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LEFT-OUTERMOST EXTENSIONS OF SOME VARIETIES

Abstract. Let $\tau : F \to \mathbb{N}$ be a type of algebras F is a nonempty set of fundamental operation symbols and $\mathbb{N}$ is the set of all positive integers. An identity $\varphi \approx \psi$ of type τ we call left-outermost if the left-outermost variables in φ and ψ are the same. For a variety V of type τ we denote by V_l the variety of type τ defined by all left-outermost identities from $\mathrm{Id}(V)$. V_l is called the left-outermost extension of V. In this paper we study minimal generics, subdirectly irreducible algebras and lattices of subvarieties in left-outermost extensions of some generalizations of the variety D of all distributive lattices.

0. Preliminaries

We shall consider algebras of type $\tau : F \to \mathbb{N}$ where F is a nonempty set of fundamental operation symbols and $\mathbb{N}$ is the set of all positive integers. It means that we do not admit nullary operation symbols. Let φ be a term of type τ. We denote by $\mathrm{Var}(\varphi)$ the set of all variables occurring in φ and we denote by $F(\varphi)$ the set of all fundamental operation symbols occurring in φ. Writing $\varphi(x_{i_1}, \ldots, x_{i_n})$ instead of φ we mean that $\mathrm{Var}(\varphi) = \{x_{i_1}, \ldots, x_{i_n}\}$. For a variety V of type τ we denote by $\mathrm{Id}(V)$ the set of all identities satisfied in every algebra from V.

An identity $\varphi \approx \psi$ is called *regular* (see [13]) if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$. We denote by $R(\tau)$ the set of all regular identities of type τ. An identity $\varphi \approx \psi$ of type τ is called *uniform* (see [16]) if it satisfies one of the following two conditions:

(0.1) $F(\varphi) = F(\psi) = F$,
(0.2) $F(\varphi) = F(\psi) \neq F$ and $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$.

We denote by $U(\tau)$ the set of all uniform identities of type τ. An identity $\varphi \approx \psi$ of type τ is called *biregular* (see [16]) if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ and $F(\varphi) = F(\psi)$. We denote by $B(\tau)$ the set of all biregular identities of type τ. Obviously each

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of the sets $R(\tau)$, $U(\tau)$ and $B(\tau)$ is an equational theory. For a variety V of type τ we denote by V_r, V_u, V_b the variety of type τ defined by all regular, all uniform, all biregular identities from $\mathrm{Id}(V)$, respectively. So $\mathrm{Id}(V_r) = \mathrm{Id}(V) \cap R(\tau)$, $\mathrm{Id}(V_u) = \mathrm{Id}(V) \cap U(\tau)$ and $\mathrm{Id}(V_b) = \mathrm{Id}(V) \cap B(\tau)$.

Let V_1 and V_2 be two varieties of type τ. We denote by $V_1 \vee V_2$ the join of V_1 and V_2 and we denote by $V_1 \times V_2$ the class of all algebras $\mathfrak{A}$ isomorphic to the direct product of some algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ where $\mathfrak{A}_1$ runs over V_1 and $\mathfrak{A}_2$ runs over V_2.

Two varieties V_0 and V_1 of type τ are called *independent* (see [9]) if there is a term $f(x_0, x_1)$ of type τ such that $x_1 \in \mathrm{Var}(f(x_0, x_1))$ and the identity $f(x_0, x_1) \approx x_k$ belongs to $\mathrm{Id}(V_k)$ for $k = 0, 1$.

The following statement was proved in [9, Theorem 1]:

(0.i) If V_0 and V_1 are independent, then $V_0 \vee V_1 = V_0 \times V_1$.

In [13] the construction $\mathcal{S}(\mathcal{A})$ was defined (quoted also in [8]) called the *sum* of a semilattice ordered system $\mathcal{A}$ of algebras $\mathfrak{A}_i$, $i \in I$.

It was shown in [13] that:

(0.ii) If $|I| > 1$ then $\mathrm{Id}(\mathcal{S}(\mathcal{A})) = (\bigcap_{i \in I} \mathrm{Id}(\mathfrak{A}_i)) \cap R(\tau)$.

It was shown in [14]:

(0.iii) If V is a variety of type τ and there is a term $x \circ y$ of type τ such that $\mathrm{Var}(x \circ y) = \{x, y\}$ and the identity $x \circ y \approx x$ belongs to $\mathrm{Id}(V)$, then V_r consists exactly of all possible sums of semilattice ordered systems of algebras from V.

For $f \in F$ we put $f' = F \setminus \{f\}$. In [22] an algebra $\mathfrak{A}_u^F$ was defined as follows: $\mathfrak{A}_u^F = (\{f'\}_{f \in F} \cup \{F\}; F^{\mathfrak{A}_u^F})$, where for $f \in F$ and $A_1, \ldots, A_{\tau(f)} \in (\{f'\}_{f \in F} \cup \{F\})$ we have $f^{\mathfrak{A}_u^F}(A_1, \ldots, A_{\tau(f)}) = A_1 \cup \ldots \cup A_{\tau(f)} \cup \{f\}$.

It was proved in [22]:

(0.iv) $\mathrm{Id}(\mathfrak{A}_u^F) = U(\tau)$.

An algebra $\mathfrak{A}$ from a variety V is called a *generic* of V (see [27]) if $\mathrm{HSP}(\mathfrak{A}) = V$, i.e. $\mathrm{Id}(\mathfrak{A}) = \mathrm{Id}(V)$. $\mathfrak{A}$ is called a *minimal generic* of V if it is a generic of a minimal possible cardinality. We denote by $g(V)$ the cardinality of a minimal generic of V.

We shall consider the following condition:

(0.v) For every $f \in F$ there exists a term $q_f(x)$ with $F(q_f(x)) = \{f\}$ such that the identity $q_f(x) \approx x$ belongs to $\mathrm{Id}(V)$.

If $\mathfrak{A} = (A; F^{\mathfrak{A}})$ is an algebra of type τ, then an element $a \in A$ is called an *idempotent* of $\mathfrak{A}$ if for every $f \in F$ we have $f^{\mathfrak{A}}(a, \ldots, a) = a$. An element

$a \in A$ is called an *absorbing element* of $\mathfrak{A}$ if for every $f \in F$, $a_1, \ldots, a_{\tau(f)} \in A$ we have: if $a \in \{a_1, \ldots, a_{\tau(f)}\}$, then $f^{\mathfrak{A}}(a_1, \ldots, a_{\tau(f)}) = a$.

