DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 1 2003

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STANDARD QBCC-ALGEBRAS

Abstract. The class of QBBC-algebras was introduced and studied by the authors in [5]. These algebras model properties of the logical connective implication "$\Rightarrow$" in which the validity of formulas $x \Rightarrow y$ and $y \Rightarrow x$ does not imply the equivalence of x and y. In the paper the properties of standard QBCC-algebras derived from qosets are studied.

1. Introduction

The notion of a BCC-algebra was introduced and studied by Y. Komori [7] when solving the problem whether the class of all BCK-algebras forms a variety. These algebras were then studied by many authors, see e.g. A.Wroński [9] and W. A. Dudek [2].

BCC-algebras serve as an algebraic model describing properties of implication reducts of certain logics containing the constant 1 meaning the logical value "true". More precisely, we use the formal definition from [2]:

DEFINITION 1. An algebra $(A, \bullet, 1)$ is a **BCC-algebra** if it satisfies the identities:

(BCC1) $(x \bullet y) \bullet [(z \bullet x) \bullet (z \bullet y)] = 1$
(BCC2) $x \bullet x = 1$
(BCC3) $x \bullet 1 = 1$
(BCC4) $1 \bullet x = x$
(BCC5) $(x \bullet y = 1 \ \&\ y \bullet x = 1) \Rightarrow\ x = y$.

The class of BCC-algebras satisfying the commutation axiom

(C) $$x \bullet (y \bullet z) = y \bullet (x \bullet z)$$

The paper was prepared under the support by Council of Czech Government No J14/98:153100011.

Key words and phrases: BCC-algebra, QBCC-algebra, standard QBCC-algebra, qoset, annihilator, congruence (kernel).

1991 *Mathematics Subject Classification*: 03B60, 06A10.

is known to be the class of all **BCK-algebras**. The class of all **Hilbert algebras**, an algebraic counterpart of the logical connective implication in intuitionistic logic, is just the class of BCK-algebras satisfying the left-distributivity axiom:

$$\text{(D)} \qquad x \bullet (y \bullet z) = (x \bullet y) \bullet (x \bullet z).$$

It is well-known that every BCC-algebra $\mathcal{A} = (A, \bullet, 1)$ can be viewed as a poset w.r.t. **natural ordering** defined on A by

$$\text{(1)} \qquad x \leq y \quad \text{iff } x \bullet y = 1.$$

A new construction of BCC-algebras derived from posets was given by the first author in [4]. In a recent paper I.Chajda and R.Halaš [3] started to study Hilbert algebras in which the axiom (BCC5) is not valid, the so-called pre-logics. This motivated us to introduce the class containing both the pre-logics and BCC-algebras as follows:

DEFINITION 2. A QBCC-algebra is an algebra $\mathcal{A} = (A, \bullet, 1)$ satisfying the axioms (BCC1)-(BCC4).

As it was shown in [5], the relation defined by (1) on any QBCC-algebra $\mathcal{A} = (A, \bullet, 1)$ is a quasiordering, the so-called **natural quasiordering** on A. For any quasiordered set $(Q, \leq)$ (qoset) we adopt the following notation: we write $a \sim b$ whenever $a \leq b$ and $b \leq a$ and call the pair a, b **indistinguishable**; the set $C(a) = \{x \in Q; x \sim a\}$ is called the **cell** of a; we shall write $a < b$ if $a \leq b$ and $a \not\sim b$, and $a \parallel b$ whenever $a \not\sim b$ and $b \not\sim a$. A pair (a, b), where $a > b$ is called a **bridge** if for each $c \in Q$ the following (dual) conditions hold:

(b1) $c > b$ implies $c \geq a$,
(b2) $c < a$ implies $c \leq b$.

2. Standard QBCC-algebras

Following the paper [5], a QBCC-algebra $\mathcal{A} = (A, \bullet, 1)$ is called **standard** if every subset of A containing the element 1 is a subalgebra of $\mathcal{A}$.

A pair (x, y) of elements $x, y \in A, x > y$, is called **normal** if $x \bullet y = y$.

