DEMONSTRATIO MATHEMATICA
Vol. XXXV No 4 2002

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AN ALMOST SURE MAXIMUM LIMIT THEOREM FOR CERTAIN CLASS OF DEPENDENT STATIONARY GAUSSIAN SEQUENCES

Abstract. Let $\{\xi_n, n \geq 1\}$ be a sequence of stationary standard normal random variables and $M_n = max(\xi_1, \dots, \xi_n)$.

Our goal is to prove that under some conditions on the covariance function $r(n) = Cov(\xi_1, \xi_{1+n})$ and for certain pair of numerical sequences (a_n), (b_n),

$$P\left(\lim_{N\to\infty} \sup_{-\infty<x<\infty} \left| \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I_{(-\infty,x]}\left(\frac{M_n - b_n}{a_n}\right) - \Lambda(x) \right| = 0 \right) = 1,$$

where: $\Lambda(x) = \exp(-e^{-x})$, for $-\infty < x < \infty$, $I_{(-\infty,x]}(\cdot)$-the indicator function of the set $(-\infty, x]$, $\log(\cdot)$-the natural logarithm.

1. Introduction

Leadbetter, Lindgren and Rootzen in [4] were concered with conditions under which, for suitable normalizing constants $a_n > 0$, b_n,

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) \xrightarrow{w} G(x),$$

where $M_n = \max(X_1, \dots, X_n)$, for some sequence $\{X_n, n \geq 1\}$ of random variables and $\xrightarrow{w}$ denotes the convergence at continuity points of G.

They showed that, if $\{X_n, n \geq 1\}$ are i.i.d., then the possible nondegenerate didstribution function G, which may appear as such a limit has (up to location and scale changes) one of the following three forms - commonly called *the* three *Extreme Value Distributions:*

$$\Lambda(x) = \exp(-e^{-x}), \quad -\infty < x < \infty;$$

Key words and phrases: maximum limit theorem; almost sure convergence; logarithmic means; dependent stationary Gaussian sequences.

2000 *Mathematics Subject Classification*: Primary 60F15; Secondary 60F05.

$$\Theta(x) = \begin{cases} 0, & x \le 0, \\ \exp(-x^{-\alpha}), \text{ for some } \alpha > 0, & x > 0; \end{cases}$$

$$\Psi(x) = \begin{cases} \exp(-(-x)^{\alpha}), \text{ for some } \alpha > 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

Natural question raised, whether assertions are possible for almost every realization of the random variables $\{X_n,\, n \ge 1\}$. In this case Cheng, Peng and Qi in [2] and simultaneously Fahrner and Stadtmuller in [3] considered the sequence $I_{(-\infty,x]}(\frac{M_n-b_n}{a_n})$, where $\{X_n,\, n \ge 1\}$ are i.i.d. and $I_{(-\infty,x]}(\cdot)$ denotes the indicator function of the set $(-\infty, x]$. Since the sequence $I_{(-\infty,x]}(\frac{M_n-b_n}{a_n})$ does not converge almost surely for any x satisfying $0 < G(x) < 1$, they investigated the limitation of $I_{(-\infty,x]}(\frac{M_n-b_n}{a_n})$ by logarithmic means and showed that, if $P(\frac{M_n-b_n}{a_n} \le x) \xrightarrow{w} G(x)$, then

$$P\Bigg(\lim_{N\to\infty} \sup_{\{x:\ 0<G(x)<1\}} \left|\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n} I_{(-\infty,x]}\left(\frac{M_n-b_n}{a_n}\right) - G(x)\right| = 0\Bigg) = 1.$$

The purpose of this paper is to prove similar result for some class of dependent stationary standard normal random variables $\{\xi_n,\, n \ge 1\}$. In this case, we assume that

$$a_n = \frac{1}{(2\log n)^{\frac{1}{2}}}, \quad b_n = (2\log n)^{\frac{1}{2}} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{\frac{1}{2}}}, \text{ for } n \ge 2,$$

$$0 < a_1, a_2 < \infty, \quad -\infty < b_1, b_2 < \infty,$$

$$G(x) = \Lambda(x) = \exp(-e^{-x}), \quad -\infty < x < \infty.$$

The choice of a_n, b_n is not accidental. Berman in [1] has given simple conditions on the covariance function $r(n) = Cov(\xi_1, \xi_{1+n})$, which ensure that for such defined a_n and b_n,

$$P\left(\frac{M_n - b_n}{a_n} \le x\right) \xrightarrow{w} \Lambda(x).$$

One of Berman's results is that it suffices that

$$\sum_{n=1}^{\infty} r^2(n) < \infty.$$

It is easily seen that this is fulfilled, if

$$\max_{n\ge1} |r(n)| = \delta < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |r(n)| < \infty.$$

Both the conditions above will be used as the assumptions of our theorem.

