{B}_1(f_{|[a,b]},\{z_n\}). \tag{1}$$

It is easy to see that $a,b \in \mathcal{B}_1(f_{|[a,b]},\{z_n\})$. So let $x \in (a,b)$. Assume for example that $|x-a| \leq |x-b|$. Let $R_x = \{y_k\}_{k=1}^{\infty}$ be the first return route to x based on the trajectory $\{x_n\}$. Let $R_x^{(z)} = \{y_k^{(z)}\}_{k=1}^{\infty}$ be the first return route to x based on the trajectory $\{z_n\}$. Then $y_1^{(z)} = a$ and $y_2^{(z)} = r(B(x,|x-a|))$. Consider the following cases:

- If $x_0 \in B(x,|x-a|)$ then $y_2^{(z)} = x_0$ and $y_k^{(z)} = y_{k-1}$ for $k \geq 3$. Hence $\lim_{k\to\infty} f(y_k^{(z)}) = \lim_{k\to\infty} f(y_k) = f(x)$, so $x \in \mathcal{B}_1(f_{|[a,b]},\{z_n\})$.

- If $|x - x_0| = |x - a|$ then $B(x, |x-a|) = B(x, |x - x_0|)$ and $y_k^{(z)} = y_k$ for $k \geq 2$. Hence $\lim_{k\to\infty} f(y_k^{(z)}) = \lim_{k\to\infty} f(y_k) = f(x)$, so $x \in \mathcal{B}_1(f_{|[a,b]}, \{z_n\})$.
- If $x_0 \notin B(x, |x-a|)$ note that $y_2^{(z)} \in \{z_n : n \geq 2\} \subset \{x_n : n \geq 1\}$. So let $y_2^{(z)} = x_m$ for some $m \geq 1$. Note that

  $$y_2^{(z)} \in \{y_k : k \geq 1\}. \tag{2}$$

  If $x_m = y_1$ the condition (2) is obvious. In the opposite case let $y_1 = x_{s_1}$, $y_2 = x_{s_2}$, ..., $y_j = x_{s_j}$ be all of elements of the sequence $\{y_k\}$ such that $s_i < m$ for $i \in \{1, 2, \ldots, j\}$. Then from the definition of $r(B(x, |x-a|))$ we infer that $x_{s_i} \notin B(x, |x-a|)$ for $i \in \{1, 2, \ldots, j\}$. Consider $y_{j+1} = r(B(x, |x - y_j|))$. Let $y_{j+1} = x_{s_{j+1}}$. Hence $s_{j+1} \geq m$, so $y_{j+1} = x_m$, which finishes the proof of (2).
  Let $y_2^{(z)} = y_{k_0}$ for some $k_0 \geq 1$. Therefore $y_k^{(z)} = y_{k_0+k-1}$ for $k \geq 3$. Hence $\lim_{k\to\infty} f(y_k^{(z)}) = \lim_{k\to\infty} f(y_k) = f(x)$, so $x \in \mathcal{B}_1(f_{|[a,b]}, \{z_n\})$.

The proof of (1) is finished. ∎

From the last proposition it is easy to deduce the following fact:

LEMMA 1. *Let $f : [0,1] \to \mathbb{R}$ and $(a,b) \subset [0,1]$. If $(a,b) \subset \mathcal{B}_1(f, \{x_n\})$ then $f_{|[a,b]}$ and $f_{|(a,b)}$ are Baire 1 functions.* ∎

It is known that

(1) $f : [0,1] \to \mathbb{R}$ is a Baire 1 function iff there exists a trajectory $\{x_n\}$ such that $\mathcal{B}_1^\wedge(f, \{x_n\}) = \emptyset$ [1];

(2) $f : [0,1] \to \mathbb{R}$ is an almost continuous Baire 1 function iff there exists a trajectory $\{x_n\}$ such that $\mathcal{C}^\wedge(f, \{x_n\}) = \emptyset$ [3].

Let $\{x_n\}$ be a fixed trajectory. By $\mathcal{B}_1(\{x_n\})$ we denote a set of all functions which are first return recoverable with respect to $\{x_n\}$. Obviously $\mathcal{B}_1(\{x_n\}) \subset \mathcal{B}_1$.

