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THE NORMALIZATION OF RECTANGULAR ALGEBRAS

Abstract. Our aim is to investigate the normalization (nilpotent shift) $(\underline{RA}_\tau)^*$ of the variety $\underline{RA}_\tau$ of rectangular algebras of a finite type τ. It can be regarded as a common generalization of various algebras studied by different authors, namely of rectangular algebras, [P&R 92, 93], rectangular bands, [Gou], medial grupoids, [J&K], or generalized diagonal algebras, [P 64, 66a, b].

We will determine all subdirectly irreducible algebras in the normalization of the variety of rectangular algebras, give a normal form for terms in the normalization and an algorithm for finding a subdirect decomposition of algberas belonging to the normalization.

1. Introduction, historical remarks

Our aim is to present here a variety of algebras which can be regarded as a common generalization of some classes of algebras studied by various authors, namely rectangular bands, [Gou], rectangular algebras, [P&R 92, 93], diagonal algebras and generalized diagonal algebras, [P 64, 66a, b], or medial grupoids, [J&K].

Fix a type $\tau : F \to \mathbb{N}$ with a family F of operation symbols of finite arities $\tau(f) \in \mathbb{N}$. Let $\mathrm{Alg}(\tau)$ be the class of all algebras of type τ (shortly, τ-algebras). The basic operation in an algebra $\mathcal{A} \in \mathrm{Alg}(\tau)$ corresponding to the operation symbol $f \in F$ will be denoted by $f^{\mathcal{A}}$, and we will keep the notation $\mathcal{A} = (A; F^{\mathcal{A}})$.

Given a set Σ of identities of type τ denote by $\mathrm{Mod}\,\Sigma$ the variety of all models of Σ, that is of all τ-algebras in which all identities from Σ are satisfied. In the algebras we are interested in, the following identities play an important role (here f, $g \in F$ with arities $\tau(f) = n$, $\tau(g) = m$):

(I_f) $\qquad f(x, \dots, x) \approx x \quad$ (*idempotency, idempotent law*)

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(D_f) $$f(f(x_{11},\dots,x_{1n}),f(x_{21},\dots,x_{2n}),\dots,f(x_{n1},\dots,x_{nn})) \approx f(x_{11},\dots,x_{nn}) \quad \textit{(diagonal law)}$$

(A_f^i) $$f(x_1,\dots,x_{i-1},f(y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n),x_{i+1},\dots,x_n) \approx f(x_1,\dots,x_n), \quad 1 \le i \le n \quad \textit{(i-th absorption, or cancellative law)}$$

$(E_{f,g})$ $$f(g(x_{11},\dots,x_{1m}),\dots,g(x_{n1},\dots,x_{nm})) \approx g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))$$

(*entropic law, or commutability, or generalized metabelian law*).

An algebra $\mathcal{A}$ is called *idempotent* if all basic operations are idempotent and is called *entropic* if $(E_{f,g})$ is satified in $\mathcal{A}$ for all couples of f, g.

REMARKS. Idempotency means that each singleton subset $\{a\} \subset A$ is actually a subalgebra of $\mathcal{A}$. An algebra $\mathcal{A} = (A; F^{\mathcal{A}})$ is entropic if and only if each basic operation $f^{\mathcal{A}}$ determines a homomorphism of algebras (denoted by the same symbol) $f^{\mathcal{A}} : (A^{\tau(f)}; F^{\mathcal{A}}) \to (A; F^{\mathcal{A}})$. The identity $(E_{f,g})$ has been given various names. If $(E_{f,g})$ holds in $\mathcal{A}$ then $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ commute which explains "commutability" . If $f = g$ are equal binary operations denoted by an infix $\circ$ then $(E_{\circ,\circ})$ becomes $(x \circ y) \circ (z \circ u) \approx (x \circ z) \circ (y \circ u)$, [Sm 99], p. 15. The notion "entropic" , in use in this context for a long time, refers to the "inner turning" of y and z, [Sm 99], p. 15; some authors prefer to use "medial" , e.g. [J&K] for grupoids.

1.1. EXAMPLE. For a single unary operation, diagonality means $f(f(x)) \approx f(x)$. On any set, constant operations and the identity belong to the family of unary operations f satisfying (D_f).

At the end of '50, E.S. Liapin studied the variety of semigroups given by

$$\underline{L} = \mathrm{Mod}((I_\circ),(AS): \;\; x \circ (y \circ z) \approx (x \circ y) \circ z, \quad (**): \;\; x \circ y \circ z \approx x \circ z)$$

and published the results in [L 60]. The variety $\underline{L}$ in fact coincides with the variety $\underline{RB}$ of rectangular bands, [Cli-P 64], [Gou 82], [P& R 93], which is usually introduced as

$$\underline{RB} = \mathrm{Mod}((AS),(*)) \quad \text{where} \quad (*): \;\; x \circ y \circ x \approx x.$$

The variety $\underline{RB}$ is generated by algebras with one binary projection as a basic operation. Every rectangular band is a direct product of two projection algebras.

In the '60, J. Płonka introduced algebras $(A; f)$ with one n-ary basic operation satisfying the identities (I_f), (D_f), as a generalization of semigroups considered by Liapin. He started to call them *n-dimensional* (=n-ary) *diagonal algebras*, [P 64], [P 66a], in a more contemporary notation,

$$\underline{DA}_{(n)} = \mathrm{Mod}((I_f),(D_f)).$$

He proved a representation theorem and also theorems on independence in the sense of Marczewski. In [P 66b], Płonka studied the variety of generalized diagonal algebras

$$\underline{GDA}_{(n)} = \mathrm{Mod}((D_f)).$$

In '90, J. Šlapal investigated algebras with a single basic operation of an arbitrary (even infinite) type satisfying diagonality only, or in combination with other properties (idempotency, mediality), [Sl 92], [Sl 94]. He also presented multiplication tables of 3-element binary generalized diagonal algebras, 4-element binary generalized diagonal algebras and one example of a generalized diagonal algebra of order 12. His further interest was motivated by the categorial view-point. He investigated especially powers and exponentiation, and not the class of diagonal algebras itself.

Being inspired by Šlapal's examples, J. Klouda constructed a computer program which was able to produce multiplication tables of binary generalized diagonal algebras up to order 10. The theoretical background used in the program was published later in [K&V]. A more up-to-date computer program which yields all binary and ternary generalized diagonal algebras up to order 20 was constructed by J. Tichý. The way how to produce examples was clear but rather mechanical.

Why on earth the algebras look like they look?

The explanation comes from considerations published in a couple of excellently written papers by R. Pöschel and M. Reichel.

2. Projection algebras and rectangular algebras

Let τ be a finite type with $F = (f_1, \ldots, f_k)$ where $\tau(f_i) = n_i \geq 1$ for $i \in I = \{1, \ldots, k\}$. Let us denote here by $e^{(n)}_{j,A} : A^n \to A$ the j-th *projection* $(a_1, \ldots, a_n) \mapsto a_j$ on a non-empty set A, $j \in \{1, \ldots, n\}$, $n \geq 1$.