Let $\mathfrak{A}_1 = (A_1; F^{\mathfrak{A}_1})$, $\mathfrak{A}_2 = (A_2; F^{\mathfrak{A}_2})$ be algebras of type τ, a_1 be an absorbing element of $\mathfrak{A}_1$, a_2 be an idempotent of $\mathfrak{A}_2$. Take the direct product $\mathfrak{A}_1 \times \mathfrak{A}_2 = (A_1 \times A_2; F^{\times})$. Let us consider a subdirect product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, namely the algebra $((A_1 \times \{a_2\}) \cup (\{a_1\} \times A_2); F^{\times}|_{(A_1 \times \{a_2\}) \cup (\{a_1\} \times A_2)})$. This algebra will be denoted by $\mathfrak{A}_1 \underset{\langle a_1, a_2 \rangle}{\times} \mathfrak{A}_2$ and will be called the $\langle a_1, a_2 \rangle$-*joining* of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (see [22]). Note that $|\mathfrak{A}_1 \underset{\langle a_1, a_2 \rangle}{\times} \mathfrak{A}_2| = |A_1| + |A_2| - 1$. Obviously, we have in $\mathfrak{A}_1 \underset{\langle a_1, a_2 \rangle}{\times} \mathfrak{A}_2$ subalgebras isomorphic with $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively. We have:

$$\mathrm{Id}(\mathfrak{A}_1 \underset{\langle a_1, a_2 \rangle}{\times} \mathfrak{A}_2) = \mathrm{Id}(\mathfrak{A}_1) \cap \mathrm{Id}(\mathfrak{A}_2).$$

Let us observe that the element F is an absorbing element of the algebra $\mathfrak{A}_u^F$.

It was proved in [22, Theorem 5.12]:

(0.vi) Let V be a variety of type τ satisfying (0.v), F be finite, $\mathfrak{A} = (A; F^{\mathfrak{A}})$ be a minimal generic of V with an idempotent a. Then $\mathfrak{A}_u^F \underset{\langle F, a \rangle}{\times} \mathfrak{A}$ is a minimal generic of V_u and $g(V_u) = |F| + g(V)$.

It was proved in [22, Corollary 5.9]:

(0.vii) If $V = V_r$, V satisfies (0.v), F is finite, $\mathfrak{A} = (A; F^{\mathfrak{A}})$ is a minimal generic of V having an idempotent i, then $\mathfrak{A}_u^F \underset{\langle F, i \rangle}{\times} \mathfrak{A}$ is a minimal generic of V_b and $g(V_b) = |F| + g(V)$.

For $|F| > 1$ we denote by $V^{c,2}$ the variety of type τ defined by all identities $\varphi \approx \psi$ from $\mathrm{Id}(V)$ satisfying one of the following two conditions:

(0.3) $F(\varphi) = F(\psi)$, $|F(\varphi)| = 1$;
(0.4) $|F(\varphi)|, |F(\psi)| \geq 2$.

In the sequel $\bigvee_{i \in I} V_i$ denotes the join of the family $\{V_i\}_{i \in I}$ of varieties. Further, $\bigotimes_{i \in I} V_i$ is the class of all algebras isomorphic to a subdirect product of the family $\{\mathfrak{A}_i\}_{i \in I}$ of algebras where $\mathfrak{A}_i$ runs over V_i for every $i \in I$.

For a variety V of type τ and for $f \in F$ we denote by $V(f)$ the variety of type τ defined by all identities $\varphi \approx \psi$ of type τ satisfying one of the following two conditions:

(0.5) $F(\varphi) \setminus \{f\} \neq \emptyset \neq F(\psi) \setminus \{f\}$;
(0.6) $(\varphi \approx \psi) \in \mathrm{Id}(V)$ and $F(\varphi) \cup F(\psi) \subseteq \{f\}$.

We denote by $V(0)$ the variety of 0-algebras of type τ, i.e. the variety defined by all identities $\varphi \approx \psi$ of type τ with $F(\varphi) \neq \emptyset \neq F(\psi)$.

It was proved in [20, Theorem 1.10]:

(0.viii) If $|F| > 1$ and the variety V satisfies (0.v), then

$$V \vee \bigvee_{f \in F} V(f) \vee V(0) = V^{c,2} = V \otimes \bigotimes_{f \in F} V(f) \otimes V(0).$$

Let F_1 and F_2 be two sets such that $F_1 \cup F_2 = F$ and $F_1 \cap F_2 = \emptyset$. We denote by V_{F_1} the variety of type $\tau_1 = \tau|_{F_1}$ defined by all identities of type τ_1 from $\mathrm{Id}(V)$. An identity $\varphi \approx \psi$ of type τ is called F_1-*regular* (see [28]) iff it is regular and of type τ_1. An identity $\varphi \approx \psi$ of type τ is called F_2-*symmetrical* (see [28]) iff $F(\varphi) \cap F_2 \neq \emptyset \neq F(\psi) \cap F_2$. For a variety V of type τ we denote by $V(F_1)$ the variety of type τ defined by all F_1-regular identities from $\mathrm{Id}(V)$ and all F_2-symmetrical identities of type τ.

Let $\mathfrak{A} = (A; F_1^{\mathfrak{A}})$ be an algebra of type τ_1. Let $c \notin A$ and put $A^* = A \cup \{c\}$. Then the algebra $\mathfrak{A}^* = (A^*; F^{\mathfrak{A}^*})$ of type τ will be called an F_2-*supalgebra* of the algebra $\mathfrak{A}$ if for every $f \in F$ and $a_1, \ldots, a_{\tau(f)} \in A^*$ we have

$$f^{\mathfrak{A}^*}(a_1, \ldots, a_{\tau(f)}) = \begin{cases} f^{\mathfrak{A}}(a_1, \ldots, a_{\tau(f)}) & \text{if } f \in F_1, \{a_1, \ldots, a_{\tau(f)}\} \subseteq A, \\ c & \text{otherwise .} \end{cases}$$

If $F_2 = \emptyset$, then an F_2-supalgebra coincides with a *supalgebra* in the sense of [12].