By $\mathcal{C}(\{x_n\})$ we denote a set of all functions which are first return continuous with respect to $\{x_n\}$. Obviously $\mathcal{C}(\{x_n\}) \subset \mathcal{B}_1 \cap \mathcal{A}$.

By $\mathcal{A}^\star(\{x_n\})$ we denote the set of all bounded functions $f : [0,1] \to \mathbb{R}$ such that $f \in \mathcal{B}_1(\{x_n\}) \cap \mathcal{A}$.

THEOREM 1. *For each trajectory $\{x_n\}$ the set $\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\})$ is porous at each point of the space $\mathcal{A}^\star(\{x_n\})$ (with the metric ρ).*

Proof. Let $\{x_n\}$ be an arbitrary fixed trajectory and without loss of generality assume that $\frac{1}{2}$ is term in the trajectory, say $x_{n_0} \in (0,1) = \frac{1}{2}$. Then $\chi_{[\frac{1}{2},1]} \in \mathcal{B}_1(\{x_n\})$ (if $f = \chi_{[\frac{1}{2},1]}$ we have $[0,1] \setminus \frac{1}{2} \subset \mathcal{B}_1(f, \{x_n\})$ and

$\lim_{k\to\infty} f(y_k) = \lim_{k\to\infty} f(\frac{1}{2}) = f(\frac{1}{2})$, where $R_{\frac{1}{2}} = \{y_k\}_{k=1}^{\infty}$ is the first return route to $\frac{1}{2}$ based on the trajectory $\{x_n\}$, so $\frac{1}{2} \in \mathcal{B}_1(f, \{x_n\})$).

Let $f \in \mathcal{A}^\star(\{x_n\})$ and let $\varepsilon > 0$. Let $\alpha = f(\frac{1}{2})$. We shall consider two following cases:

1^0 The function f is not first return continuous from the left at $\frac{1}{2}$ with respect to the trajectory $\{x_n\}$.

Then there exists $n_0 \in \mathbb{N}$ and a subsequence $\{z_n\}$ of a sequence $P^l_{\frac{1}{2}}$ such that $\lim_{n\to\infty} z_n = \frac{1}{2}$ and $f(z_n) \notin [\alpha - \frac{1}{n_0}, \alpha + \frac{1}{n_0}]$ for each $n \in \mathbb{N}$. We shall show that

$$B(f, \frac{1}{3n_0}) \cap (\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\})) = \emptyset. \tag{3}$$

Indeed, if $g \in B(f, \frac{1}{3n_0})$, $g(z_n) \notin [\alpha - \frac{2}{3n_0}, \alpha + \frac{2}{3n_0}]$ for each $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} z_n = \frac{1}{2}$ and $\{z_n\}$ is a subsequence of a sequence $P^l_{\frac{1}{2}}$, the function g is not first return continuous from the left at $\frac{1}{2}$ with respect to the trajectory $\{x_n\}$. Hence $g \notin \mathcal{C}(\{x_n\})$. The proof of (3) is finished.

Hence for $R < \frac{1}{3n_0}$

$$B(f, R) \cap (\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\})) = \emptyset,$$

so

$$p(\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\}), f) = 1.$$

2^0 The function f is first return continuous from the left at $\frac{1}{2}$ with respect to the trajectory $\{x_n\}$.

$2a$) The function f is continuous from left at $\frac{1}{2}$.

Then there exists $\delta_0 > 0$ such that

$$f\left(\left[\frac{1}{2} - \delta_0, \frac{1}{2}\right]\right) \subset \left(\alpha - \frac{\varepsilon}{8}, \alpha + \frac{\varepsilon}{8}\right). \tag{4}$$