Under a *projection τ-algebra* we will understand a τ-algebra for which each basic operation $f_i^{\mathcal{A}}$ is a projections $e^{(n_i)}_{q_i,A}$ onto the carrier set, $1 \leq q_i \leq n_i$, $i = 1, \ldots, k$.

2.1. LEMMA. ([P&R 93], 2.6. Lemma, p. 186) *Every projection τ-algebra $\mathcal{A} = (A; (e^{(n_i)}_{q_i,A})_{i=1,\ldots,k})$ is isomorphic to a subalgebra of the direct power $\mathcal{B}^r$ of the two-element projection algebra $\mathcal{B} = (\{0,1\}; (e^{(n_i)}_{q_i,\{0,1\}})_{i=1,\ldots,k})$ where r is the least natural number such that $|A| \leq 2^r$. An isomorphism is induced by any injection $h : A \to \{0,1\}^r$.*

We will use the notation $\mathcal{A}_{\underline{q}} = (A; (e^{(n_i)}_{q_i,A})_{i=1,\ldots k})$ where $\underline{q} = (q_1, \ldots q_k)$ and $1 \leq q_i \leq n_i$. Let us denote

(2.1) $$n_0 = n_k \cdot \ldots \cdot n_1 = \tau(f_k) \cdot \ldots \cdot \tau(f_1).$$

2.2. LEMMA. *For a given finite type τ with $\tau(f_i) = n_i$, $i = 1, \ldots, k$, there are exactly n_0 projection algebras.*

Proof. For the i-th projection, there are n_i choices $e^{(n_i)}_{1,A}, \ldots, e^{(n_i)}_{n_i,A}$. ■

The family of all projection τ-algebras with the same carrier set A can be ordered linearly as follows. On the set of all possible k-tuples $(q_1, \ldots q_k)$ with $1 \leq q_i \leq n_i$, let us take the lexicographical order from right to left, i.e. $(q_1, \ldots, q_k) < (q'_1, \ldots, q'_k)$ if and only if there is an index i such that $q_i < q'_i$ and for $j > i$, $q_j = q'_j$. We obtain a sequence $\underline{q}^1, \ldots, \underline{q}^{n_0}$. The sequence $\mathcal{A}_{\underline{q}^1}, \ldots, \mathcal{A}_{\underline{q}^{n_0}}$ defines a linear order on the set of all projection τ-algebras on the given carrier set A. The following formula will be useful in the sequel. If we define a function $\mu(q)$, ([P&R 93] p. 190, 187), by

$$(2.2)\quad \mu(q) = q_1 + (q_2 - 1)n_1 + (q_3 - 1)n_2 \cdot n_1 + \ldots + (q_k - 1)n_{k-1} \cdot \ldots \cdot n_1$$

then $\mu(\underline{q}^j) = j$ as can be checked.

Let P_τ denote the class of all projection algebras of a given finite type τ. In [P&R 92, 93], the variety $\underline{RA}_\tau = Var(P_\tau)$ generated by the class P_τ was investigated. Its elements have been called *rectangular τ-algebras.* Among others, the authors proved the following. An algebra is rectangular if and only if it is isomorphic to a direct product of projection algebras. The variety of rectangular algebras is finitely based and can be given as

$$\underline{RA}_\tau = \mathrm{Mod}(\{(I_f), (D_f), (E_f); f \in F\})$$

with the generating system of identities $\Sigma_\tau = \{(I_f), (D_f), (E_{f,h}); f, h \in F\}$.

The authors also proved decomposition theorems for rectangular algebras, derived normal forms for terms in this variety, described an algoritm for finding normal forms and used it for a representation of a rectangular algebra as a subdirect product, proved solidity and investigated generating algebras.

In many proofs, it might be reasonable to substitute the diagonality condition by the family of absorption laws ([P 66 b], p. 19, 3°). Let us give an alternative proof.

2.3. LEMMA. *The identity (D_f) is equivalent to the system of identities (A^i_f), $1 \leq i \leq n$.*

Proof. Let $f \in F$. If the identities (A^i_f) hold for all $i \in \{1, \ldots, n\}$ then

$$f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})) \underset{(A^1_f)}{\approx} f(x_{11}, f(x_{21}, \ldots, x_{2n}), \ldots,$$

$$f(x_{n1}, \ldots, x_{nn})) \underset{(A^2_f)}{\approx} \cdots \underset{(A^n_f)}{\approx} f(x_{11}, \ldots, x_{nn}).$$

Vice versa, by diagonality (used either "outside", or "inside" the term)

$f(x_1,\dots,x_{i-1},f(y_1,\dots,x_i,\dots,y_n),x_{i+1},\dots,x_n)\approx$
$f(f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n),f(f(y_1,\dots,x_i,\dots,y_n),\dots,$
$f(y_1,\dots,x_i,\dots,y_n)),f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n))$
$\approx f(f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n),$
$f(y_1,\dots,x_i,\dots,y_n),f(x_1,\dots,x_n),\dots,f(x_1,\dots,x_n))\approx f(x_1,\dots,x_n)$
for any $i\in\{1,\dots,n\}$. ∎

If $\tau=(n)$ we obtain Płonka's diagonal algebras, $\underline{RA}_{(n)}=\underline{DA}_{(n)}$. The special case with $\tau=(2)$ is known as the variety of rectangular bands, $\underline{RA}_{(2)}=\underline{RB}$.

2.4. LEMMA. *The following varieties coincide*: $\underline{RB}=\underline{L}=\underline{DA}_{(2)}=\underline{RA}_{(2)}$.

Proof. The proof is based on the fact that in the class of all grupoids, the following holds:

$$\mathrm{Mod}((AS),(*))\subset\mathrm{Mod}((I_\circ),(AS),(**))\subset\mathrm{Mod}((I_\circ),(D_\circ))\subset$$
$$\subset\mathrm{Mod}((I_\circ),(A^1_\circ),(A^2_\circ),(E_\circ))\subset\mathrm{Mod}((AS),(*)).$$

In fact, let $\mathcal{A}=(A;\circ)$ be a semigroup satisfying $(*)$. Then $(**)$ also holds since $x\circ z\underset{(*)}{\approx}(x\circ z)\circ(x\circ y\circ z)\circ(x\circ z)\underset{(AS)}{\approx}(x\circ z\circ x)\circ y\circ(z\circ x\circ z)\underset{(*)}{\approx}x\circ y\circ z$, and $\mathcal{A}$ is idempotent as well: $x\circ x\underset{(*)}{\approx}(x\circ x)\circ x\circ(x\circ x)\underset{(AS)}{\approx}x\circ(x\circ x\circ x)\circ x\underset{(*)}{\approx}x\circ x\circ x\underset{(*)}{\approx}x$. Further, $(D_\circ)$ holds in the variety of idempotent semigroups satisfying $(**)$ since $x\circ z\underset{(**)}{\approx}x\circ(y\circ u)\circ z\underset{(AS)}{\approx}(x\circ y)\circ(u\circ z)$. $(D_\circ)$ is equivalent with a couple of identities $(A^1_\circ)$, $(A^2_\circ)$ by 2.3.Lemma. Let us show that $(E_\circ)$ follows by $(D_\circ)$. $(x\circ y)\circ(u\circ z)\underset{(D_\circ)}{\approx}x\circ z\underset{(D_\circ)}{\approx}(x\circ u)\circ(y\circ z)$. Now let $\mathcal{A}$ be a binary rectangular algebra. Then $\mathcal{A}$ is a semigroup since $(A^1_\circ)$ together with $(A^2_\circ)$ imply associativity. By (AS) and $(I_\circ)$, the identity $(*)$ also holds in $\mathcal{A}$. Hence the corresponding varieties coincide. ∎

Now we may ask which (reasonable) varieties include $\underline{RA}_\tau$ *as a subvariety, and which generalizations of rectangular algebras are at the same time generalizations of* $\underline{GDA}_{(n)}$.