Consider the following condition:

(0.ix) There exists a term $\varphi(x, y)$ such that $F(\varphi(x, y)) \subseteq F_1$ and $(\varphi(x, y) \approx x) \in \mathrm{Id}(V)$.

The following two facts were proved in [26, Corollary 2.3, Corollary 2.11, respectively]:

(0.x) Let V be a variety of type τ, V be trivial or V satisfy (0.ix). Then $\mathfrak{A} = (A; F^{\mathfrak{A}})$ belongs to V_r and is subdirectly irreducible iff $\mathfrak{A}$ belongs to V and is subdirectly irreducible , or $\mathfrak{A}$ is a supalgebra of a subdirectly irreducible algebra from the variety V.

(0.xi) Let $F_1 \neq \emptyset \neq F_2$, let V be a variety of type τ and let V satisfy (0.ix). Then an algebra $\mathfrak{A} = (A; F^{\mathfrak{A}})$ belongs to $V(F_1)$ and is subdirectly irreducible iff $\mathfrak{A}$ is trivial or $\mathfrak{A}$ is an F_2-supalgebra of a subdirectly irreducible algebra from the variety V_{F_1}.

In (0.x) and (0.xi) a 1-element algebra is considered to be subdirectly irreducible. However in the sequel we do not do that.

We shall denote by T the trivial variety of type τ, i.e. defined by $x \approx y$.

An identity $\varphi \approx \psi$ of type τ is called *left-outermost* if the left-most variables in φ and ψ are the same. For example $x + x \approx x$ is left-outermost but $x \cdot y \approx y \cdot x$ is not. This notion was considered in [6] and [4] where the terminology "first-regular" was used. We denote by $L(\tau)$ the set of all left-outermost identities of type τ. For a variety V of type τ we denote by V_l the variety of type τ defined by all left-outermost identities from $\mathrm{Id}(V)$. The variety V_l will be called the *left-outermost extension* of V.

We have (see [4]):

(0.xii) $\mathrm{Id}(T_l) = L(\tau)$ and the variety T_l is nontrivial.

(0.xiii) For a variety V of type τ we have $V_l = T_l \vee V$, $\mathrm{Id}(V_l) = L(\tau) \cap \mathrm{Id}(V)$.

For a variety V of type τ we denote by $\mathcal{L}(V)$ the lattice of all subvarieties of V ordered by inclusion understand by formula: $V_1 \subseteq V_2$ iff $\mathrm{Id}(V_2) \subseteq \mathrm{Id}(V_1)$.

In this paper we want to find minimal generics and lattices of subvarieties of the varieties D_l, $(D_r)_l$, $(D_u)_l$ and $(D_b)_l$ where D is the variety of all distributive lattices.

We hope that the next results of this paper present a good example, how different constructions can cooperate with one another in explaining properties and structures of algebras.

1. Minimal generics

From now on we restrict our considerations to a type $\tau_2 : F_2 \longrightarrow \mathbb{N}$ where $F_2 = \{+, \cdot\}$ and $\tau_2(+) = \tau_2(\cdot) = 2$. Let D denote the variety of all distributive lattices of type τ_2. We have:

(1.i) $D_l = T_l \vee D = T_l \times D$.

REMARK 1.1. The property (1.i) was proved in [4] using (0.xiii) and using (0.i) for $f(x_0, x_1) = x_0 \cdot x_1 + x_1$.

It is known that the 2-element lattice $\mathbf{2} = (\{0,1\}; +, \cdot)$ with $0 < 1$ is a minimal generic of D.

Consider an algebra $\mathfrak{L} = (\{l_1, l_2\}; +, \cdot)$ where $a + b = a \cdot b = a$ for every $a, b \in \{l_1, l_2\}$. We have:

(1.ii) The algebra $\mathfrak{L}$ is a minimal generic of T_l.

In fact, the identities $x + y \approx x \cdot y \approx x$ form an equational base of T_l.

THEOREM 1.2. *The algebra* $\mathfrak{L} \times \mathbf{2}$ *is a minimal generic of* D_l *and* $g(D_l) = 4$.

Proof. We have $\mathrm{Id}(\mathfrak{L} \times \mathbf{2}) = L(\tau_2) \cap \mathrm{Id}(\mathbf{2}) = L(\tau_2) \cap \mathrm{Id}(D) = \mathrm{Id}(D_l)$ by (0.xiii). Consequently, $\mathfrak{L} \times \mathbf{2}$ is a generic of D_l. If $\mathfrak{B}$ is a generic of D_l then by (1.i) $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$ where $\mathfrak{B}_1 \in T_l$ and $\mathfrak{B}_2 \in D$. So it must be $|\mathfrak{B}_1|, |\mathfrak{B}_2| > 1$. Otherwise $\mathfrak{B} \in T_l$ or $\mathfrak{B} \in D$. Consequently $\mathfrak{B}$ is not a generic of D_l since both T_l and D are proper subvarieties of D_l. In fact, the identity $x + y \approx x \cdot y$ belongs to $L(\tau_2)$ and does not belong to $\mathrm{Id}(D)$. Similarly $x \cdot y + y \approx y$ belongs to $\mathrm{Id}(D)$ and does not belong to $L(\tau_2)$. ∎

Let us consider the following semilattice ordered system

$$\mathcal{A}_2 = ((\{1,2\};\leq), \{\mathfrak{A}_1, \mathfrak{A}_2\}, \{h_1^1, h_1^2, h_2^2\})$$

where $1 \leq 1 < 2 \leq 2$, $\mathfrak{A}_1 = \mathfrak{L}$, $\mathfrak{A}_2 = \mathbf{2}$, $h_1^2(l_1) = h_1^2(l_2) = 0$.