Recall the main result of [5] describing all standard QBCC-algebras:

PROPOSITION 1. *Let $(Q, \leq, 1)$ be a qoset with a greatest element 1 and $C(1) = \{1\}$. Let us define the operation $\bullet$ on Q as follows:*

(q1) $x \bullet y = 1$ *if* $x \leq y$,
(q2) $1 \bullet x = x$,
(q3) $x \bullet y = y$ *if* $x \parallel y$,
(q4) $x \bullet y = y$ *if* $x > y$ *and* (x, y) *is not a bridge,*

(q5) *if* (x,y) *is a bridge in* Q *and* $x \neq 1$ *one can set* $x \bullet y = y$ *or* $x \bullet y = x$*; in the latter case for each* $z \geq x$ *we have either* $z \sim x$ *and* $z \bullet y = z$ *or* $z > x$ *and* $z \bullet x = x$, $z \bullet y = y$*; for each* $z \leq y$ *we have either* $z \sim y$ *and* $x \bullet z = x$ *or* $z < y$ *and* $x \bullet z = y \bullet z = z$.

Then $(Q, \bullet, 1)$ *is a standard QBCC-algebra and each standard QBCC-algebra is of this form.*

Hence, Proposition 1 allows to construct all standard QBCC-algebras starting from a given qoset.

EXAMPLE 1. Let us consider a qoset Q with the diagram in Fig.1.

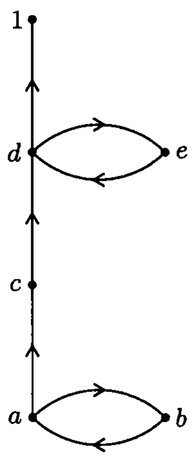

Fig. 1

By setting $d \bullet c = d$ we get by (q5) $e \bullet c = e, c \bullet a = a, d \bullet a = a, c \bullet b = b$. The rest of cases is given by (q1), (q2) and (q4), hence the operation $\bullet$ is completely determined.

In the following we will describe algebraic properties of standard QBCC-algebras.

3. Congruences, ideals and annihilators in standard QBCC-algebras

DEFINITION 3. A subset $\emptyset \neq I \subseteq Q$ of a standard QBCC-algebra $\mathcal{Q} = (Q, \leq, 1)$ satisfying the conditions

(I1) $x \in I, y \in Q, x \leq y$ imply $y \in I$,
(I2) (x,y) being a bridge and $x \bullet y = x \in I$ imply $y \in I$,

is called an **ideal** of $\mathcal{Q}$.

The set of all ideals of $\mathcal{Q}$ will be denoted by $Id(\mathcal{Q})$. For a congruence θ on $\mathcal{Q}$ ($\theta \in Con(\mathcal{Q})$) denote by $[1]_\theta$ its congruence class containg the element 1, the so-called **kernel** of θ. The following lemma shows that the set of congruence kernels of standard QBCC-algebras coincide with their ideals.

LEMMA 1. *Let $\mathcal{Q} = (Q, \leq, 1)$ be a standard QBCC-algebra, $\theta \in Con(\mathcal{Q})$, $I \in Id(\mathcal{Q})$. Then*

(a) *$I_\theta := [1]_\theta$ is an ideal of $\mathcal{Q}$,*
(b) *the relation θ_I on $\mathcal{Q}$ defined by*

$$\langle x, y\rangle \in \theta_I \text{ iff } x \sim y \text{ or } x, y \in I$$

is the greatest congruence θ on $\mathcal{Q}$ with $[1]_\theta = I$.

Proof. (a) Suppose $\langle x, 1\rangle \in \theta, y \geq x$. By compatibility we obtain $\langle x \bullet y, 1 \bullet y\rangle = \langle 1, y\rangle \in \theta$. If (x, y) is a bridge and $x \bullet y = x \in I_\theta$, then again by compatibility $\langle x \bullet y, 1 \bullet y\rangle = \langle x, y\rangle \in \theta$. Hence $[x]_\theta = [y]_\theta = [1]_\theta$ and $y \in I_\theta$.

(b) We will show that θ_I is a congruence on $\mathcal{Q}$.

Reflexivity and symmetry of θ_I are clear. To prove transitivity, let $\langle x, y\rangle$, $\langle y, z\rangle \in \theta_I$. If $x \sim y \sim z$ or $x, y, z \in I$ occur, we have $\langle x, z\rangle \in \theta_I$. In the remaining case $x \sim y$, $y, z \in I$ we obtain by (I1) $x \in I$, hence also $\langle x, z\rangle \in \theta_I$.