E.g. we can drop diagonality. Elements of the variety $\underline{M}_\tau=\mathrm{Mod}(\{(I_f),(E_f);f\in F\})$ are known as *modes*. The theory of modes has been developed e. g. by J.D.H. Smith, [Sm 99], and by A. Romanowska who has given an encyclopedic survey in [R 92], [R&S]. An interesting characterization of modes is due to K. Kearnes:

A τ*-algebra* $\mathcal{A}$ *is a mode if and only if each polynomial function of* $\mathcal{A}$ *is a homomorphism*, [Sm 99], p. 16, [Ke].

From this view-point, rectangular algebras are exactly diagonal modes. Modes are not oblighed to satisfy diagonality (D_f) in general as can be easily seen.

2.5. EXAMPLE. A binary algebra $(\{0,1\};f)$ given by $f(0,0)=0$, $f(x,y)=1$ otherwise is a mode since polynomials are homomorphisms, namely, projections and constant operations, $f(0,y)=y=e_2^{(2)}(x,y)$, $f(x,1)=x=e_1^{(2)}(x,y)$, $f(1,y)=f(x,1)=1$, but is not diagonal since $f(f(0,1),f(1,0))=1$ while $f(0,0)=0$.

Another possibility is to consider diagonality only, and some investigations in this direction appear in the papers by J. Šlapal. We have seen that n-ary algebras from $\underline{GDA}_{(n)}$ are characterized by the identity (D_f). But this is not the whole truth. It can be checked that generalized diagonal algebras satisfy not only (D_f), but also (E_f), and in fact all normal identities of the variety $\underline{RA}_{(n)}$. That is, $\underline{GDA}_{(n)}$ is the so-called normalization of $\underline{RA}_{(n)}$.

This motivates the study of normalization of the variety of rectangular algebras of type τ, which will be our way of generalization. For this purpose, we will need some results the proofs of which can be found in the paper [P&R 93]. As far as decomposition properties of rectangular algebras are concerned, the following is known:

2.6. LEMMA. ([P&R 93], 2.6.Lemma, 2.7.Th., p. 186) *Rectangular algebras can be characterized either as isomorphic images of finite products of projection algebras of type τ, or as isomorphic images of subalgebras of direct products of 2-element projection algebras of type τ:*

$$\underline{RA}_\tau = \mathbb{IP}_{fin}(P_\tau), \qquad \underline{RA}_\tau = \mathbb{ISP}\,(P_\tau^{[2]}). \tag{2.3}$$

Here $\mathbb{I}$ means isomorphic images, $\mathbb{S}$ subalgebras, and $\mathbb{P}$ products, [Ih].

Denote by SI $(\mathcal{V})$ all subdirectly irreducible algebras, shortly SI-algebras, of a given variety $\mathcal{V}$. Let $P_\tau^{[2]}$ denote the class of all two-element projection τ-algebras.

2.7. LEMMA. ([P&R 93], 2.8. Corollary, p. 186) *All subdirectly irreducible algebras in $\underline{RA}_\tau$ are precisely the two-element projection τ-algebras,*

$$\mathrm{SI}\,(\underline{RA}_\tau) = P_\tau^{[2]}. \tag{2.4}$$

2.8. EXAMPLE. Let $A=\{0,1,2,3\}$. Then $(A;e_{1,A}^{(2)}) \in \underline{RA}_{(2)}$ is isomorphic to the direct product of $(\{0,1\};e_{1,\{0,1\}}^{(2)})$ with itself.

Since there are n_0 projection τ-algebras on a two-element set we obtain as an immediate consequence:

2.9. LEMMA ([P&R 93], 2.9.Corollary, p. 187) *Let the type* τ *be finite,* $F = (f_1, \ldots, f_k)$. *Then there are exactly* n_0 *isomorphism classes of rectangular SI-*τ*-algebras and* 2^{n_0} *subvarieties in* $\underline{RA}_\tau$.

Given terms p, q of type τ, an identity $p \approx q$ in a variety $\mathcal{V}$ is called a *hyperidentity* for $\mathcal{V}$ if it is satisfied no matter which term operations of the same arity are substituted for the operation symbols in the identity. A variety $\mathcal{V}$ is called *solid* if all identities satisfied in $\mathcal{V}$ are hyperidentities. Equivalently, a variety is solid iff the identities from the generating system are hyperidentities, [Gr 88]. By ([DLPS 91], 4.13 Prop., p. 110), any non-trivial solid variety of type τ contains all projection τ-algebras.

2.10. LEMMA ([P&R 93], 5.1.Th, p. 192) *All identities of the generating system* Σ_τ *of* $\underline{RA}_\tau$ *are hyperidentities. Consequently, all identities valid in* $\underline{RA}_\tau$ *are hyperidentities, and* $\underline{RA}_\tau$ *is a solid variety.*

So in the lattice L_τ of all varieties of type τ, $\underline{RA}_\tau$ is the least solid variety. Especially, for $\mathcal{A} \in \underline{RA}_\tau$, each term operation is idempotent and diagonal, and any couple of term operations commute. This enables us to construct new rectangular algebras from already known ones.

2.11. PROPOSITION. *Let* $\mathcal{A} \in \underline{RA}_\tau$. *Let* $(t_j)_{1 \le j \le k}$ *be any finite collection of terms of a given type. Then* $(A; (t_1^A, \ldots, t_k^A))$ *is also a rectangular algebra.*

Proof. A consequence of the above arguments: all term operations satisfy all identities of the generating system Σ_τ. ■

3. Normalization of a variety

Given a class K of algebras of the type τ denote by $Id(K)$ the set of all identities valid in K and by $Id_N(K)$ the family of all normal identities satisfied in K. Note that the identity $p \approx q$ is called *normal* if the terms $p(x_1, \ldots, x_n)$, $q(x_1, \ldots, x_n)$ are of the same arity and either both are equal to the same variable, or both are not variables (= proper terms), and *non-normal* otherwise. We can write $Id(K) = Id_N(K) \cup \Xi(K)$ where $\Xi(K)$ is the set of all non-normal identities.

Let $\mathcal{V}$ be a variety of type τ with $\tau(f) \ge 1$ for at least one $f \in F$. If $\Xi(\mathcal{V})$ is non-empty and contains some identity $p(x_1, \ldots, x_n) \approx x_i$ where p is not a variable then $\Xi(\mathcal{V})$ contains also the identity $v(x) \approx x$ where $v(x) = p(x, \ldots, x)$ is a unary term. Note that $v(x)$ is determined uniquely up to identity. So we can call $v(x)$ an *assigned term* of $\mathcal{V}$, ([Ch 95a], p. 35), ([Ch&G 99], p. 50).