THEOREM 1.3. *The algebra $\mathcal{S}(\mathcal{A}_2)$ is a minimal generic of $(D_r)_l$ and $g((D_r)_l) = 4$.*

Proof. We have $\mathrm{Id}((D_r)_l) = \mathrm{Id}(D) \cap R(\tau_2) \cap L(\tau_2) = \mathrm{Id}(D) \cap L(\tau_2) \cap R(\tau_2) = \mathrm{Id}((D_l)_r)$. By (0.ii) we have $\mathrm{Id}(\mathcal{S}(\mathcal{A}_2)) = \mathrm{Id}(\mathfrak{A}_1) \cap \mathrm{Id}(\mathfrak{A}_2) \cap R(\tau_2) = \mathrm{Id}(\mathfrak{L}) \cap \mathrm{Id}(\mathbf{2}) \cap R(\tau_2) = \mathrm{Id}(\mathbf{2}) \cap \mathrm{Id}(\mathfrak{L}) \cap R(\tau_2) = \mathrm{Id}(D_l) \cap R(\tau_2) = \mathrm{Id}((D_l)_r)$. So $\mathcal{S}(\mathcal{A}_2)$ is a generic of $(D_r)_l$. Since D_l satisfies the identity

$$x \cdot y + x \approx x, \tag{1}$$

so by (0.iii) every generic $\mathfrak{B}$ of $(D_l)_r$ must be a sum of a semilattice ordered system of algebras $\mathfrak{B}_j$ for $j \in J$ for some set J of indices, where $\mathfrak{B}_j = \mathfrak{B}_j^1 \times \mathfrak{B}_j^2$, $\mathfrak{B}_j \in D_l$, $\mathfrak{B}_j^1 \in T_l$, $\mathfrak{B}_j^2 \in D$. Since a 1-element algebra satisfies all identities of type τ_2, D_l satisfies (1) so it must be at least two indices j_1 and j_2 with $j_1 \neq j_2$ such that $|\mathfrak{B}_{j_1}|, |\mathfrak{B}_{j_2}| > 1$. Consequently, $|\mathfrak{B}| \geq 4$. ∎

For $\tau = \tau_2$ the algebra $\mathfrak{A}_u^F$ from Section 0 is of the form $\mathfrak{A}_u^{F_2} = (U_2; +, \cdot)$ where $U_2 = \{\{\cdot\}, \{+\}, \{+,\cdot\}\}$ and for $A, B \in U_2$ we have $A + B = A \cup B \cup \{+\}$, $A \cdot B = A \cup B \cup \{\cdot\}$.

THEOREM 1.4. *The algebra $\mathfrak{A}_u^{F_2} \underset{\langle F_2, \langle l_1, 0\rangle\rangle}{\times} (\mathfrak{L} \times \mathbf{2})$ is a minimal generic of $(D_u)_l$ and $g((D_u)_l) = 6$.*

Proof. We have $(D_u)_l = (D_l)_u$. Now by Theorem 1.2 and by the fact that $\langle l_1, 0\rangle$ is an idempotent of $\mathfrak{L} \times \mathbf{2}$, the assumptions of (0.vi) are satisfied and we get the statement. ∎

THEOREM 1.5. *The algebra $\mathfrak{A}_u^{F_2} \underset{\langle F_2, l_1\rangle}{\times} \mathcal{S}(\mathcal{A}_2)$ is a minimal generic of $(D_b)_l$ and $g((D_b)_l) = 6$.*

Proof. We have $(D_b)_l = ((D_l)_r)_u = (D_l)_b$. Now the statement holds by Theorem 1.3 and (0.vii). ∎

2. Subdirectly irreducible algebras

In this section we find subdirectly irreducible algebras in left-outermost extensions of the varieties D, D_r, D_u and D_b.

Let us consider the following 10 algebras:

$\mathfrak{A}_1 = (\{a_1, b_1\}; +, \cdot)$ where $x + y = x \cdot y = x$ for $x, y \in \{a_1, b_1\}$.

$\mathfrak{A}_2 = (\{a_2, b_2\}; +, \cdot)$ where for $x, y \in \{a_2, b_2\}$ we have:

$$x + y = \begin{cases} b_2 \text{ if } b_2 \in \{x, y\} \\ a_2 \text{ otherwise} \end{cases}$$

$$x \cdot y = \begin{cases} a_2 \text{ if } a_2 \in \{x, y\} \\ b_2 \text{ otherwise} \end{cases}$$

$\mathfrak{A}_3 = (\{a_3, b_3\}; +, \cdot)$ where for $x, y \in \{a_3, b_3\}$ we have:

$$x + y = \begin{cases} a_3 \text{ if } x = y = a_3 \\ b_3 \text{ otherwise} \end{cases}$$

$$x \cdot y = b_3$$

$\mathfrak{A}_4 = (\{a_4, b_4, c_4\}; +, \cdot)$ where for $x, y \in \{a_4, b_4, c_4\}$ we have:

$$x + y = \begin{cases} x \;\text{ if } x, y \in \{a_4, c_4\} \\ b_4 \text{ otherwise} \end{cases}$$

$$x \cdot y = b_4$$

$\mathfrak{A}_5 = (\{a_5, b_5\}; +, \cdot)$ where for $x, y \in \{a_5, b_5\}$ we have:

$$x + y = b_5$$

$$x \cdot y = \begin{cases} a_5 \text{ if } x = y = a_5 \\ b_5 \text{ otherwise} \end{cases}$$

$\mathfrak{A}_6 = (\{a_6, b_6, c_6\}; +, \cdot)$ where for $x, y \in \{a_6, b_6, c_6\}$ we have:

$$x + y = b_6$$

$$x \cdot y = \begin{cases} x \;\text{ if } x, y \in \{a_6, c_6\} \\ b_6 \text{ otherwise} \end{cases}$$

$\mathfrak{A}_7 = (\{a_7, b_7\}; +, \cdot)$ where for $x, y \in \{a_7, b_7\}$ we have: $x + y = x \cdot y = b_7$

$\mathfrak{A}_8 = (\{a_8, b_8\}; +, \cdot)$ where for $x, y \in \{a_8, b_8\}$ we have:

$$x + y = x \cdot y = \begin{cases} x \;\text{ if } x = y \\ b_8 \text{ otherwise} \end{cases}$$

$\mathfrak{A}_9 = (\{a_9, c_9, b_9\}; +, \cdot)$ where for $x, y \in \{a_9, c_9, b_9\}$ we have:

$$x + y = \begin{cases} a_9 \text{ if } x = y = a_9 \\ c_9 \text{ if } c_9 \in \{x, y\} \text{ and } b_9 \notin \{x, y\} \\ b_9 \text{ otherwise} \end{cases}$$

$$x \cdot y = \begin{cases} c_9 \text{ if } x = y = c_9 \\ a_9 \text{ if } a_9 \in \{x, y\} \text{ and } b_9 \notin \{x, y\} \\ b_9 \text{ otherwise} \end{cases}$$

$\mathfrak{A}_{10} = (\{a_{10}, c_{10}, b_{10}\}; +, \cdot)$ where for $x, y \in \{a_{10}, c_{10}, b_{10}\}$ we have:

$$x + y = \begin{cases} x \quad \text{if } b_{10} \notin \{x, y\} \\ b_{10} \text{ otherwise} \end{cases}$$

$$x \cdot y = \begin{cases} x \quad \text{if } b_{10} \notin \{x, y\} \\ b_{10} \text{ otherwise} \end{cases}$$

Obviously $\mathfrak{A}_1$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{A}_2$ is isomorphic to **2**.