Let $\{t_k\}$ be a sequence such that $\lim_{k\to\infty} t_k = \frac{1}{2}$ and $t_k \in [\frac{1}{2} - \delta_0, \frac{1}{2}] \cap P^l_{\frac{1}{2}}$ for each $k \in \mathbb{N}$. We define a function $g : [0,1] \to \mathbb{R}$ in the following way

$$g(x) = \begin{cases} f(x) + \frac{\varepsilon}{4} & \text{if } x \in [0, \frac{1}{2} - \delta_0]; \\ l_0(x) & \text{if } x \in [\frac{1}{2} - \delta_0, t_1]; \\ l_k(x) & \text{if } x \in [t_k, \frac{t_k + t_{k+1}}{2}], k = 1, 2, \ldots; \\ \tilde{l}_k(x) & \text{if } x \in [\frac{t_k + t_{k+1}}{2}, t_{k+1}], k = 1, 2, \ldots; \\ f(x) & \text{if } x \in [\frac{1}{2}, 1]; \end{cases}$$

where l_0 is a linear function such that $l_0(\frac{1}{2} - \delta_0) = f(\frac{1}{2} - \delta_0) + \frac{\varepsilon}{4}$ and $l_0(t_1) = \alpha + \frac{\varepsilon}{8}$; l_k $(k = 1, 2, \ldots)$ is a linear function such that $l_k(t_k) = \alpha + \frac{\varepsilon}{8}$ and $l_k(\frac{t_k + t_{k+1}}{2}) = \alpha$; $\tilde{l}_k$ $(k = 1, 2, \ldots)$ is a linear function such that $\tilde{l}_k(\frac{t_k + t_{k+1}}{2}) = \alpha$ and $\tilde{l}_k(t_{k+1}) = \alpha + \frac{\varepsilon}{8}$.

The function g is continuous at each point of $[\frac{1}{2}-\delta_0, \frac{1}{2})$, so

$$\left(\frac{1}{2}-\delta_0, \frac{1}{2}\right) \subset \mathcal{B}_1(g, \{x_n\}).$$

Moreover

$$\left[0, \frac{1}{2}-\delta_0\right) \cup \left(\frac{1}{2}, 1\right] \subset \mathcal{B}_1(g, \{x_n\}),$$

because $g_{|[0,\frac{1}{2}-\delta_0)} = f_{|[0,\frac{1}{2}-\delta_0)} + \frac{\varepsilon}{4}$, $g_{|(\frac{1}{2},1]} = f_{|(\frac{1}{2},1]}$ and $f \in \mathcal{B}_1(\{x_n\})$.

Since $g_{|[0,\frac{1}{2}-\delta]} = f_{|[0,\frac{1}{2}-\delta]} + \frac{\varepsilon}{4}$, $f \in \mathcal{B}_1(\{x_n\})$ and $g_{|[\frac{1}{2}-\delta,\frac{1}{2})}$ is continuous,

$$\frac{1}{2}-\delta_0 \in \mathcal{B}_1(g, \{x_n\}).$$

Now, note that

(5) $$\frac{1}{2} \in \mathcal{B}_1(g, \{x_n\}).$$

Indeed, let $R_{\frac{1}{2}} = \{y_k\}_{k=1}^{\infty}$ be the first return route to $\frac{1}{2}$ (based on the trajectory $\{x_n\}$). Since $\chi_{[\frac{1}{2},1]} \in \mathcal{B}_1(\{x_n\})$, $\lim_{k\to\infty} \chi_{[\frac{1}{2},1]}(y_k) = \chi_{[\frac{1}{2},1]}(\frac{1}{2}) = 1$. Hence almost all of points of $\{y_k\}$ belongs to $[\frac{1}{2}, 1]$, so $g(y_k) = f(y_k)$, for almost all $k \in \mathbb{N}$. Therefore (by the fact that $f \in \mathcal{B}_1(\{x_n\})$)

$$\lim_{k\to\infty} g(y_k) = \lim_{k\to\infty} f(y_k) = f(\frac{1}{2}) = g(\frac{1}{2}),$$

so $\frac{1}{2} \in \mathcal{B}_1(g, \{x_n\})$. The proof of (5) is finished.

We have just showed that $x \in \mathcal{B}_1(g, \{x_n\})$ for each $x \in [0, 1]$, so

$$g \in \mathcal{B}_1(\{x_n\}).$$

Since $f \in \mathcal{A}^{\star}(\{x_n\}) = \mathcal{B}_1(\{x_n\}) \cap \mathcal{A} \subset \mathcal{B}_1 \cap \mathcal{A}$ and $g_{|[\frac{1}{2}-\delta_0,\frac{1}{2}]} \in \mathcal{B}_1 \cap \mathcal{D} = \mathcal{B}_1 \cap \mathcal{A}$ we can infer that

$$g \in \mathcal{A}.$$

Moreover

(6) $$B\left(g, \frac{\varepsilon}{32}\right) \subset B(f, \varepsilon).$$

In fact, let $h \in B(g, \frac{\varepsilon}{32})$. Then

(7) $$\rho(h, f) \le \rho(h, g) + \rho(g, f) < \frac{\varepsilon}{32} + \rho(g, f).$$

Consider the following cases:

1) $x \in [0, \frac{1}{2}-\delta_0]$. Then $|g(x) - f(x)| = \frac{\varepsilon}{4}$.