For a variety $\mathcal{V}$ consider the variety $\mathcal{V}^*$ of all τ-algebras given by all normal identities of $\mathcal{V}$:

$$\text{(3.1)} \quad \mathcal{V}^* = \mathrm{Mod}(Id_N(\mathcal{V})) = \{\mathcal{A} \mid p \approx q \text{ holds in } \mathcal{A} \text{ for all } p \approx q \text{ from } Id_N(\mathcal{V})\}.$$

$\mathcal{V}^*$, denoted also by $\mathcal{N}(\mathcal{V})$, is the so-called *normalization* or *nilpotent shift* of $\mathcal{V}$, [M 73], [Ch&G 99], [Ch 95a], and $\mathcal{V}$ is a subvariety in the variety $\mathcal{V}^*$ of type τ. A variety $\mathcal{V}$ will be called *normally presentable* if $\mathcal{V}^* = \mathcal{V}$ and *non-normally presentable* otherwise. Given a normally presentable variety $\mathcal{W}$ and a unary term $v(x)$ (both of a given type) there exists a unique variety $\mathcal{V}$ with an assigned term $v(x)$ such that $\mathcal{V}^* = \mathcal{W}$, ([M 73], Prop. 1, p. 704). We will need the following.

3.1. LEMMA. *Let $\mathcal{V}$ be a non-normally presentable variety with an assigned term $v(x)$. Then $\mathcal{V} = \mathrm{Mod}(Id_N(\mathcal{V}) \cup \{v(x) \approx x\})$. Moreover, if $\mathcal{V} = \mathrm{Mod}(\Sigma)$ for some system of identities Σ then there exists a system of normal identities $\Sigma_N \subset Id_N(\mathcal{V})$ such that $\mathcal{V} = \mathrm{Mod}(\Sigma_N \cup \{v(x) \approx x\})$. Especially, if $\mathcal{V}^* = \mathrm{Mod}(\tilde{\Sigma})$ then $\mathcal{V} = \mathrm{Mod}(\tilde{\Sigma} \cup \{v(x) \approx x\})$.*

Proof. Similar arguments as in ([M 73], p. 704) can be used for the proof. The following is crucial: any non-normal identity $p(x_1, \ldots, x_m) \approx x_i$ can be replaced by a normal identity $p(v(x_1), \ldots, v(x_m)) \approx v(x_i)$ which, together with $v(x) \approx x$, gives back the original one. ■

3.2. LEMMA. *Let $\mathcal{V} = \mathrm{Mod}(\Sigma_N \cup \{v(x) \approx x\})$ be a non-normally presentable variety and $\Sigma_N \subset Id_N(\mathcal{V})$. Then the normalization can be described by identities as follows:*

$$\mathcal{V}^* = \mathrm{Mod}(\Sigma_N \cup \Sigma_v)$$

where

$$\Sigma_v = \{f(x_1, \ldots, x_j, \ldots, x_n) \approx f(x_1, \ldots, x_{j-1}, v(x_j), x_{j+1}, \ldots, x_n), \quad v(f(x_1, \ldots, x_n)) \approx f(x_1, \ldots, x_n) \mid f \in F, j = 1, \ldots, n\}.$$

Proof. The proof follows from ([M 73], Lemma and Theorem 2, p. 705). ■

3.3. PROPOSITION. *If $\mathcal{V}$ is a solid variety then the normalization $\mathcal{V}^*$ is also solid.*

Proof. In a solid variety $\mathcal{V}$, all identities, especially all normal identities, are hyperidentities. By definition, $Id(\mathcal{V}^*) = Id_N(\mathcal{V})$, so that all identities in $\mathcal{V}^*$ are hyperidentities as well. ■

On a two-element set $\{0, 1\}$, consider a constant τ-algebra $\mathcal{C} = (\{0, 1\}; F^{\mathcal{C}})$ with operations given by $f^{\mathcal{C}}(a_1, \ldots, a_n) = 0$ for all $f \in F$ and $a_1, \ldots, a_n \in \{0, 1\}$. All two-element constant τ-algebras are isomorphic to $\mathcal{C}$ and form an isomorphism class denoted by $C_\tau^{[2]}$.

3.4. LEMMA. ([Ch 95a], Theorem 4, p. 42) *Let $\mathcal{V}$ be a non-normally presentable variety and $\mathcal{V}^*$ the corresponding normalization. Then all subdi-*

rectly irreducible algebras of $\mathcal{V}^*$ are two-element constant τ-algebras together with all SI*-algebras of $\mathcal{V}$,*

$$\mathrm{SI}(\mathcal{V}^*) = \mathrm{SI}(\mathcal{V}) \cup C_\tau^{[2]}. \tag{3.2}$$

Let $\mathcal{A} \in \mathcal{V}^*$ and let v denote an assigned term. The set

$$Sk\,\mathcal{A} = \{d \in A \mid v^{\mathcal{A}}(d) = d\} \tag{3.3}$$

will be called a *skeleton* of $\mathcal{A}$ and its elements will be called *skeletal elements.* If we introduce a relation Φ on A by: $(a,b) \in \Phi$ if and only if for each n-ary $f \in F$ and $a_2,\dots, a_n \in A$, $f^{\mathcal{A}}(b, a_2, \dots, a_n) = f^{\mathcal{A}}(a, a_2, \dots, a_n)$, then $\Phi \in Con\mathcal{A}$, i.e. Φ is a congruence on $\mathcal{A}$, each congruence class $[a]_\Phi$ contains exactly one skeletal element d, and the congruence class $[a]_\Phi$ is formed by all elements $b \in A$ for which $v^{\mathcal{A}}(b) = d$. Obviously, $(a,b) \in \Phi$ iff a can be replaced by b at each place in each term. Hence we conclude

3.5. LEMMA. *The equivalence relation given above can be characterized as follows:*

$$(a,b) \in \Phi \quad \text{if and only if} \quad v^{\mathcal{A}}(a) = v^{\mathcal{A}}(b). \tag{3.4}$$

3.6. LEMMA. *An element $a \in A$ belongs to $Sk\,\mathcal{A}$ if and only if a is a result of some term operation on A,*

$$Sk\,\mathcal{A} = \{t^{\mathcal{A}}(a_1, \dots, a_n) \mid a_i \in A,\ t \in W_\tau\}. \tag{3.5}$$

Proof. If $a \in Sk\,\mathcal{A}$ then a is a result of $v^{\mathcal{A}}$. Vice versa, let $a = t^{\mathcal{A}}(a_1, \dots, a_m)$ for some m-ary term t and $a_1,\dots, a_m \in A$. Since $v(x) \approx x$ holds in $\mathcal{V}$ the identity $v(t(x_1, \dots, x_m)) \approx t(x_1, \dots, x_m)$ belongs to $Id_N(\mathcal{V})$, that is, must be satisfied in $\mathcal{V}^*$, and $v^{\mathcal{A}}(a) = v^{\mathcal{A}}(t^{\mathcal{A}}(a_1, \dots, a_m)) = t^{\mathcal{A}}(a_1, \dots, a_m)$. ∎

3.7. LEMMA. ([Ch 95a], 37–38) *If $\mathcal{A} \in \mathcal{V}^*$ then the map $\iota : A/\Phi \to Sk\,\mathcal{A}$, $[a]_\Phi \mapsto v^{\mathcal{A}}(a)$ is an isomorphism, $A/\Phi \simeq Sk\,\mathcal{A}$, and the skeleton $Sk\,\mathcal{A}$ is a maximal subalgebra of $\mathcal{A}$ belonging to $\mathcal{V}$.*

It can be observed that a map $F : \mathcal{A} \mapsto Sk\,\mathcal{A}$ induces a functor. Consider the category $\underline{\underline{\mathcal{V}}}$ formed by algebras of the class $\mathcal{V}$ together with homomorphisms as morphisms, similarly for $\underline{\underline{\mathcal{V}}}^*$.