We have:

(2.i) The algebra $\mathfrak{A}_1$ is (up to isomorphism) the unique subdirectly irreducible algebra from T_l.

This follows from the fact that if $\mathfrak{A} = (A; +, \cdot)$ belongs to T_l, then every equivalence relation is a congruence of $\mathfrak{A}$. Consequently, if $|A| > 2$, then for every $a, b \in A$ with $a \neq b$ there is a nontrivial congruence separating a and b. Thus $\mathfrak{A}$ is subdirectly reducible.

It is known that:

(2.ii) The algebra $\mathfrak{A}_2$ is (up to isomorphism) the unique subdirectly irreducible algebra of D.

(2.iii) Algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are (up to isomorphism) the unique subdirectly irreducible algebras in D_l.

This follows at once from (1.i), (2.i) and (2.ii).

(2.iv) Algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_8$, $\mathfrak{A}_9$ and $\mathfrak{A}_{10}$ are (up to isomorphism) the unique subdirectly irreducible algebras in $(D_r)_l$.

In fact, we have $\mathrm{Id}((D_r)_l) = \mathrm{Id}((D_l)_r)$. But D_l satisfies (1) so (0.iii) is satisfied for D_l and we can use (0.x) and then (2.iii). Recall that in (0.x) a 1-element algebra is considered to be subdirectly irreducible.

(2.v)

We denote by $T(0)$ the variety satisfying $x + y \approx x \cdot y \approx u \cdot v$.

We shall consider the following condition for a variety V of type τ_2.

(2.vi) If $\varphi \approx \psi$ is an identity from $\mathrm{Id}(V)$ such that $|F(\varphi) \cup F(\psi)| \leq 1$, then $\varphi \approx \psi$ is regular.

We have:

(2.vii) If V is the variety satisfying (2.vi), then $V^{c,2} = V_u$.

According to the notation from Section 0 the variety $(T_l)_{\{+\}}$ is of type $\tau_2|_{\{+\}}$ and is defined by $x + y \approx x$. Analogously the variety $(T_l)_{\{\cdot\}}$ is of type $\tau_2|_{\{\cdot\}}$ and is defined by $x \cdot y \approx x$. Arguing similarly as in (2.i) we get that

(2.viii) If an algebra $\mathfrak{A}$ belongs to $(T_l)_{\{+\}}$, then $\mathfrak{A}$ is subdirectly irreducible iff $|\mathfrak{A}| = 2$. If an algebra $\mathfrak{A}$ belongs to $(T_l)_{\{\cdot\}}$, then $\mathfrak{A}$ is subdirectly irreducible iff $|\mathfrak{A}| = 2$.

(2.ix) An algebra $\mathfrak{A}$ belongs to $D_l(+)$ and is subdirectly irreducible iff $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_3$ or $\mathfrak{A}_4$. An algebra $\mathfrak{A}$ belongs to $D_l(\cdot)$ and is subdirectly irreducible iff $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_5$ or $\mathfrak{A}_6$.

We prove the first sentence, the proof of the second one is similar. $D_l(+)$ satisfies (2.vi), so it is defined by the identities:

$$x + x \approx x, (x + y) + z \approx x + (y + z), x + y + z \approx x + z + y, \tag{2}$$
$$x \cdot y \approx u \cdot v, x + (u \cdot v) \approx (u \cdot v) + x \approx u \cdot v.$$

By the same identities the variety $T_l(\{+\})$ is defined. Since T_l satisfies $x+y \approx x$, by (0.ix), (2.viii) and (0.xi) we get the statement.

(2.x) The statements (2.ix) are true if we substitute $D_l(+)$, $D_l(\cdot)$ by $(D_r)_l(+)$, $(D_r)_l(\cdot)$, respectively.

This follows from the fact $(D_r)_l(+) = D_l(+)$, $(D_r)_l(\cdot) = D_l(\cdot)$ and we argue as in (2.ix).

(2.xi) An algebra $\mathfrak{A}$ belongs to $(D_u)_l$ and is subdirectly irreducible iff $\mathfrak{A}$ is isomorphic to one of algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_7$.

In fact, we have $(D_u)_l = (D_l)_u$. The variety D_l satisfies (0.v) since we can put $q_+(x) = x + x$ and $q_\cdot(x) = x \cdot x$. Further, D_l satisfies (2.vi) so by (2.vii) using (0.viii) for $V = D_l$ we get $(D_u)_l = (D_l)_u = D_l \otimes D_l(+) \otimes D_l(\cdot) \otimes T(0)$. Now we get the statement by (2.iii), (2.v), (2.ix).

(2.xii) An algebra $\mathfrak{A}$ belongs to $(D_b)_l$ and is subdirectly irreducible iff $\mathfrak{A}$ is isomorphic to one of algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_{10}$.

In fact, we have $(D_b)_l = ((D_l)_r)_u = (D_l)_b$. The variety $(D_l)_r$ satisfies (0.v) and (2.vi). So putting in (0.viii) $(D_l)_r$ for V we get by (2.vii) $(D_b)_l = ((D_l)_r)_u = (D_l)_r \otimes (D_l)_r(+) \otimes (D_l)_r(\cdot) \otimes T(0)$. Now we use (2.iv), (2.v) and (2.x).

For $V \subseteq (D_b)_l$ let us put $\mathrm{Ir}(V) = V \cap \{\mathfrak{A}_1, \ldots, \mathfrak{A}_{10}\}$. Consequently, by (2.xii), we have:

$$\mathrm{Ir}((D_b)_l) = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_{10}\}.$$

Denote $M = \{\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7\}$.

LEMMA 2.1. $\mathrm{Ir}((T_u)_l) = M$.