Now we prove compatibility of θ_I. Suppose $\langle x, y\rangle \in \theta_I$ and $u \in Q$ be an arbitrary element. It is enough to prove $\langle u \bullet x, u \bullet y\rangle, \langle x \bullet u, y \bullet u\rangle \in \theta_I$.

First let $x, y \in I$. Applying (I1) one gets $u \bullet x, u \bullet y \in I$, and so $\langle u \bullet x, u \bullet y\rangle \in \theta_I$.

If $x \bullet u = 1$, then $x \bullet u \in I$ and $x \leq u$ so by (I1) $u \in I$. This yields $y \bullet u \in I$. Suppose further $x \bullet u = u$. The case $y \bullet u = u$ gives us $\langle x \bullet u, y \bullet u\rangle = \langle u, u\rangle \in \theta_I$. For $y \bullet u = y \neq u, 1$ due to (I2) $u \in I$ and $\langle x \bullet u, y \bullet u\rangle = \langle u, y\rangle \in \theta_I$. If $x \bullet u = x \neq u, 1$ then $u \in I$ according to (I2) and hence $y \bullet u \in I$ yielding $\langle x \bullet u, y \bullet u\rangle = \langle x, y \bullet u\rangle \in \theta_I$.

Secondly let us suppose that $x \sim y$. We will show that $u \bullet x \sim u \bullet y$ and $x \bullet u \sim y \bullet u$ hold. Let $u \parallel x$. By (q3) $u \bullet x = x \sim y = u \bullet y$, $x \bullet u = u \sim u = y \bullet u$. The case $u \sim x$ leads to $u \bullet x = 1 \sim 1 = u \bullet y$, $x \bullet u = 1 \sim 1 = y \bullet u$. Suppose further $u > x$. The following two subcases can occur. If both pairs $(u, x), (u, y)$ are normal we get $u \bullet x = x \sim y = u \bullet y$, $x \bullet u = 1 \sim 1 = y \bullet u$. If they are non-normal, then $u \bullet x = u \sim u = u \bullet y$, $x \bullet u = 1 \sim 1 = y \bullet u$. The last possibility is $u < x$. In this case we have also two subcases. The both pairs (x, u), (y, u) are either normal, then $u \bullet x = 1 \sim 1 = u \bullet y$, $x \bullet u = u \sim u = y \bullet u$, or non-normal and $u \bullet x = 1 \sim 1 = u \bullet y$, $x \bullet u = x \sim y = y \bullet u$.

The equality $[1]_{\theta_I} = I$ follows directly from the definition of θ_I.

Let us show that θ_I is the greatest congruence on $\mathcal{Q}$ with kernel I.

Suppose $\theta \in Con(\mathcal{Q}), [1]_\theta = I$, and let $\langle x, y\rangle \in \theta$. This by using of substitution property of θ leads to $\langle x \bullet y, 1\rangle, \langle y \bullet x, 1\rangle \in \theta$, i.e. $x \bullet y, y \bullet x \in I$. Assume further that $x \not\sim y$. If x, y are incomparable, we get $x \bullet y = y \in I$ and $y \bullet x = x \in I$. In the case of comparability let e.g. $x \leq y$. Then $y \not\leq x$ (otherwise $x \sim y$), hence $y \bullet x \in \{x, y\}$. Having $y \bullet x = x \in I$ we obtain with respect to $x \leq y$ and (I1) also $y \in I$. For $y \bullet x = y \in I$ the pair (y, x) is a bridge, and so by (I2) also $x \in I$. In the summary, we have proved that $x \not\sim y$ implies $x, y \in I$. Finally, we have got $\theta \subseteq \theta_I$. ∎

EXAMPLE 2. In contrast to BCC-algebras, standard QBBC-algebras need not be congruence distributive:

Consider a qoset $(Q, \leq)$ given in Fig. 2 and the corresponding standard QBCC-algebra $\mathcal{Q} = (Q, \leq, 1)$.