2. Main result

We shall consider the limitation of $I_{(-\infty,x]}(\frac{M_n-b_n}{a_n})$ by logarithmic means and prove the following theorem.

THEOREM 1. *Let $\{\xi_n, n \geq 1\}$ be a sequence of stationary standard normal random variables, $M_n = \max(\xi_1, \dots, \xi_n)$. Assume that the covariance function $r(n) = Cov(\xi_1, \xi_{1+n})$ is such that*

$$\max_{n\geq 1} |r(n)| = \delta < 1, \tag{1}$$

$$\sum_{n=1}^{\infty} |r(n)| < \infty. \tag{2}$$

Then, we have

$$P\Big(\lim_{N\to\infty} \sup_{-\infty<x<\infty} \Big|\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n} I_{(-\infty,x]}\Big(\frac{M_n-b_n}{a_n}\Big) - \Lambda(x)\Big| = 0\Big) = 1, \tag{3}$$

where

$$a_n = \frac{1}{(2\log n)^{\frac{1}{2}}}, \quad b_n = (2\log n)^{\frac{1}{2}} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{\frac{1}{2}}}, \text{ for } n \geq 2,$$
$$0 < a_1, a_2 < \infty, \ -\infty < b_1, b_2 < \infty$$

and

$$\Lambda(x) = \exp(-e^{-x}), \quad -\infty < x < \infty,$$
$$I_{(-\infty,x]}(\cdot) - \textit{the indicator function of the set } (-\infty, x].$$

Proof. In our derivations C, $C(x)$ denote some non-negative constants, which can vary from line to line ($C(x)$ depends on the fixed real number x) and $I_x(\cdot)$ stands for the indicator function $I_{(-\infty,x]}(\cdot)$.

Set

$$K(N) = \sum_{n=1}^{N}\frac{1}{n}, \tag{4}$$

$$S_N(x) = \frac{1}{K(N)}\sum_{n=1}^{N}\frac{1}{n} I_x\Big(\frac{M_n-b_n}{a_n}\Big). \tag{5}$$

Obviously, we have that

$$\sup_{-\infty<x<\infty}\Big|\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n} I_x\Big(\frac{M_n-b_n}{a_n}\Big) - S_N(x)\Big|$$
$$\leq \Big|\frac{K(N)}{\log N} - 1\Big| \to 0 \quad \text{a.s. (as } N\to\infty).$$

Thus, to show (3), it suffices to prove that

$$\lim_{N\to\infty} \sup_{-\infty<x<\infty} |S_N(x) - \Lambda(x)| = 0 \quad \text{a.s..} \tag{6}$$

Let us notice that, as Λ is continous, condition (6) is equivalent to

$$P\{\omega : \lim_{N\to\infty} S_N(x) = \Lambda(x), \text{ for all } -\infty < x < \infty\} = 1. \tag{7}$$

We now fix $-\infty < x < \infty$. Let for $1 \le j \le n$,

$$g_{jn} = E\Big[I_x\Big(\frac{M_j - b_j}{a_j}\Big) - \Lambda(x)\Big]\Big[I_x\Big(\frac{M_n - b_n}{a_n}\Big) - \Lambda(x)\Big]. \tag{8}$$

We shall write g_{jn} in the following way

$$g_{jn} = E\Big\{\Big[I_x\Big(\frac{M_j - b_j}{a_j}\Big) - I_x\Big(\frac{\widehat{M}_j - b_j}{a_j}\Big)\Big] + \Big[I_x\Big(\frac{\widehat{M}_j - b_j}{a_j}\Big) - \Lambda(x)\Big]\Big\}$$
$$\times \Big\{\Big[I_x\Big(\frac{M_n - b_n}{a_n}\Big) - I_x\Big(\frac{\widehat{M}_n - b_n}{a_n}\Big)\Big] + \Big[I_x\Big(\frac{\widehat{M}_n - b_n}{a_n}\Big) - \Lambda(x)\Big]\Big\},$$

where $\widehat{M}_n = \max(\widehat{\xi}_1, \ldots, \widehat{\xi}_n)$ and $\{\widehat{\xi}_n, n \ge 1\}$ are i.i.d. standard normal random variables.