3.8. PROPOSITION. *The map F given by $F(\mathcal{A}) = Sk\,\mathcal{A}$, $F(\varphi) = \varphi|_{Sk\,\mathcal{A}}$ for $\varphi \in Mor(\underline{\underline{\mathcal{V}}}^*)$, $\varphi : \mathcal{A} \to \mathcal{B}$, $\mathcal{A}$, $\mathcal{B} \in Ob(\underline{\underline{\mathcal{V}}}^*)$ is a covariant functor from the category $\underline{\underline{\mathcal{V}}}^*$ to the category $\underline{\underline{\mathcal{V}}}$.*

Proof. Since $Sk\,\mathcal{A} \leq \mathcal{A}$ we obtain $id_{\mathcal{A}}|_{Sk\,\mathcal{A}} = id_{Sk\,\mathcal{A}}$. Given a couple of homomorphisms $\varphi \in Hom(\mathcal{A},\mathcal{B})$, $\psi \in Hom(\mathcal{B},\mathcal{C})$, $\mathcal{A}$, $\mathcal{B}$, $C \in \mathcal{V}^*$ then for the assigned term v of $\mathcal{V}$, $\varphi(v^{\mathcal{A}}(x)) = v^{\mathcal{B}}(\varphi(x))$ holds since homomorphisms preserve terms. Hence homomorphisms preserve skeletons, $\varphi|_{Sk\,\mathcal{A}} : Sk\,\mathcal{A} \to$

$Sk\,\mathcal{B}$. For a couple of homomorphisms, $(\psi \circ \varphi)|_{Sk\,\mathcal{A}} = \psi|_{Sk\,\mathcal{B}} \circ \varphi|_{Sk\,\mathcal{A}}$. We conclude $F(id_{\mathcal{A}}) = id_{F(\mathcal{A})}$ and $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$. ∎

REMARK. In [Ch 95a, b], the so-called choice algebras are used to give an explicite construction of the nilpotent shift. If $\mathcal{V}$ is a variety of type τ denote by $E(\tau)$ the family consisting of all nullary terms of $\mathcal{V}$ and all nullary operations of type τ if there are any. Let $\mathcal{A} = (A; F^{\mathcal{A}}) \in \mathcal{V}$. Let $\theta \in Con\,\mathcal{A}$ be a congruence. Let $\kappa : Exp\,A \to A$ be a choice function compatible with θ, i.e. $\kappa([a]_\theta) \in [a]_\theta$ and if $[a]_\theta \cap E^{\mathcal{A}}(\tau) \neq \emptyset$ then $\kappa([a]_\theta) \in [a]_\theta \cap E^{\mathcal{A}}(\tau)$. For any n-ary operation $f^{\mathcal{A}} \in F^{\mathcal{A}}$ we can create a new n-ary operation $f^{\mathcal{A}^*}$ by $f^{\mathcal{A}^*}(a_1, \dots, a_n) := \kappa([f^{\mathcal{A}}(a_1, \dots, a_n)]_\theta)$ such that the algebra $\mathcal{A}^* = (A; \{f^{\mathcal{A}^*}; f \in F\})$, called a *$(\theta, \kappa)$-choice algebra*, belongs to $\mathcal{N}(\mathcal{V})$, and $Sk\,\mathcal{A}^* = \{\kappa([a]_\theta); a \in A\}$. If we denote by $\mathfrak{C}(\mathcal{V})$ the class of all (θ, κ)-choice algebras for all algebras $\mathcal{A} \in \mathcal{V}$ and all cogruences θ on $\mathcal{A}$ then the nilpotet shift of $\mathcal{V}$ consists exactly from all homomorphic images of algebras from the class $\mathfrak{C}(\mathcal{V})$,

$$\mathcal{N}(\mathcal{V}) = \mathbb{H}\,\mathfrak{C}(\mathcal{V}). \tag{3.6}$$

Up to isomorphism, $\mathcal{N}(\mathcal{V})$ is unique.

3.9. EXAMPLE. Let $A = \{0, 1, 2, 3, \}$. The algebra $\mathcal{A} = (A; f) \in \underline{RA}_{(2)}$ with f given below is isomorphic to the direct product of the algebra $(\{0,1\}; e^{(2)}_{1,\{0,1\}})$ with $(\{0,1\}; e^{(2)}_{2,\{0,1\}})$. Let $\theta \in Con\,\mathcal{A}$ be given by $\theta = \Delta_A \cup \{(0,1), (2,3)\}$. Let us choose a θ-function $\varkappa$ by $\varkappa(\{0,1\}) = 0$, $\varkappa(\{2,3\}) = 2$. Then the $(\theta, \varkappa)$-choice algebra $\mathcal{A}^* = (A; f^*)$ of $\mathcal{A}$ belongs to $(\underline{RA}_{(2)})^*$, $Sk\,\mathcal{A}^* = \{1, 2\}$. The corresponding multiplication tables are

f	0	1	2	3
0	0	1	0	1
1	0	1	0	1
2	2	3	2	3
3	2	3	2	3

f^*	0	1	2	3
0	0	0	0	0
1	0	0	0	0
3	2	2	2	2
3	2	2	2	2

3.10. EXAMPLE. Now let us consider $\mathcal{B}^* = (A; f^*) \in (\underline{RA}_{(2)})^*$ with the binary operation given below and A as above. The term $v(x) = f^*(x, x)$ is an assigned term. Obviously, a maximal idempotent subalgebra of $\mathcal{B}^*$ is $Sk\,\mathcal{B}^* = \{0, 2\}$, and the binary relation Φ on A given by: $(a, b) \in \Phi$ if and only if $f^*(a, a) = f^*(b, b)$, is a congruence on A with the congruence classes $\{0\}$, $\{1, 2, 3\}$. Now it is natural to assume a Φ-function given by $\kappa([0]_\Phi) = 0$, $\kappa([2]_\Phi) = 2$. $\mathcal{B}^* = (A; f^*)$ arises as a (Φ, κ)-choice algebra from $\mathcal{B} = (A; f)$ with $f = e^{(2)}_{1,A}$.