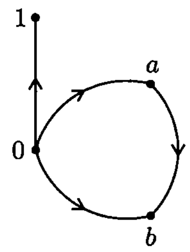

Fig. 2

Then $Con(\mathcal{Q})$ is visualized in Fig. 3,

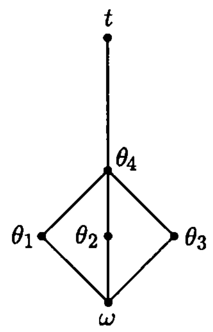

Fig.3

where

$\theta_1 \ldots \{\{0, b\}, \{a\}, \{1\}\}$,
$\theta_2 \ldots \{\{0, a\}, \{b\}, \{1\}\}$,
$\theta_3 \ldots \{\{a, b\}, \{0\}, \{1\}\}$,
$\theta_4 \ldots \{\{0, a, b\}, \{1\}\}$.

This example also shows that ideals of QBCC-algebras can be kernels of more than one congruence.

We will show that congruence kernels in arbitrary QBCC-algebras (i.e. not necessarily standard ones) correspond to deductive systems:

DEFINITION 4. A subset $D \subseteq A$ of a QBCC-algebra $\mathcal{A} = (A, \bullet, 1)$ is called a **deductive system** if

(D1) $1 \in D$
(D2) $x \bullet (y \bullet z) \in D$ and $y \in D$ imply $x \bullet z \in D$.

Denote by $Ck(\mathcal{A})$ or $Ded(\mathcal{A})$ the set of all congruence kernels of $\mathcal{A}$ or the set of all deductive systems of $\mathcal{A}$, respectively.

LEMMA 2. *For an arbitrary QBCC-algebra $\mathcal{A}$ it holds $Ck(\mathcal{A})$=$Ded(\mathcal{A})$.*

Proof. It is easy to prove that $Ck(\mathcal{A}) \subseteq \mathcal{D}\rceil\lceil(\mathcal{A})$.

Conversely, let $D \in Ded(\mathcal{A})$. Define the relation θ_D on A by

$$\langle x, y\rangle \in \theta_D \text{ iff } x \bullet y, y \bullet x \in D.$$

We show that $\theta_D \in Con(\mathcal{A})$ with $[1]_{\theta_D} = D$.

Reflexivity and symmetry of θ_D are clear. To prove transitivity of θ_D, assume that $\langle x, y\rangle, \langle y, z\rangle \in \theta_D$, i.e. $x \bullet y, y \bullet x, y \bullet z, z \bullet y \in D$. Then by (D1) and (BCC1) we have $1 = (y \bullet z) \bullet [(x \bullet y) \bullet (x \bullet z)] \in D$, and since $x \bullet y \in D$, (D2) yields $(y \bullet z) \bullet (x \bullet z) \in D$. Applying (D2) once more to $1 \bullet ((y \bullet z) \bullet (x \bullet z)) \in D$ with $y \bullet z \in D$, we obtain $x \bullet z = 1 \bullet (x \bullet z) \in D$. The validity of $z \bullet x \in D$ can be proved analogously, and so θ_D is transitive.

Further, let $\langle x, y\rangle \in \theta_D$ and $u \in A$ be an arbitrary element. Then $x \bullet y$, $y \bullet x \in D$ and, by (BCC1),

$$(u \bullet x) \bullet (u \bullet y) \geq x \bullet y \in D.$$

This leads by (D2) to $(u \bullet x) \bullet (u \bullet y) \in D$ (observe that $a \in D, b \in A$ and $a \leq b$ imply $1 \bullet (a \bullet b) = a \bullet b = 1 \in D$ and hence $b = 1 \bullet b \in D$ according to (D2)). Analogously we prove $(u \bullet y) \bullet (u \bullet x) \in D$ and $\langle u \bullet x, u \bullet y\rangle \in \theta_D$. Applying (D2) again for $1 = (x \bullet u) \bullet [(y \bullet x) \bullet (y \bullet u)] \in D$ and $y \bullet x \in D$ one gets $(x \bullet u) \bullet (y \bullet u) \in D$. Interchanging x and y we have also $(y \bullet u) \bullet (x \bullet u) \in D$ and $\langle x \bullet u, y \bullet u\rangle \in \theta_D$. Finally, using transitivity of θ_D this gives $\theta_D \in Con(\mathcal{A})$ with $[1]_{\theta_D} = D$. ∎