Let us denote

$$u_n(x) = a_n x + b_n.$$

It is easy to check that

$$\begin{aligned} g_{jn} &= [P(M_j \le u_j(x), M_n \le u_n(x)) - P(\widehat{M}_j \le u_j(x), \widehat{M}_n \le u_n(x))] \\ &\quad + \Lambda(x)[P(\widehat{M}_j \le u_j(x)) - P(M_j \le u_j(x))] \\ &\quad + \Lambda(x)[P(\widehat{M}_n \le u_n(x)) - P(M_n \le u_n(x))] + \widehat{g}_{jn}, \end{aligned}$$

where

$$\widehat{g}_{jn} := E\Big[I_x\Big(\frac{\widehat{M}_j - b_j}{a_j}\Big) - \Lambda(x)\Big]\Big[I_x\Big(\frac{\widehat{M}_n - b_n}{a_n}\Big) - \Lambda(x)\Big].$$

Hence, we have that

$$\begin{aligned} |g_{jn}| &\le |P(M_j \le u_j(x), M_n \le u_n(x)) - P(\widehat{M}_j \le u_j(x), \widehat{M}_n \le u_n(x))| \\ &\quad + |P(M_j \le u_j(x)) - P(\widehat{M}_j \le u_j(x))| \\ &\quad + |P(M_n \le u_n(x)) - P(\widehat{M}_n \le u_n(x))| + |\widehat{g}_{jn}| \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{9}$$

On the other hand, from definition of $u_n(x)$ and sequences (a_n), (b_n), we have that for $n \ge 2$,

$$u_n(x) = \frac{x}{(2\log n)^{\frac{1}{2}}} + (2\log n)^{\frac{1}{2}} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{\frac{1}{2}}}$$

and it is easy to check that $u_n(x) \geq u_j(x)$, for $n \geq j > h(x)$, where

$$h(x) = \left[\max\left\{\exp\left(\frac{x}{2} + \frac{1}{2} - \frac{\log 4\pi}{4}\right), 2\right\}\right] \tag{10}$$

and [ ] denotes in this case the integer part of the number.

Hence, for $n \geq j > h(x)$, A_1 in (9) can be written as follows

$$A_1 = |P(\xi_1 \leq u_j(x), \ldots, \xi_j \leq u_j(x), \xi_{j+1} \leq u_n(x), \ldots, \xi_n \leq u_n(x)) \\ - P(\widehat{\xi}_1 \leq u_j(x), \ldots, \widehat{\xi}_j \leq u_j(x), \widehat{\xi}_{j+1} \leq u_n(x), \ldots, \widehat{\xi}_n \leq u_n(x))|.$$

Due to the fact that $\{\xi_n, n \geq 1\}$ are standard normal and stationary, we have from Theorem 4.2.1 in [4] that, if δ is such as in assumption (1), then

$$A_1 \leq C \sum_{1 \leq k < l \leq j} |r(l-k)| \exp\left(-\frac{u_j^2(x)}{1+\delta}\right) \\ + C \sum_{\substack{1 \leq k \leq j \\ j < l \leq n}} |r(l-k)| \exp\left(-\frac{1}{2}\frac{u_j^2(x) + u_n^2(x)}{1+\delta}\right) \\ + C \sum_{j < k < l \leq n} |r(l-k)| \exp\left(-\frac{u_n^2(x)}{1+\delta}\right).$$

Set

$$\gamma = \frac{1}{1+\delta}. \tag{11}$$

Using notation (11), we can write that

$$A_1 \leq Cj \exp(-\gamma u_j^2(x)) \sum_{t=1}^{j-1} |r(t)| \\ + Cj \exp\left(-\frac{1}{2}\gamma u_j^2(x)\right) \exp\left(-\frac{1}{2}\gamma u_n^2(x)\right) \sum_{t=1}^{n-1} |r(t)| \\ + Cn \exp(-\gamma u_n^2(x)) \sum_{t=1}^{n-j-1} |r(t)|$$

and by assumption (2),

$$A_1 \leq C\Big[j \exp\left(-\gamma u_j^2(x)\right) + j \exp\left(-\frac{1}{2}\gamma u_j^2(x)\right) \exp\left(-\frac{1}{2}\gamma u_n^2(x)\right) \\ + n \exp(-\gamma u_n^2(x))\Big]. \tag{12}$$

On the other hand, from definition of $u_j(x)$, we have that

$$\exp(-\gamma u_j^2(x)) \leq C(x) \frac{1}{j^{2\gamma}} (\log j)^{\gamma}. \tag{13}$$

(12) and (13) imply that

$$(14)\quad A_1 \le C(x)\Big[j\frac{1}{j^{2\gamma}}(\log j)^\gamma + j\frac{1}{j^\gamma}(\log j)^{\frac{\gamma}{2}}\frac{1}{n^\gamma}(\log n)^{\frac{\gamma}{2}} + n\frac{1}{n^{2\gamma}}(\log n)^\gamma\Big].$$