f^*	0	1	2	3
0	0	0	0	0
1	2	2	2	2
2	2	2	2	2
3	2	2	2	2

f	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

4. Normalization of rectangular algebras

Our aim is to pay attention to the normalization of the variety $\underline{RA}_\tau$ which is worth considering since $Id(\underline{RA}_\tau) \neq Id_N(\underline{RA}_\tau)$:

4.1. LEMMA. *Let t be an n-ary term, with $n \geq 1$, of type τ which is not a variable. Then $t(\underbrace{x, \ldots, x}_{n\text{-times}}) \approx x$ is the non-normal identity in $\underline{RA}_\tau$. Consequently, the variety $\underline{RA}_\tau$ is not normally presentable.*

Proof. Due to solidity of $\underline{RA}_\tau$ (2.10. Lemma) all term operations in rectangular algebras are idempotent. ∎

4.2. PROPOSITION. *The normalization $\mathcal{A} \in (\underline{RA}_\tau)^*$ is a solid variety.*

Proof. By 3.3.Prop., the normalization of a solid variety is always solid. ∎

4.3. LEMMA *The skeleton $Sk\,\mathcal{A}$ of $\mathcal{A} \in (\underline{RA}_\tau)^*$ is a maximal idempotent subalgebra in $\mathcal{A}$.*

Proof. A consequence of 3.7.Lemma. ∎

4.4. LEMMA. *The congruence classes $[a]_\Phi$, $a \in A$ are constant subalgebras of $\mathcal{A}$.*

Proof. Let $f_i \in F$ and let $a_1, \ldots, a_{n_i}$ for some $d \in Sk\,\mathcal{A}$. Then

$$f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i}) = f_i^{\mathcal{A}}(d, \ldots, d) = v^{\mathcal{A}}(d) = d. \;∎$$

4.5. PROPOSITION. *The map F given by $F(\mathcal{A}) = Sk\,\mathcal{A}$, $F(\varphi) = \varphi|_{Sk\,\mathcal{A}}$ for $\varphi \in Mor((\underline{RA}_\tau)^*)$, $\varphi : \mathcal{A} \to \mathcal{B}$, $\mathcal{A}$, $\mathcal{B} \in Ob((\underline{RA}_\tau)^*)$ is a covariant functor from the category of generalized diagonal algebras and their homomorphisms to the category of rectangular algebras and their homomorphisms.*

Proof. A consequence of 3.8. Prop. ∎

We will be interested in the following problems:

(i) *how to decide whether a given τ-algebra belongs to the variety $(\underline{RA}_\tau)^*$?*

(ii) *what do subdirectly irreducible algebras in $(\underline{RA}_\tau)^*$ look like?*

(iii) *how to find a representation of* $\mathcal{A} \in (\underline{RA}_\tau)^*$ *as a subdirect product of subdirectly irreducible ones?*

(iv) *given an* n*-ary term* $t \in W_\tau^{(n)}$, *how to define and find its normal form in* $(\underline{RA}_\tau)^*$*?*

(v) *is there some "especially nice" choice for the assigned term of* $(\underline{RA}_\tau)^*$*?*

4.6. PROPOSITION. *Let* τ *be a finite type with* $F = (f_1, \dots, f_k)$. *Then every identity which holds in the nilpotent shift* $(\underline{RA}_\tau)^*$ *of the variety of rectangular algebras can be derived from the finite system of identities, namely*

$$(\underline{RA}_\tau)^* = \mathrm{Mod}(\{(D_{f_i}), (E_{f_i}), i = 1, \dots, k\} \cup \Sigma)$$

where

$$\begin{aligned}\Sigma = \{&f_i(x_1, \dots, x_{n_i}) \approx f_i(f_j(x_1, \dots, x_1), x_2, \dots, x_{n_i}), f_i(x_1, \dots, x_{n_i})\\ &\approx f_i(x_1, f_j(x_2, \dots, x_2), \dots, x_{n_i}), \dots, f_i(x_1, \dots, x_{n_i})\\ &\approx f_i(x_1, x_2, \dots, f_j(x_{n_i}, \dots, x_{n_i}), f_i(x_1, \dots, x_{n_i})\\ &\approx f_j(f_i(x_1, \dots, x_{n_i}), \dots, f_i(x_1, \dots, x_{n_i})) \mid i \neq j,\ i, j = 1, \dots, k\}.\end{aligned}$$

Proof. A consequence of 3.2.Lemma. If $i = j$ the corresponding identities follow from absorption laws, or from diagonality respectively. ∎

There is an easy answer to question (i): it is sufficient to verify the above identities.

Now it is also obvious that $(\underline{RA}_{(n)})^* = \underline{GDA}_{(n)} = \mathrm{Mod}((D_f))$.

4.7. EXAMPLE. On a two-element set there are only two isomorphism classes of generalized diagonal n-ary algebras, each consisting of n elements, namely an idempotent class of projection algebras and a non-idempotent class of constant algebras.

4.8. PROPOSITION. *All subdirectly irreducible algebras in the variety* $(\underline{RA}_\tau)^*$ *of a finite type* τ *are two-element constant algebras and two-element projection algebras,*

$$\mathrm{SI}((\underline{RA}_\tau)^*) = P_\tau^{[2]} \cup C_\tau^{[2]}. \tag{4.1}$$

Proof. An immediate consequence of 3.4.Lemma and 2.7.Lemma. ∎

So we can answer the question (ii): any generalized diagonal algebra of finite type is isomorphic to a subdirect product of 2-element projection and 2-element constant algebras. Answering of both (iii) and (iv) requires some knowledge of normal forms for terms and is postponed to the next paragraph.

5. Normal forms for terms in $\underline{RA}_\tau$ and in $(\underline{RA}_\tau)^*$

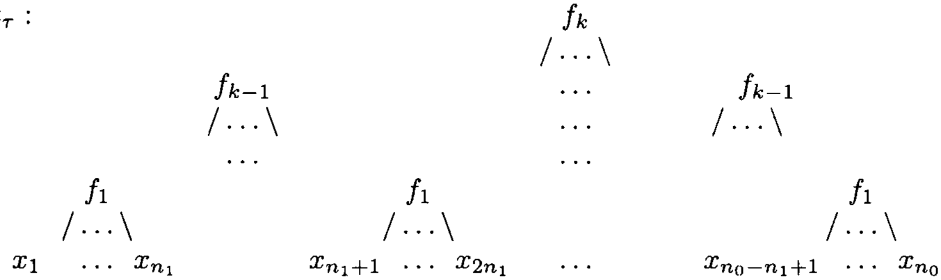
Given a variety $\mathcal{V}$ of type τ, a *normal form* for terms in $\mathcal{V}$ is a map of the word algebra into itself $\mathrm{NF} : W_\tau \to W_\tau$, $t \mapsto \mathrm{NF}(t)$ such that both terms t, $\mathrm{NF}(t)$ are of the same arity, t is equivalent with $\mathrm{NF}(t)$ in $\mathcal{V}$, $\mathcal{V} \models t \approx \mathrm{NF}(t)$, and the term t is equivalent to another term t' iff they have the same normal forms:

$$\mathcal{V} \models t \approx t' \iff \mathrm{NF}(t) = \mathrm{NF}(t').$$

We are going to find normal form for terms in $(\underline{RA}_\tau)^*$.