Proof. It is easy to check that $\mathrm{Id}((T_u)_l) = \mathrm{Id}(\mathfrak{A}_1) \cap \mathrm{Id}(\mathfrak{A}_3) \cap \mathrm{Id}(\mathfrak{A}_4) \cap \mathrm{Id}(\mathfrak{A}_5) \cap \mathrm{Id}(\mathfrak{A}_6) \cap \mathrm{Id}(\mathfrak{A}_7) = \mathrm{Id}(\mathrm{HSP}(M))$ but $M \subseteq \mathrm{Ir}((D_b)_l)$ by (2.xii). So $\mathrm{HSP}(M) \subseteq (D_b)_l$ and consequently $\mathrm{Ir}(\mathrm{HSP}(M)) \subseteq \mathrm{Ir}((D_b)_l)$.

Each of algebras $\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7$ satisfies the identity

$$(x+y)\cdot(x+y) \approx (x+x)\cdot(x+x). \tag{3}$$

So $\mathrm{HSP}(M)$ satisfies (3). However none of the algebras $\mathfrak{A}_2, \mathfrak{A}_8, \mathfrak{A}_9, \mathfrak{A}_{10}$ satisfies (3), so none of them belongs to $\mathrm{HSP}(M)$. Consequently, $M = \mathrm{Ir}(\mathrm{HSP}(M)) = \mathrm{Ir}((T_u)_l)$. ∎

3. Lattices of subvarieties

Observe that no two algebras from $\mathrm{Ir}((D_b)_l)$ are isomorphic. However, some subdirectly irreducible algebras can generate others. So first we have to find some connections between algebras from $\mathrm{Ir}((D_b)_l)$.

LEMMA 3.1. $\mathfrak{A}_3 \in \mathrm{HSP}(\{\mathfrak{A}_4\})$.

Proof. Put $h(a_4) = h(c_4) = a_3$, $h(b_4) = b_3$. Then h is a homomorphism. ∎

LEMMA 3.2. $\mathfrak{A}_5 \in \mathrm{HSP}(\{\mathfrak{A}_6\})$.

Proof. The proof is similar to that of Lemma 3.1. ∎

LEMMA 3.3. $\mathfrak{A}_7 \in \mathrm{HSP}(\{\mathfrak{A}_3, \mathfrak{A}_5\})$.

Proof. Take the direct product of $\mathfrak{A}_3$ and $\mathfrak{A}_5$ and put: $h(\langle a_3, a_5\rangle) = a_7$ and $h(\langle x, y\rangle) = b_7$ otherwise. ∎

LEMMA 3.4. $\mathfrak{A}_4 \in \mathrm{HSP}(\{\mathfrak{A}_1, \mathfrak{A}_3\})$.

Proof. Take the direct product of $\mathfrak{A}_1$ and $\mathfrak{A}_3$ and put: $h(\langle a_1, a_3\rangle) = a_4$, $h(b_1, a_3) = c_4$ and $h(\langle x, y\rangle) = b_4$ otherwise. ∎

LEMMA 3.5. $\mathfrak{A}_6 \in \mathrm{HSP}(\{\mathfrak{A}_1, \mathfrak{A}_5\})$.

LEMMA 3.6. $\mathfrak{A}_9 \in \mathrm{HSP}(\{\mathfrak{A}_2, \mathfrak{A}_8\})$.

LEMMA 3.7. $\mathfrak{A}_{10} \in \mathrm{HSP}(\{\mathfrak{A}_1, \mathfrak{A}_8\})$.

LEMMA 3.8. $\mathfrak{A}_2, \mathfrak{A}_8 \in \mathrm{HSP}(\{\mathfrak{A}_9\})$.

Proof. Obviously $\mathfrak{A}_2$ is isomorphic to the subalgebra $(\{a_9, c_9\}; +, \cdot)$ of $\mathfrak{A}_9$. Further put $h(a_9) = h(c_9) = a_8$ and $h(b_9) = b_8$. ∎

LEMMA 3.9. $\mathfrak{A}_1, \mathfrak{A}_8 \in \mathrm{HSP}(\{\mathfrak{A}_{10}\})$.

The set $S \subseteq \mathrm{Ir}((D_b)_l)$ will be called $(D_b)_l$-closed if it satisfies the following conditions (c_1)–(c_9):

(c_1) If $\mathfrak{A}_4 \in S$, then $\mathfrak{A}_3 \in S$.
(c_2) If $\mathfrak{A}_6 \in S$, then $\mathfrak{A}_5 \in S$.
(c_3) If $\{\mathfrak{A}_3, \mathfrak{A}_5\} \subseteq S$, then $\mathfrak{A}_7 \in S$.

(c_4) If $\{\mathfrak{A}_1, \mathfrak{A}_3\} \subseteq S$, then $\mathfrak{A}_4 \in S$.
(c_5) If $\{\mathfrak{A}_1, \mathfrak{A}_5\} \subseteq S$, then $\mathfrak{A}_6 \in S$.
(c_6) If $\{\mathfrak{A}_2, \mathfrak{A}_8\} \subseteq S$, then $\mathfrak{A}_9 \in S$.
(c_7) If $\{\mathfrak{A}_1, \mathfrak{A}_8\} \subseteq S$, then $\mathfrak{A}_{10} \in S$.
(c_8) If $\mathfrak{A}_9 \in S$, then $\{\mathfrak{A}_2, \mathfrak{A}_8\} \subseteq S$.
(c_9) If $\mathfrak{A}_{10} \in S$, then $\{\mathfrak{A}_1, \mathfrak{A}_8\} \subseteq S$.

LEMMA 3.10. *Let S be a $(D_b)_l$-closed subset of* $\mathrm{Ir}((D_b)_l)$. *If $\mathfrak{A}_k \notin S$, then $\mathfrak{A}_k \notin \mathrm{HSP}(S)$, $k = 1, \ldots, 10$.*

Proof. Let $k = 1$. Then by (c_9) we get $\mathfrak{A}_{10} \notin S$ and $S \subseteq \{\mathfrak{A}_2, \ldots, \mathfrak{A}_9\}$. Take an identity

$$(x \cdot y) + (x \cdot y) \approx (y \cdot x) + (y \cdot x). \tag{4}$$

So (4) is satisfied in every algebra $\mathfrak{A}_2, \ldots, \mathfrak{A}_9$. So it is satisfied in $\mathrm{HSP}(S)$. However (4) is not satisfied in $\mathfrak{A}_1$ and $\mathfrak{A}_1 \notin \mathrm{HSP}(S)$.