2) $x \in [\frac{1}{2}-\delta_0, t_1]$. Then (by (4) and the definition of l_0) $0 < \alpha + \frac{\varepsilon}{8} - f(x) < g(x) - f(x) = l_0(x) - f(x) < \frac{\varepsilon}{2}$, so $|g(x) - f(x)| < \frac{\varepsilon}{2}$.

3) $x \in [t_k, \frac{t_k+t_{k+1}}{2}]$, $k = 1, 2, \ldots$ Then (by (4) and the definition of l_k) $-\frac{\varepsilon}{8} = \alpha - \alpha - \frac{\varepsilon}{8} < l_k(x) - f(x) = g(x) - f(x) < \frac{\varepsilon}{4}$, so $|g(x) - f(x)| < \frac{\varepsilon}{4}$.

4) $x \in [\frac{t_k+t_{k+1}}{2}, t_{k+1}]$, $k = 1, 2, \dots$ Then (by (4) and the definition of $\tilde{l}_k$) $-\frac{\varepsilon}{8} = \alpha - \alpha - \frac{\varepsilon}{8} < \tilde{l}_k(x) - f(x) = g(x) - f(x) < \frac{\varepsilon}{4}$, so $|g(x) - f(x)| < \frac{\varepsilon}{4}$.

5) $x \in [\frac{1}{2}, 1]$. Then $|g(x) - f(x)| = |f(x) - f(x)| = 0$.

We have just showed that $|g(x) - f(x)| < \frac{\varepsilon}{2}$ for each $x \in [0,1]$, so $\rho(g, f) \le \frac{\varepsilon}{2}$. Hence (by (7))

$$\rho(h, f) < \varepsilon,$$

which finishes the proof of (6).

Now, note that

$$B\left(g, \frac{\varepsilon}{32}\right) \cap (\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\})) = \emptyset. \tag{8}$$

Indeed, if $h \in B(g, \frac{\varepsilon}{32})$,

$$h\left(\frac{1}{2}\right) < g\left(\frac{1}{2}\right) + \frac{\varepsilon}{32} = \alpha + \frac{1}{32}\varepsilon$$

and for each $k \in \mathbb{N}$

$$h(t_k) > g(t_k) - \frac{\varepsilon}{32} = \alpha + \frac{\varepsilon}{8} - \frac{\varepsilon}{32} = \alpha + \frac{3}{32}\varepsilon.$$

Since moreover $\lim_{k\to\infty} t_k = \frac{1}{2}$ and $\{t_k\}$ is the subsequence of $P^l_{\frac{1}{2}}$, the function h is not first return continuous from the left at $\frac{1}{2}$ with respect to the trajectory $\{x_n\}$. Hence $h \notin \mathcal{C}(\{x_n\})$, which finishes the proof of (8).

From (8) and (6) we deduce that

$$p(\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star, f)) \ge \frac{1}{32} > 0.$$

2*b*) The function f is not continuous from the left at $\frac{1}{2}$.

Then

$$\limsup_{x\to\frac{1}{2}^-} f(x) > \alpha \text{ or } \liminf_{x\to\frac{1}{2}^-} f(x) < \alpha.$$

Suppose that $\limsup_{x\to\frac{1}{2}^-} f(x) > \alpha$. Let $\varepsilon > 0$ be an arbitrary real number such that $\alpha + \frac{\varepsilon}{4} < \limsup_{x\to\frac{1}{2}^-} f(x)$.