In ([P&R 93], p. 187–9), an n_0-ary term t_τ (called the *general* NF*-term* for $\underline{RA}_\tau$) is introduced inductively. Also a formation tree diagram for t_τ is presented in which leaves correspond to variables while non-leave nodes correspond to operation symbols. Schematically,

t_τ :



It is proven that any term $t \in W_\tau$ has a unique normal form $\mathrm{NF}(t)$ in the variety $\underline{RA}_\tau$ (called NF*-term* for $\underline{RA}_\tau$) which differs from t_τ at most in variables. Obviously, we can answer (v) as follows:

5.1. LEMMA. *Up to equivalence, the assigned term for the normalization* $(\underline{RA}_\tau)^*$ *is given by*

$$v(x) = t_\tau(x, \ldots, x). \tag{5.1}$$

Keeping the above notation let us introduce a map $\mathrm{nf} : W_\tau \to W_\tau$ by

$$\mathrm{nf}(x) := x \qquad \text{for a variable } x,$$
$$\mathrm{nf}(t) := \mathrm{NF}(t) \quad \text{for a non-trivial term } t \in W_\tau.$$

5.2. PROPOSITION. $\mathrm{nf}(t)$ *introduced above is a normal form for terms* $t \in W_\tau$ *in the variety* $(\underline{RA}_\tau)^*$ *of generalized diagonal algebras of a given type* τ. *For a non-trivial term t there is a unique set of variables (not necessarily distinct)* $x_{u_1}, \ldots, x_{u_0}$ *such that* $\mathrm{nf}(t) = t_\tau(x_{u_1}, \ldots, x_{u_0})$.

P r o o f. According to the definition, each variable x is equivalent to $\mathrm{nf}(x)$, and two variables are equal iff they have the same nf-term. Obviously, a non-trivial term t is never equivalent with a variable x in $(\underline{RA}_\tau)^*$ since a non-normal identity $t \approx x$ cannot be satisfied in the normalization. Now let t be a non-trivial n-ary term from W_τ. Let $\mathcal{A} \in (\underline{RA}_\tau)^*$ and let $a_1, \ldots, a_n \in A$.

Let v be an assigned term of $(\underline{RA}_\tau)^*$, (5.1). Since the results of all term operations belong to the skeleton it holds $t^{\mathcal{A}}(a_1,\dots,a_n) \in Sk\,\mathcal{A}$ and also $\mathrm{NF}^{\mathcal{A}}(t)(a_1,\dots,a_n) \in Sk\,\mathcal{A}$ which means

$$\mathrm{NF}^{\mathcal{A}}(t)(a_1,\dots,a_n) = v^{\mathcal{A}}(\mathrm{NF}^{\mathcal{A}}(t)(a_1,\dots,a_n)),$$
$$v^{\mathcal{A}}(t^{\mathcal{A}}(a_1,\dots,a_n)) = t^{\mathcal{A}}(a_1,\dots,a_n).$$

Now we use the fact that $\underline{RA}_\tau \models \mathrm{NF}(t) \approx t$ as well as the fact that the map $x \mapsto v(x)$ determines an endomorphism of $\mathcal{A}$ into $Sk\,\mathcal{A} \in \underline{RA}_\tau$ ([Ch 95a], Lemma 1) to verify that the following holds:

$$v^{\mathcal{A}}(\mathrm{NF}^{\mathcal{A}}(t)(a_1,\dots,a_n)) = \mathrm{NF}^{\mathcal{A}}(t)(v^{\mathcal{A}}(a_1),\dots,v^{\mathcal{A}}(a_n)) =$$
$$= t^{\mathcal{A}}(v^{\mathcal{A}}(a_1),\dots,v^{\mathcal{A}}(a_n)) = v^{\mathcal{A}}(t^{\mathcal{A}}(a_1,\dots,a_n)).$$

Together, we conclude $(\underline{RA}_\tau)^* \models t \approx \mathrm{NF}(t) = \mathrm{nf}(t)$. According to ([P&R 93], p. 187–8) there exists a suitable (uniquely determined) family of variables (some of which might coincide) $x_{u_1},\dots,x_{u_{n_0}}$ such that

$$\mathrm{nf}(t) = \mathrm{NF}(t) \equiv t_\tau(x_{u_1},\dots,x_{u_{n_0}}). \tag{5.2}$$

Now let t, t' be non-trivial terms such that $(\underline{RA}_\tau)^* \models t \approx t'$. This normal identity must be satisfied also in the variety of rectangular algebras, $\underline{RA}_\tau \models t \approx t'$, which implies that the corresponding NF-terms are uniquely determined and $\mathrm{NF}(t) = \mathrm{NF}(t')$ holds. That is, the nf-terms coincide, $\mathrm{nf}(t) = \mathrm{nf}(t')$. Vice versa, if the equality $\mathrm{nf}(t) = \mathrm{nf}(t')$ holds for non-trivial terms t and $t' \in W_\tau$ then we use $(\underline{RA}_\tau)^* \models \mathrm{nf}(t) \approx t$ (and similarly for t') to prove that $(\underline{RA}_\tau)^* \models t \approx t'$. ■

So the problem (iv) is reduced to the same problem for $\underline{RA}_\tau$ the solution of which is given in ([P&R 93], p. 190): $\mathrm{NF}(t) = t_\tau(x_{\lambda(\underline{q}^1)},\dots,x_{\lambda(\underline{q}^{n_0})})$ where

$$x_{\lambda(\underline{q}^j)} = t^{\mathcal{D}_{\underline{q}^j}}(x_1,\dots,x_n), \qquad 1 \le j \le n_0,$$
$$\mathcal{D}_{\underline{q}^j} = (\{x_1,\dots,x_n\}; (e^{(n_i)}_{q_i^j,\{x_1,\dots,x_n\}})_{i=1,\dots,k}).$$

To describe an algorithm for finding the decomposition of algebras from the normalization $(\underline{RA}_\tau)^*$, we can use a similar procedure as in [P&R 93], only we must add constant algebras where necessary.