Let $k = 2$. Then by (c_8) $\mathfrak{A}_9 \notin S$. We take the identity

$$(x + y) \cdot (x + y) \approx (x \cdot y) + (x \cdot y). \tag{5}$$

Then we argue as in case $k = 1$.

Let $k = 3$. Then $\mathfrak{A}_4 \notin S$ by (c_1). We take the identity

$$(x + x) \cdot (x + x) \approx x + x. \tag{6}$$

Let $k = 4$. Then by (c_4) it can not be $\{\mathfrak{A}_1, \mathfrak{A}_3\} \subseteq S$. If $\mathfrak{A}_3 \notin S$ we take (6). If $\mathfrak{A}_1 \notin S$, then also $\mathfrak{A}_{10} \notin S$ by (c_9). We take the identity

$$x + y \approx y + x.$$

Let $k = 5$. Then $\mathfrak{A}_6 \notin S$ by (c_2). We take

$$(x \cdot x) + (x \cdot x) \approx x \cdot x. \tag{7}$$

Let $k = 6$. Then by (c_4) it can not be $\{\mathfrak{A}_1, \mathfrak{A}_5\} \subseteq S$. If $\mathfrak{A}_5 \notin S$ we take (7). If $\mathfrak{A}_1 \notin S$, then by (c_9) we have $\mathfrak{A}_{10} \notin S$. We take the identity

$$x \cdot y \approx y \cdot x.$$

Let $k = 7$. By (c_1), (c_2) and (c_3) it can not be $S \cap \{\mathfrak{A}_3, \mathfrak{A}_4\} \neq \emptyset \neq S \cap \{\mathfrak{A}_5, \mathfrak{A}_6\}$. If $S \cap \{\mathfrak{A}_3, \mathfrak{A}_4\} = \emptyset$ we take $x \cdot x \approx x$. If $S \cap \{\mathfrak{A}_5, \mathfrak{A}_6\} = \emptyset$ we take $x + x \approx x$.

Let $k = 8$. Then $\mathfrak{A}_9 \notin S$ and $\mathfrak{A}_{10} \notin S$ by (c_8) and (c_9). We take the identity

$$(x \cdot y) + x \approx (x \cdot z) + x. \tag{8}$$

Let $k = 9$. By (c_8) it can not be $\{\mathfrak{A}_2, \mathfrak{A}_8\} \subseteq S$. If $\mathfrak{A}_8 \notin S$, then $\mathfrak{A}_{10} \notin S$ by (c_9). We take (8). If $\mathfrak{A}_2 \notin S$ we take (5).

Let $k = 10$. It can not be $\{\mathfrak{A}_1, \mathfrak{A}_8\} \subseteq S$ by (c_9). If $\mathfrak{A}_8 \notin S$, then $\mathfrak{A}_9 \notin S$ by (c_8). We take (8). If $\mathfrak{A}_1 \notin S$ we take (4). ■

LEMMA 3.11. *If a variety V belongs to $\mathcal{L}((D_b)_l)$ and $\mathfrak{A} \in V$, then $\mathfrak{A}$ is isomorphic to a subdirect product of a family of subdirectly irreducible algebras belonging to* $\mathrm{Ir}(V)$.

Proof. By Birkhoff's Subdirect Representation Theorem (see [2]), if $\mathfrak{A} \in V$, then it is isomorphic to an algebra $\mathfrak{A}'$ being a subdirect product of a family $\{\mathfrak{A}_j\}_{j\in J}$ of subdirectly irreducible algebras from V. By (2.xii) each $\mathfrak{A}_j$ is isomorphic to an algebra $\mathfrak{A}_j^*$ from $\mathrm{Ir}((D_b)_l)$. Thus $\mathfrak{A}_j^*$ belongs to V and belongs to $\mathrm{Ir}((D_b)_l)$, hence $\mathfrak{A}_j^*$ belongs to $\mathrm{Ir}(V)$. Consequently, $\mathfrak{A}'$ is isomorphic to an algebra $\mathfrak{A}^*$ being a subdirect product of the family $\{\mathfrak{A}_j^*\}_{j\in J}$ and $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^*$. ■

We denote by $\mathcal{C}((D_b)_l)$ the set of all $(D_b)_l$-closed sets.

LEMMA 3.12. (i) *For every variety $V \in \mathcal{L}((D_b)_l)$ the set $\mathrm{Ir}(V)$ is $(D_b)_l$-closed.*

(ii) *For every variety $V \in \mathcal{L}((D_b)_l)$ we have $V = \mathrm{HSP}(\mathrm{Ir}(V))$.*

(iii) *If $S \in \mathcal{C}((D_b)_l)$, then $S = \mathrm{Ir}(\mathrm{HSP}(S))$.*

(iv) *If $V_1, V_2 \in \mathcal{L}((D_b)_l)$, then $V_1 \subseteq V_2$ iff $\mathrm{Ir}(V_1) \subseteq \mathrm{Ir}(V_2)$.*

Proof. (i) follows from Lemmas 3.1–3.9.

(ii) Since $\mathrm{Ir}(V) \subseteq V$, $\mathrm{HSP}(\mathrm{Ir}(V)) \subseteq V$. The converse inclusion follows at once from Lemma 3.11.

(iii) If an algebra $\mathfrak{A}$ belongs to S, then $\mathfrak{A} \in \mathrm{HSP}(S)$. But $\mathfrak{A} \in \mathrm{Ir}((D_b)_l)$ since $S \subseteq \mathrm{Ir}((D_b)_l)$, so $\mathfrak{A} \in \mathrm{Ir}(\mathrm{HSP}(S))$. If $\mathfrak{A} \notin S$, then $\mathfrak{A} \notin \mathrm{HSP}(S)$ by Lemma 3.10, hence $\mathfrak{A} \notin \mathrm{Ir}(\mathrm{HSP}(S))$.