We define a function $g : [0,1] \to \mathbb{R}$ in the following way:

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, \frac{1}{2}); \\ f(x) + \frac{\varepsilon}{4} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Since $f \in \mathcal{B}_1(\{x_n\}) \cap \mathcal{A} \subset \mathcal{B}_1 \cap \mathcal{A} = \mathcal{B}_1 \cap \mathcal{D}$, it is obvious that the function g satisfies the Young's condition at each point $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Moreover, there exists a sequence $\{w_n\}$ such that $w_n \searrow \frac{1}{2}$ and $f(w_n) \to f(\frac{1}{2}) = \alpha$. Next, by the assumption that $\limsup_{x\to\frac{1}{2}^-} f(x) > \alpha + \frac{\varepsilon}{4}$ and by the fact $f \in \mathcal{B}_1 \cap \mathcal{D}$, there exists a sequence $\{z_n\}$ such that $z_n \nearrow \frac{1}{2}$ and $\lim_{n\to\infty} f(z_n) = \alpha + \frac{\varepsilon}{4}$, so

$$w_n \searrow \frac{1}{2} \text{ and } \lim_{n\to\infty} g(w_n) = \lim_{n\to\infty} f(w_n) + \frac{\varepsilon}{4} = g(\frac{1}{2})$$

and

$$z_n \nearrow \frac{1}{2} \text{ and } \lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} f(z_n) = g(\frac{1}{2}).$$

Hence the function g satisfies the Young condition at $\frac{1}{2}$.

The above considerations show that

$$g \in \mathcal{B}_1 \cap \mathcal{D} \subset \mathcal{A}.$$

It is easy to note that

$$g \in \mathcal{B}_1(\{x_n\}).$$

Indeed, obviously $x \in \mathcal{B}_1(g, \{x_n\})$, for each $x \in [0,1] \setminus \{\frac{1}{2}\}$. For $x = \frac{1}{2}$ the proof is analogous as the proof of (5).

It is not difficult to show that

$$B(g, \frac{\varepsilon}{16}) \subset B(f, \varepsilon). \tag{9}$$

Now note that

$$B(g, \frac{\varepsilon}{16}) \cap (\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\})) = \emptyset. \tag{10}$$

Indeed, if $h \in B(g, \frac{\varepsilon}{16})$, for a sequence $\{t_k\}$ such that $t_k \nearrow \frac{1}{2}$ and $t_k \in P^l_{\frac{1}{2}}$ for each $k \in \mathbb{N}$,

$$\begin{aligned}\limsup_{k\to\infty} h(t_k) &\le \lim_{k\to\infty}\left(g(t_k) + \frac{\varepsilon}{16}\right) = \lim_{k\to\infty}\left(f(t_k) + \frac{\varepsilon}{16}\right) \\ &= f\left(\frac{1}{2}\right) + \frac{\varepsilon}{16} = \alpha + \frac{\varepsilon}{16}.\end{aligned}$$

On the other hand

$$h\left(\frac{1}{2}\right) > g\left(\frac{1}{2}\right) - \frac{\varepsilon}{16} = \alpha + \frac{3}{16}\varepsilon,$$

so the function h is not first return continuous from the left at $\frac{1}{2}$ with respect to the trajectory $\{x_n\}$. Hence $h \notin \mathcal{C}(\{x_n\})$. This finishes the proof of (10).

From (9) and (10) we obtain that

$$p(\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\}), f) \ge \frac{1}{16} > 0,$$

so the set $\mathcal{C}(\{x_n\}) \cap \mathcal{A}^\star(\{x_n\})$ is porous at f. ∎

PROPOSITION 2. *Let $f \in \mathcal{B}_1(\{x_n\})$. Then $f \in \mathcal{A}$ iff there exists a trajectory $\{y_n\}$ such that $[0,1] \setminus \mathcal{C}(f, \{x_n\}) \subset \mathcal{C}(f, \{y_n\})$.*

Proof. Let $f \in \mathcal{A}$. Then $f \in \mathcal{A} \cap \mathcal{B}_1(\{x_n\}) \subset \mathcal{A} \cap \mathcal{B}_1$. Then there exists a trajectory $\{y_n\}$ such that $\mathcal{C}(f, \{y_n\}) = [0,1]$.

Now let $\{y_n\}$ be a trajectory such that $[0,1] \setminus \mathcal{C}(f, \{x_n\}) \subset \mathcal{C}(f, \{y_n\})$. Let $x_0 \in [0,1]$.