COROLLARY 1. *For an arbitrary standard QBCC-algebra $\mathcal{Q} = (Q, \bullet, 1)$ it holds $Id(\mathcal{Q}) = Ck(\mathcal{Q}) = Ded(\mathcal{Q})$.*

In what follows it is shown that congruences on standard QBCC-algebras are of a very special type:

LEMMA 3. *Let $\mathcal{Q} = (Q, \bullet, 1)$ be a standard QBCC-algebra, $\theta \in Con(\mathcal{Q})$. If $\langle x, y\rangle \in \theta$ and $x, y \notin I = [1]_\theta$, then $x \sim y$.*

Proof. It results from Lemma 1. ∎

DEFINITION 5. Let $\mathcal{Q} = (Q, \leq, 1)$ be a standard QBCC-algebra and B, C be non-void subsets of Q. The set

$$\langle C \rangle = \{x \in Q; x \bullet c = c \text{ for each } c \in C\}$$

is called the **annihilator** of C. The set

$$\langle C, B \rangle = \{x \in Q; (x \bullet c) \bullet c \in B \text{ for each } c \in C\}$$

is called the **relative annihilator** of C with respect to B.

If $C = \{c\}$ is a singleton then $\langle C \rangle$ will be briefly denoted by $\langle c \rangle$. For a qoset $(Q, \leq)$ and $\emptyset \neq M \subseteq Q$ put $U(M) = \{x \in Q; m \leq x \text{ for each } m \in M\}$. In case $M = \{a_1, \ldots, a_n\}$ we also write $U(a_1, \ldots, a_n)$ instead of $U(M)$.

THEOREM 1. *Let $\mathcal{Q} = (Q, \leq, 1)$ be a standard QBCC-algebra and I be an ideal of $\mathcal{Q}$. Then $\langle I \rangle$ is also an ideal and a pseudocomplement of I in the lattice $Id(\mathcal{Q})$. Moreover,*

$$\langle I \rangle = \{x \in Q; x \parallel i \text{ for all } i \in I \backslash \{1\}\} \cup \{1\}.$$

Proof. At first we prove that $\langle I \rangle = \{x \in Q; x \parallel i \text{ for all } i \in I \backslash \{1\}\} \cup \{1\}$. Suppose $a \in \langle I \rangle$, i.e. $a \bullet i = i$ for all $i \in I$. Evidently, for every $i \in I \backslash \{1\}$ either $a \parallel i$ or $a \geq i$. If there exists $i \in I \backslash \{1\}$ with $a \geq i$ then we have by (I1) $a \in I$ and so $1 = a \bullet a = a$, proving that $a \in \{x \in Q; x \parallel i$ for all $i \in I \backslash \{1\}\} \cup \{1\}$. The converse inclusion is clear.

Further let us prove that $\langle I \rangle \in Id(\mathcal{Q})$. Suppose $x \in \langle I \rangle$ and $x \leq y \neq 1$. Then $y \leq i$ for some $i \in I \backslash \{1\}$ leads to $x \leq i$, contadicting $x \in \langle I \rangle$. The case $y \geq i$ for some $i \in I \backslash \{1\}$ means $y \in I$ which is also impossible due to $x \leq y$. This shows $y \in \langle I \rangle$. We have to show that (I2) holds. For this let $x \bullet y = x \in \langle I \rangle$ for some bridge (x, y). Let us note that $x = 1$ would imply $y = 1 \bullet y = 1$, hence it holds $x \neq 1$. Since (x, y) forms a bridge, the property $x \parallel i$ for each $i \in I \backslash \{1\}$ yields also $y \parallel i$ for each $i \in I \backslash \{1\}$, and so $y \in \langle I \rangle$.