Let $\mathcal{A} \in (\underline{RA}_\tau)^*$. According to 2.6.Lemma and ([P&R 93], p. 190), the skeleton $Sk\,\mathcal{A} \in \underline{RA}_\tau$ is isomorphic to a subalgebra of the direct product of two-element projection algebras. In more details, let $a_0 \in Sk\,\mathcal{A}$ be a fixed element and let t_τ be the general NF-term. Let $A_j := \{a_j := (a_0,\dots,a_0,d,a_0,\dots,a_0) \mid d \in Sk\,\mathcal{A}\}$, $j \in J := \{1,2,\dots,n_0\}$. Let $r(j)$ denote the least integer for which the cardinality $|A_j| \le 2^{r(j)}$, $j \in J$. Let $h_j : A_j \to \{0,1\}^{r(j)}$ be a fixed embedding for $j \in J$. Let $\underline{q}^j = (q_1^j,\dots,q_k^j)$

denote the j-th k-tuple determined by $\mu(\underline{q}^j) = j$ (μ from (2.2) corresponds to the lexicographic order). Finally let $\mathcal{B}_j$ denote the algebra

$$\mathcal{B}_j = (\{0,1\}; (e^{(n_i)}_{q_i^j,\{0,1\}})_{i=1,\dots,k}), \quad j \in J.$$

Then $h : Sk\,\mathcal{A} \to \prod_{j\in J}\mathcal{B}_j^{r(j)}$, $h(d) = (h_1(d), \dots, h_{n_0}(d))$, $d \in Sk\,\mathcal{A}$ is the desired isomorphism, and $h(Sk\,\mathcal{A}) \leq \prod_{j\in J}\mathcal{B}_j^{r(j)}$. For any $d \in Sk\,\mathcal{A}$, denote $\nu(d) = |\{a \in A \,|\, v^{\mathcal{A}}(a) = v^{\mathcal{A}}(d)\}|$. Let p be the minimal integer such that

$$\max\{\nu(d) \,|\, d \in Sk\,\mathcal{A}\} \leq 2^p.$$

Let us choose injections

$$b_d : [d]_\Phi \to \{0,1\}^p.$$

Denote $\mathcal{C} = (\{0,1\}; (c_0^{(n_i)})_{i=1,\dots,k})$ where $c_0^{(n_i)}(a_1, \dots, a_{n_i}) = 0$ for any choice $a_1, \dots, a_{n_i}$ from $\{0,1\}$.

Now we can answer the question (iii) as follows:

5.3. PROPOSITION. *An algebra $\mathcal{A} \in (\underline{RA}_\tau)^*$ is isomorphic to a subalgebra of the direct product $\prod_{j\in J}\mathcal{B}_j^{r(j)} \times \mathcal{C}^p$. In the above notation, the corresponding injective homomorphism (embedding) $H : \mathcal{A} \to \prod_{j\in J}\mathcal{B}_j^{r(j)} \times \mathcal{C}^p$ is given by*

$$H(a) = (h(v^{\mathcal{A}}(a)), b_{v^{\mathcal{A}}(a)}(a)) \quad \text{for } a \in A.$$

Proof. The decomposition of the skeleton and the isomorphism h of $Sk\,\mathcal{A}$ into a product of two-element projection algebras is described in ([P&R 93], p. 190). The rest follows from the fact that $[d]_\Phi$ are constant subalgebras of $\mathcal{A}$ and $b_d : [d]_\Phi \to \mathcal{C}^p$ are injective homomorphisms of constant algebras. ∎

Let us give a couple of examples for illustration.

5.4. EXAMPLE. Let $M = \{a,b,c\}$ and let a binary operation f on M be given by the multiplication table

f	a	b	c
a	a	c	c
b	a	c	c
c	a	c	c

Then $(M; f) \in (\underline{RA}_{(2)})^*$ is isomorphic to a subdirect product of the 2-element projection grupoid $(\{0,1\}; e^{(2)}_{1,\{0,1\}})$ with the 2-element constant grupoid $(\{0,1\}; c_0^{(2)})$, $c_0^{(2)}(u,v) = 0$ for u, $v \in M$. Similarly, the algebra $(M; e_{1,M}^{(2)}, f) \in \underline{RA}^*{}_{(2,2)}$ is isomorphic to a subdirect product of the algebra $(\{0,1\}; e_{1,\{0,1\}}^{(2)}, e_{2,\{0,1\}}^{(2)})$ with $(\{0,1\}; e_{1,\{0,1\}}^{(2)}, c_0^{(2)})$.

5.5. Example. Let f_1, f_2 be a couple of binary operations on $A = \{a, b, c, e, j, k\}$ given as follows:

f_1	a	b	c	e	j	k
a	a	a	a	a	a	a
b	a	a	a	a	a	a
c	c	c	c	c	c	c
e	e	e	e	e	e	e
j	j	j	j	j	j	j
k	j	j	j	j	j	j

f_2	a	b	c	e	j	k
a	a	a	c	a	c	c
b	a	a	c	a	c	c
c	a	a	c	a	c	c
e	e	e	j	e	j	j
j	e	e	j	e	j	j
k	e	e	j	e	j	j

Then the algebra $\mathcal{A} = (A; f_1, f_2)$ belongs to the normalization $(\underline{RA}_{(2,2)})^*$. The corresponding skeleton is $Sk\,\mathcal{A} = \{a, c, e, j\} \in \underline{RA}_{(2,2)}$, and the assigned term $t_\tau = f_2(f_1(x_1, x_2), f_1(x_3, x_4))$ is in the normal form. Let us choose $a_0 := a \in Sk\,\mathcal{A}$. Then $A_1 = \{a, e\}$, $A_3 = \{a, c\}$, $A_2 = A_4 = \{a\}$, $r(1) = r(3) = 1$, $r(2) = r(4) = 0$. Denote $\mathcal{B}_1 = (\{0,1\}; e_1^{(2)}, e_1^{(2)})$, $\mathcal{B}_3 = (\{0,1\}; e_1^{(2)}, e_2^{(2)})$. Then $Sk\,\mathcal{A} \simeq \mathcal{B}_1 \times \mathcal{B}_3$. The bijections

$$h_1 : A_1 \to \{0,1\},\ a \mapsto 0,\ e \mapsto 1, \qquad h_3 : A_3 \to \{0,1\},\ a \mapsto 0,\ c \mapsto 1$$

are components of the isomorphism $h : Sk\,\mathcal{A} \to (\{0,1\}^2; e_1^{(2)} \times e_1^{(2)}, e_1^{(2)} \times e_2^{(2)})$

$$a \mapsto (0,0),\quad c \mapsto (0,1),\quad e \mapsto (1,0),\quad j \mapsto (1,1).$$

The congruence classes are

$$[a]_\Phi = \{a, b\},\quad [j]_\Phi = \{j, k\},\quad [c]_\Phi = \{c\},\quad [e]_\Phi = \{e\}$$

so that $\max\{\nu(d) \mid d \in Sk\,\mathcal{A}\} = 2$, and $p = 1$. Let us choose the injections $b_a : [a]_\Phi \to \{0,1\}$, $a \mapsto 0$, $b \mapsto 1$, $b_c : [c]_\Phi \to \{0,1\}$, $c \mapsto 0$, $b_e : [e]_\Phi \to \{0,1\}$, $e \mapsto 0$, $b_j : [a]_\Phi \to \{0,1\}$, $j \mapsto 0$, $k \mapsto 1$. We get the induced injective homomorphism $H : \mathcal{A} \to (\{0,1\}^3; e_1^{(2)} \times e_1^{(2)} \times c_0^{(2)}, e_1^{(2)} \times e_2^{(2)} \times c_0^{(2)})$ of $\mathcal{A}$ into $\mathcal{B}_1 \times \mathcal{B}_3 \times \mathcal{C}$ given by

$$a \mapsto (0,0,0),\ b \mapsto (0,0,1),\ c \mapsto (0,1,0),\ e \mapsto (1,0,0),$$
$$j \mapsto (1,1,0),\ k \mapsto (1,1,1).$$

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