- If $x_0 \in \mathcal{C}(f, \{x_n\})$, there exist sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha_n \searrow x_0$, $\beta_n \nearrow x_0$ and $\lim_{n\to\infty} f(\alpha_n) = f(x_0) = \lim_{n\to\infty} f(\beta_n)$.
- If $x_0 \notin \mathcal{C}(f, \{x_n\})$, $x_0 \in \mathcal{C}(f, \{y_n\})$ (by the assumption) and hence there exist sequences $\{\gamma_n\}$, $\{\delta_n\}$ such that $\gamma_n \searrow x_0$, $\delta_n \nearrow x_0$ and $\lim_{n\to\infty} f(\gamma_n) = f(x_0) = \lim_{n\to\infty} f(\delta_n)$.

Hence, by the Young condition and the fact that $f \in \mathcal{B}_1(\{x_n\}) \subset \mathcal{B}_1$,

$$f \in \mathcal{B}_1 \cap \mathcal{D} \subset \mathcal{A}. \square$$

Now we will define some classes of functions wider than the class $\mathcal{B}_1(\{x_n\})$.

Let $\mathcal{F}$ be an ideal of subsets of real line such that: if $A \in \mathcal{F}$, then $\operatorname{int}(A) = \emptyset$. A function $f : [0,1] \to \mathbb{R}$ is $\mathcal{F}-$ almost everywhere first return recoverable with respect to the trajectory $\{x_n\}$ if $\mathcal{B}_1^{\wedge}(f, \{x_n\}) \in \mathcal{F}$.

A function $f : [0,1] \to \mathbb{R}$ is strongly $\mathcal{F}-$ almost everywhere first return recoverable with respect to the trajectory $\{x_n\}$ if $\overline{\mathcal{B}_1^{\wedge}(f, \{x_n\})} \in \mathcal{F}$.

We will consider the second class. Let us denote this class (of strongly $\mathcal{F}-$ almost everywhere first return recoverable with respect to the trajectory $\{x_n\}$ by the symbol $\mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. We will say that a function $f : [0,1] \to \mathbb{R}$ is strongly $\mathcal{F}-$ almost everywhere first return recoverable if there exists a trajectory $\{x_n\}$ such that $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. Let $\mathcal{B}_1^{\mathcal{F}}$ denote a set of all strongly $\mathcal{F}-$ almost everywhere first return recoverable functions $f : [0,1] \to \mathbb{R}$.

Note that $\mathcal{B}_1^{\mathcal{F}}$ contains a large class of functions. For example, all functions $f : [0,1] \to \mathbb{R}$ such that D_f is a nowhere dense set, are strongly $\mathcal{F}-$ almost everywhere first return recoverable. Moreover the following fact is obvious:

PROPOSITION 3. *If $f : [0,1] \to \mathbb{R}$ is Baire 1 function, $f \in \mathcal{B}_1^{\mathcal{F}}$.* ■

The following three propositions show that functions in $\mathcal{B}_1^{\mathcal{F}}$ have properties similar to those of Baire one functions.

PROPOSITION 4. *The set of all points of discontinuity of an arbitrary function $f \in \mathcal{B}_1^{\mathcal{F}}$ is meager.*

Proof. Let $\{x_n\}$ be a trajectory such that the set $\mathcal{B}_1(f, \{x_n\})$ has a dense interior. Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be a sequence of all components of the set $\operatorname{int}(\mathcal{B}_1(f, \{x_n\}))$. Then $f_{|(a_n,b_n)}$ is Baire 1 function for each $n \in \mathbb{N}$ (Lemma 1), so $D_{f_{|(a_n,b_n)}}$ is a meager set for each $n \in \mathbb{N}$. Hence

$$D_f \subset \bigcup_{n=1}^{\infty} D_{f_{|(a_n,b_n)}} \cup \{a_n : n = 1, 2, \ldots\} \cup \{b_n : n = 1, 2, \ldots\} \cup \overline{\mathcal{B}_1^{\wedge}(f, \{x_n\})}$$

is a meager set, too. ∎

PROPOSITION 5. *If $f : [0,1] \to \mathbb{R}$, $g : [0,1] \to \mathbb{R}$, $f, g \in \mathcal{B}_1^{\mathcal{F}}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in \mathcal{B}_1^{\mathcal{F}}$.*