It is evident that $I \cap \langle I \rangle = \{1\}$. Suppose that J is any ideal of $\mathcal{Q}$ with the property $I \cap J = \{1\}$. If $j \in J \backslash \{1\}, i \in I \backslash \{1\}$ then $i \parallel j$ otherwise either $i \leq j \in I \cap J$ or $j \leq i \in I \cap J$, a contradiction. This means that $J \subseteq \langle I \rangle$ and hence $\langle I \rangle$ is a pseudocomplement of I in $Id(\mathcal{Q})$. ∎

THEOREM 2. *Let B, C be ideals of a standard QBCC-algebra $\mathcal{Q} = (Q, \leq, 1)$. Then $\langle C, B \rangle$ is a relative pseudocomplement of C with respect to B in the lattice $Id(\mathcal{Q})$. Moreover,*

$$\langle C, B \rangle = \{x \in Q; x \parallel c \text{ for each } c \in C \backslash B\} \cup B.$$

Proof. At first we show that $\langle C,B\rangle = \{x \in Q; x \parallel c$ for each $c \in C\backslash B\} \cup B$. It is easily seen that $B \subseteq \langle C,B\rangle$. Suppose $x \parallel c$ for each $c \in C\backslash B$. Then $(x \bullet c) \bullet c = c \bullet c = 1 \in B$ for all $c \in C\backslash B$. In the remaining case we have also $(x \bullet d) \bullet d \in B$ whenever $d \in C \cap B$, and altogether $x \in \langle C,B\rangle$.

Conversely, suppose $y \in \langle C,B\rangle\backslash B$ and assume $y \not\parallel c$ for some $c \in C\backslash B$. If $y \leq c$ then $(y \bullet c) \bullet c = 1 \bullet c = c \in B$, a contradiction. In the case $y \geq c$ we conclude $y \in C$ and, moreover, $y = 1 \bullet y = (y \bullet y) \bullet y \in B$, which is also a contradiction. This proves $y \parallel c$ for each $c \in C\backslash B$.

Now we show that $\langle C,B\rangle$ is an ideal of $\mathcal{Q}$. Let $x \in \langle C,B\rangle$ and $x \leq y$. We have $y \in B$ whenever $x \in B$. Suppose further $x \parallel c$ for each $c \in C\backslash B$. It is clear that $y \not\leq c$ for any $c \in C\backslash B$, otherwise $x \leq c$. So let $y \geq c$ for some $c \in C\backslash B$. Then by (I1) also $y \in C$. Moreover, $y \in B \subseteq \langle C,B\rangle$ since in the opposite case we would have $x \parallel y$. In the remaining case $y \parallel c$ for each $c \in C\backslash B$, hence also $y \in \langle C,B\rangle$.

We prove that $\langle C,B\rangle$ satisfies (I2). Let $x \bullet y = x \in \langle C,B\rangle$ for some bridge (x,y). Then $y \in B$ whenever $x \in B$. Suppose further that $x \parallel c$ for each $c \in C\backslash B$ and assume $y \not\parallel c$ for some $c \in C\backslash B$. If $y \leq c$ we get $x > c$ or $x \leq c$ (since (x,y) is a bridge), a contradiction. Similarly, $c \leq y < x$ contradicts $c \parallel x$ and, finally, $\langle C,B\rangle$ is an ideal of $\mathcal{Q}$.

It is clear that $C \cap \langle C,B\rangle \subseteq B$. Let J be an ideal of $\mathcal{Q}$ with the property $C \cap J \subseteq B$ and assume $j \in J\backslash B$. Suppose further $j \not\parallel c$ for some $c \in C\backslash B$. If $c \leq j$, then $j \in C \cap J \subseteq B$, a contradiction. The case $j \leq c$ leads to the contradiction $c \in J \cap C \subseteq B$. This means that $J \subseteq \langle C,B\rangle$ and $\langle C,B\rangle$ is the relative pseudocomplement of C with respect to B in $Id(\mathcal{Q})$. ∎

There is a natural question to find conditions under which the annihilator of every non-void subset M of Q is equal to the annihilator of the ideal generated by M. We will show that the answer is closely connected with the following example:

EXAMPLE 3. Consider a qoset Q with a greatest element 1 where $Q\backslash\{1\}$ is composed by pairwise incomparable blocks $B_i, i \in \Omega$, being either a cell or $B_i = C(a_i) \cup C(b_i)$ for $a_i < b_i$, with $b_i \bullet a_i = b_i$. Such a standard QBCC-algebra will be called a **quasiimplication algebra**.