(iv) If $V_1 \subseteq V_2$, then $\mathrm{Ir}(V_1) \subseteq \mathrm{Ir}(V_2)$ by the definition of $\mathrm{Ir}(V)$. The converse implication follows at once from Lemma 3.11. ■

THEOREM 3.13. *The set $S \subseteq \mathrm{Ir}((D_b)_l)$ is equal to $\mathrm{Ir}(V)$ for some variety $V \in \mathcal{L}((D_b)_l)$ iff S is $(D_b)_l$-closed.*

Proof. It follows from Lemma 3.12 (i) and (iii). ■

THEOREM 3.14. *The lattice $\mathcal{L}((D_b)_l)$ as a poset is isomorphic to the poset $(\mathcal{C}((D_b)_l); \subseteq)$, so the lattice $\mathcal{L}((D_b)_l)$ is isomorphic to the lattice $(\mathcal{C}((D_b)_l); \subseteq)$.*

Proof. For $V \in \mathcal{L}((D_b)_l)$ put $\varphi(V) = \mathrm{Ir}(V)$. Then φ is well defined by the definition of $\mathrm{Ir}(V)$. φ maps $\mathcal{L}((D_b)_l)$ into $\mathcal{C}((D_b)_l)$ by Lemma 3.12 (i). If $\mathrm{Ir}(V_1) = \mathrm{Ir}(V_2)$, then by Lemma 3.12 (ii) $V_1 = \mathrm{HSP}(\mathrm{Ir}(V_1)) = \mathrm{HSP}(\mathrm{Ir}(V_2)) = V_2$. Thus φ is 1–1. By Lemma 3.12 (iii), φ is onto. If $V_1 \subseteq V_2$, then $\mathrm{Ir}(V_1) \subseteq \mathrm{Ir}(V_2)$ by the definition of $\mathrm{Ir}(V)$. The converse implication follows at once from Lemma 3.11. ■

For $V \in \mathcal{L}((D_b)_l)$ let us put $\mathcal{C}(V) = \{S \subseteq V : S \in \mathcal{C}((D_b)_l)\}$.

COROLLARY 3.15. *If $V \in \mathcal{L}((D_b)_l)$, then the lattice $\mathcal{L}(V)$ is isomorphic to the lattice $(\mathcal{C}(V); \subseteq)$.*

Let $\mathbf{A}$ be a family of sets and B be a set with $B \notin \mathbf{A}$. Denote $B \cup_\times \mathbf{A} = \{B \cup A : A \in \mathbf{A}\}$, $\mathbf{A} \cup_\times B = \{A \cup B : A \in \mathbf{A}\}$.

For a set $A \subseteq \mathrm{Ir}((D_b)_l)$ we denote $\mathrm{Ind}(A) = \{i : \mathfrak{A}_i \in A\}$. We have:

(3.i) To every $A \subseteq \mathrm{Ir}((D_b)_l)$ the set $\mathrm{Ind}(A)$ is assigned into 1–1 way and for every $A, B \subseteq \mathrm{Ir}((D_b)_l)$ we have $\mathrm{Ind}(A) \subseteq \mathrm{Ind}(B)$ iff $A \subseteq B$.

Put $P = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

COROLLARY 3.16. *The lattice $\mathcal{L}(D_l)$ is isomorphic to the lattice $(P; \subseteq)$.*

Proof. This follows at once from (2.iii), Corollary 3.15 and (3.i). ∎

COROLLARY 3.17. *The lattice $\mathcal{L}((D_r)_l)$ is isomorphic to the lattice $(P \cup \{\{8\}, \{1,8,10\}, \{2,8,9\}, \{1,2,8,9,10\}\}; \subseteq)$.*

Proof. This follows at once from (2.iv), Corollary 3.15 and (3.i). ∎

COROLLARY 3.18. *The lattice $\mathcal{L}((D_r)_l)$ is isomorphic to the lattice $\mathcal{L}(D_l) \times \mathbf{2}$.*

Proof. This follows at once from Corollary 3.17. ∎

REMARK 3.19. Corollary 3.18 follows also from the main result of [5] since D_l satisfies (1).

Let M be the set from Lemma 2.1. Put $\mathbf{M} = \{\mathrm{Ind}(S) : S \subseteq M,\ S$ is $(D_b)_l$-closed$\}$. One can check that $\mathbf{M} = \mathbf{M}_0 \cup \mathbf{M}_1$ where

$$\mathbf{M}_0 = \{\emptyset, \{3\}, \{3,4\}, \{5\}, \{5,6\}, \{7\}, \{3,7\}, \{3,4,7\}, \{5,7\}, \{5,6,7\}, \{3,5,7\}, \{3,5,6,7\}, \{3,4,5,7\}, \{3,4,5,6,7\}\},$$

$$\mathbf{M}_1 = \{\{1\}, \{1,3,4\}, \{1,5,6\}, \{1,7\}, \{1,3,4,7\}, \{1,5,6,7\}, \{1,3,4,5,6,7\}\}.$$

COROLLARY 3.20. *The lattice $\mathcal{L}((D_u)_l)$ is isomorphic to the lattice $(\mathbf{M} \cup (\{2\} \cup_\times \mathbf{M}); \subseteq)$.*

Proof. It follows from (2.xi), Corollary 3.15, (3.i) and (c_1)–(c_5). ∎

COROLLARY 3.21. *The lattice $\mathcal{L}((D_u)_l)$ is isomorphic to the lattice $\mathbf{2} \times \mathcal{L}((T_u)_l)$.*

Proof. We have $\mathcal{L}((T_u)_l)$ is isomorphic to $(\mathbf{M}; \subseteq)$ by Lemma 2.1, Corollary 3.15 and (3.i). Now we use Corollary 3.20. ∎

COROLLARY 3.22. *The lattice $\mathcal{L}((D_b)_l)$ is isomorphic to the lattice* $(\mathbf{M} \cup (\{2\} \cup_\times \mathbf{M}) \cup (\mathbf{M}_0 \cup_\times \{8\}) \cup (\mathbf{M}_1 \cup_\times \{8,10\}) \cup (\mathbf{M}_0 \cup_\times \{8,2,9\}) \cup (\mathbf{M}_1 \cup_\times \{8,2,9,10\}); \subseteq)$.

Proof. This follows from (2.xii), Corollary 3.15 and (3.i). ∎

COROLLARY 3.23. *The lattice* $\mathcal{L}((D_b)_l)$ *is isomorphic to the lattice* $\mathcal{L}((D_u)_l) \times \mathbf{2}$.

Proof. This follows from Corollary 3.22. ∎

Note that $|\mathcal{L}((D_u)_l)| = 42$ and $|\mathcal{L}((D_b)_l)| = 84$.

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