Proof. Let $\{x_n^{(f)}\}$, $\{x_n^{(g)}\}$ be trajectories such that $\overline{\mathcal{B}_1^{\wedge}(f, \{x_n^{(f)}\})} \in \mathcal{F}$ and $\overline{\mathcal{B}_1^{\wedge}(g, \{x_n^{(g)}\})} \in \mathcal{F}$. Then $U = \mathrm{int}(\mathcal{B}_1(f, \{x_n^{(f)}\}))$ and $W = \mathrm{int}(\mathcal{B}_1(g, \{x_n^{(g)}\}))$ are dense in $[0,1]$. Hence $V = U \cap W$ is open and dense in $[0,1]$. Moreover it is easy to note that $f_{|V}$ and $g_{|V}$ are Baire 1 functions (Lemma 1). Then obviously $(\alpha f + \beta g)_{|V}$ is Baire 1 function. We define a function $\phi : [0,1] \to \mathbb{R}$ in the following way:

$$\phi(x) = \begin{cases} \alpha f(x) + \beta g(x) & \text{if } x \in V; \\ 0 & \text{if } x \in [0,1] \setminus V. \end{cases}$$

It is not difficult to show that

$$\phi \in \mathcal{B}_1.$$

Hence there exists a trajectory $\{z_n\}$ such that $\phi \in \mathcal{B}_1(\{x_n\})$. Note that

(11) $$V \subset \mathcal{B}_1(\alpha f + \beta g, \{x_n\}).$$

Indeed, let $x_0 \in V$. Then $\lim_{k\to\infty} \phi(y_k) = \phi(x_0)$, where the sequence $R_{x_0} = \{y_k\}_{k=1}^{\infty}$ is the first return route to x_0 based on the trajectory $\{x_n\}$. Then $y_k \in V$ for k large enough. Hence $\lim_{k\to\infty}(\alpha f + \beta g)(y_k) = (\alpha f + \beta g)(x_0)$, so $x_0 \in \mathcal{B}_1(\alpha f + \beta g, \{z_n\})$. The proof of (11) is finished.

By inclusion (11) it is to easy to infer that $\overline{\mathcal{B}_1^{\wedge}(\alpha f + \beta g, \{z_n\})} \in \mathcal{F}$, so $\alpha f + \beta g \in \mathcal{B}_1^{\mathcal{F}}(\{z_n\}) \subset \mathcal{B}_1^{\mathcal{F}}$. ∎

In the analogous way we can prove the following fact:

PROPOSITION 6. *If $f, g \in \mathcal{B}_1^{\mathcal{F}}$, $fg \in \mathcal{B}_1^{\mathcal{F}}$.* ∎

It turns out that in the space $\mathcal{B}_1^{\mathcal{F}}$ the set of measurable functions is small in category sense.

THEOREM 2. *In the space $\mathcal{B}_1^{\mathcal{F}}$ a set $\mathcal{L}$ of all function $f \in \mathcal{B}_1^{\mathcal{F}}$ measurable in the sense of Lebesgue is superporous at each point of this space.*

Proof. Let $f \in \mathcal{B}_1^{\mathcal{F}}$ and let $\Phi \subset \mathcal{B}_1^{\mathcal{F}}$ be a porous set at f. Let $R > 0$ and let $r_1' = \frac{\gamma(f,R,\Phi)}{2} > 0$. Then there exists $r_1 > r_1'$ and $h \in \mathcal{B}_1^{\mathcal{F}}$ such that

(12) $$B(h, r_1) \subset B(f, R) \setminus \Phi.$$

We shall show that

(13) $$\text{there exists } g \in \mathcal{B}_1^{\mathcal{F}} \text{ such that } B\left(g, \frac{r_1}{3}\right) \subset B(h, r_1) \setminus \mathcal{L}.$$

Let $\{x_n\}$ be a trajectory such that $\overline{\mathcal{B}_1^{\wedge}(h, \{x_n\})} \in \mathcal{F}$. There exists $x_0 \in C_h$ (Proposition 4). Let $\delta > 0$ be a number such that $[x_0 - \delta, x_0 + \delta] \subset [0,1]$ and