THEOREM 3. *For a standard QBCC-algebra* $\mathcal{Q} = (Q, \leq, 1)$ *the following conditions are equivalent:*

(a) *for each* $\emptyset \neq M \subseteq Q$ *it holds* $\langle M\rangle = \langle I(M)\rangle$,

(b) $\mathcal{Q}$ *is a quasiimplication algebra.*

Proof. (a)⇒(b): Take $M = \{c\}$ for $c \in Q\backslash\{1\}$. We know that $I(c)$, the principal ideal generated by $\{c\}$, is equal to $U(c)$ if there is no non-normal pair (c,d) with $c > d$ in Q or $I(c) = U(c) \cup \{d\}$ if such a pair exists (see [5]).

We will show that in both cases

$$(*)\qquad \langle I(c)\rangle = \{x \in Q; U(x,c) = \{1\}\}.$$

Suppose $x \in \langle I(c)\rangle$ and let $y \in U(x,c)$ be an arbitrary element. Then $y \in I(c)$, hence $1 = x \bullet y = y$ proving that $U(x,c) = \{1\}$.

Suppose conversely that $U(x,c) = \{1\}$ for some $x \in Q$ and let $y \in U(c)$. If $x \leq y$, then $y \in U(x,c) = \{1\}$ and hence $x \bullet y = x \bullet 1 = 1 = y$. Otherwise we have either $x \parallel y$ and $x \bullet y = y$ or $y < x$ and $x \in U(x,c) = \{1\}$ and $x \bullet y = 1 \bullet y = y$. Altogether we proved that $x \in \langle U(c)\rangle$. Finally, let (c,d) be a non-normal pair of Q with $c > d$, so $c \bullet d \neq d$. Let us prove that $x \in \langle d\rangle$. We have either $x = 1$ and $x \bullet d = 1 \bullet d = d$ or $x \parallel c$. In the latter case since (c,d) is a bridge, also $x \parallel d$ and $x \bullet d = d$, hence the equality $(*)$ is proved.

Consider now $b > c$ for some $b \in Q$. If the pair (b,c) is normal, then $b \bullet c = c$, hence $b \in \langle c\rangle = \langle I(c)\rangle$ which, by $(*)$, gives $U(b,c) = U(b) = \{1\}$ and $b = 1$. By Proposition 1 this means that Q contains at most three-element chains otherwise it would contain a normal pair (x,y) with $x,y \neq 1$. If $1 > b > c$ is a three-element chain of Q, the pair (b,c) cannot be normal, hence $b \bullet c = b$, i.e. $b \notin \langle c\rangle$ verifying that Q is a quasiimplication algebra.

(b)⇒(a): Suppose that $c \in B_i$ for B_i being a cell. Then we have $\langle c\rangle = Q \backslash B_i = \langle I(c)\rangle = \langle U(c)\rangle$.

Further let $B_i = C(a_i) \cup C(b_i)$ with $a_i < b_i$, and $b_i \bullet a_i = b_i$. In this case we have $I(x) = B_i \cup \{1\} = U(a_i)$ for each $x \in B_i$, hence $\langle I(x)\rangle = Q \backslash B_i = \langle x\rangle$.

Since the join of ideals is their set-theoretical union and $\langle C\rangle = \cap\{\langle c\rangle; c \in C\}$ for each $C \subseteq Q$, we are done. ∎

References

[1] W. J. Blok and D. Pigozzi, *Algebraizable logics*, Memoirs of the American Math. Soc. No. 396, Providence, Rhode Island 1989.

[2] W. A. Dudek, *The number of subalgebras of finite* BCC-*algebras*, Bull. Inst. Math., Academia Sinica 20(2) (1992), 129–135.

[3] I. Chajda and R. Halaš, *Pre-logics*, Math. Slovaca, to appear.

[4] R. Halaš, BCC-*algebras inherited from posets*, Multiple Valued Logic, to appear.

[5] R. Halaš and J. Ort, QBCC-*algebras inherited from qosets*, submitted.

[6] Y. Imai and K. Iséki, *On axiomatic system of propositional calculi* XIV, Proc. Japan Acad. 42 (1966), 19–22.

[7] Y. Komori, *The class of* BCC-*algebras do not form a variety*, Math. Japon. 29 (1984), 391–394.

[8] A. Wroński, *An algebraic motivation for* BCK-*algebras*, Math. Japon. 30 (1985), 183–193.
[9] A. Wroński BCK-*algebras do not form a variety*, Math. Japon. 28 (1983), 211–213.

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Received March 29, 2002; revised version June 6, 2002.