(14) $$h([x_0 - \delta, x_0 + \delta]) \subset \left(h(x_0) - \frac{r_1}{3}, h(x_0) + \frac{r_1}{3}\right).$$

Denote by C the Cantor-like set such that $C \subset [x_0 - \delta, x_0 + \delta]$ and $m_1(C) > 0$. Then there exists a non-measurable set $C^* \subset C$. We define a function $g : [0,1] \to \mathbb{R}$ in the following way:

$$g(x) = \begin{cases} h(x) & \text{if } x \in [0, x_0 - \delta) \cup (x_0 + \delta, 1]; \\ h(x_0) & \text{if } x \in [x_0 - \delta, x_0 + \delta] \setminus C; \\ h(x_0) + \frac{r_1}{3} & \text{if } x \in C^*; \\ h(x_0) - \frac{r_1}{3} & \text{if } x \in C \setminus C^*. \end{cases}$$

Note that

$$g \in \mathcal{B}_1^{\mathcal{F}}.$$

Indeed,

$\mathcal{B}_1(g, \{x_n\}) \supset (\mathcal{B}_1(h, \{x_n\}) \cap ([0, x_0 - \delta) \cup (x_0 + \delta, 1])) \cup ([x_0 - \delta, x_0 + \delta] \setminus C)$, so the set $\mathcal{B}_1(g, \{x_n\})$ has a dense interior. Hence $\overline{\mathcal{B}_1^{\wedge}(g, \{x_n\}))} \in \mathcal{F}$ and $g \in \mathcal{B}_1^{\mathcal{F}}$.

It is easy to observe that

(15) $$B\left(g, \frac{r_1}{3}\right) \subset B(h, r_1).$$

Now, we shall show that

(16) $$B\left(g, \frac{r_1}{3}\right) \cap \mathcal{L} = \emptyset.$$

Let $\psi \in B(g, \frac{r_1}{3})$. Then

$$\psi^{-1}((h(x_0), +\infty)) \cap C = C^*.$$

Indeed, if $x \in C^*$, $\psi(x) > g(x) - \frac{r_1}{3} = h(x_0)$ and $x \in \psi^{-1}((h(x_0), +\infty)) \cap C$. If $x \in \psi^{-1}((h(x_0), +\infty)) \cap C$, $g(x) > h(x_0) - \frac{r_1}{3}$ and $x \in C$. Hence, by the definition of the function g, $x \in C^*$.

The equality we have just proved shows that $\psi \notin \mathcal{L}$. The proof of (16) is finished.

From (15) and (16) it follows (13). From (13) and (12) we infer that

$$B\left(g, \frac{r_1}{3}\right) \subset B(f, R) \setminus (\Phi \cup \mathcal{L}).$$

Hence

$$\gamma(f, R, \Phi \cup \mathcal{L}) \geq \frac{r_1}{3} > \frac{r_1'}{3} = \frac{\gamma(f, R, \Phi)}{3},$$

so

$$p(\Phi \cup \mathcal{L}, f) \geq 2 \limsup_{R \to 0^+} \frac{\gamma(f, R, \Phi)}{3R} > 0.$$

Hence the set $\Phi \cup \mathcal{L}$ is superporous at f. ■

EXAMPLE. There exists a function $f \in \mathcal{B}_1^{\mathcal{F}}$ which satisfies Young condition and $f([0,1]) = \{0, 1\}$ (so f is not almost continuous).

Let $\hat{C}$ be a sum of closures of components of complement of Cantor set C "removed" at odd steps of the construction of C. We define a function $f : [0,1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in \hat{C}; \\ 1 & \text{if } x \in [0,1] \setminus \hat{C}. \end{cases}$$

Then $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$, where $\{x_n\}$ is an arbitrary trajectory contained in $[0,1] \setminus C$, because $\overline{\mathcal{B}_1^{\wedge}(f, \{x_n\})} \subset C \in \mathcal{F}$. The fact that f satisfies Young condition follows from the fact that both sets $\hat{C}$ and $[0,1] \setminus \hat{C}$ is bilateral dense in itself. ■

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Received November 12, 2001; revised version May 6, 2002.