Using Theorem 4.2.1 in [4] again and similar calculations as by the estimation of A_1, it is easy to find the estimates for A_2 and A_3 in (9). We have that

$$(15)\qquad A_2 \le C(x)j\frac{1}{j^{2\gamma}}(\log j)^\gamma$$

and

$$(16)\qquad A_3 \le C(x)n\frac{1}{n^{2\gamma}}(\log n)^\gamma.$$

(14)–(16) yield that for $n \ge j > h(x)$,

$$(17)\quad A_1 + A_2 + A_3 \le C\Big[j\frac{1}{j^{2\gamma}}(\log j)^\gamma + j\frac{1}{j^\gamma}(\log j)^{\frac{\gamma}{2}}\frac{1}{n^\gamma}(\log n)^{\frac{\gamma}{2}} + n\frac{1}{n^{2\gamma}}(\log n)^\gamma\Big].$$

Thus, it remains to estimate $A_4 = |\widehat{g}_{jn}|$. As

$$A_4 = \Big|E\Big[I_x\Big(\frac{\widehat{M}_j - b_j}{a_j}\Big) - \Lambda(x)\Big]\Big[I_x\Big(\frac{\widehat{M}_n - b_n}{a_n}\Big) - \Lambda(x)\Big]\Big|,$$

it is easy to check that for $n \ge j > h(x)$,

$$(18)\quad \begin{aligned} A_4 &\le |P(\widehat{\xi}_1 \le u_j(x), \ldots, \widehat{\xi}_j \le u_j(x), \widehat{\xi}_{j+1} \le u_n(x), \ldots, \widehat{\xi}_n \le u_n(x)) \\ &\qquad - \Lambda(x)P(\widehat{M}_j \le u_j(x))| + |P(\widehat{M}_n \le u_n(x)) - \Lambda(x)| \\ &= D_1 + D_2. \end{aligned}$$

First, we estimate D_1. Using the fact that $\{\widehat{\xi}_n, n \ge 1\}$ are i.i.d. standard normal random variables, we obtain that

$$\begin{aligned} D_1 &= |P(\widehat{\xi}_1 \le u_j(x), \ldots, \widehat{\xi}_j \le u_j(x))P(\widehat{\xi}_{j+1} \le u_n(x), \ldots, \widehat{\xi}_n \le u_n(x)) \\ &\qquad\qquad - \Lambda(x)P(\widehat{M}_j \le u_j(x))| \\ &\le |P(\widehat{\xi}_{j+1} \le u_n(x), \ldots, \widehat{\xi}_n \le u_n(x)) - \Lambda(x)| \\ &= |\Phi^{n-j}(u_n(x)) - \Lambda(x)| \\ &= |[\Phi^{n-j}(u_n(x)) - \Phi^{n-j}(u_{n-j}(x))] + [\Phi^{n-j}(u_{n-j}(x)) - \Lambda(x)]|, \end{aligned}$$

where Φ is the standard normal distribution function. Hence

$$(19)\quad D_1 \le |\Phi^{n-j}(u_n(x)) - \Phi^{n-j}(u_{n-j}(x))| + |\Phi^{n-j}(u_{n-j}(x)) - \Lambda(x)|.$$

From derivation on p. 39 in [4], we have

$$\Phi(u_n(x)) = 1 - \frac{e^{-x}}{n}\Big[1 - \frac{(\log\log n)^2}{16\log n}(1 + o(1))\Big].$$

This implies that for $n-j>1$,

$$(20)\quad |\Phi(u_n(x))-\Phi(u_{n-j}(x))| \le C(x)\left[\frac{j}{(n-j)n}+\frac{(\log\log n)^2}{n\log n}+\frac{(\log\log(n-j))^2}{(n-j)\log(n-j)}\right].$$

Besides, it is easily seen that

$$(21)\ |\Phi^{n-j}(u_n(x))-\Phi^{n-j}(u_{n-j}(x))| \le |\Phi(u_n(x))-\Phi(u_{n-j}(x))|(n-j).$$

From (20) and (21), we obtain that for $n-j>1$,

$$(22)\quad |\Phi^{n-j}(u_n(x))-\Phi^{n-j}(u_{n-j}(x))| \le C(x)\left[\frac{j}{n}+\frac{(\log\log n)^2}{\log n}+\frac{(\log\log(n-j))^2}{\log(n-j)}\right].$$

By (2.4.8) p. 39 in [4],

$$\Phi^{n-j}(u_{n-j}(x))-\Lambda(x)\sim\frac{\exp(-e^{-x})e^{-x}}{16}\frac{(\log\log(n-j))^2}{\log(n-j)}.$$

Hence, we can write that for $n-j>1$,

$$(23)\quad |\Phi^{n-j}(u_{n-j}(x))-\Lambda(x)|\le C(x)\frac{(\log\log(n-j))^2}{\log(n-j)}.$$

From (19), (22) and (23), we have that for $n-j>1$,

$$(24)\quad D_1\le C(x)\left[\frac{j}{n}+\frac{(\log\log n)^2}{\log n}+\frac{(\log\log(n-j))^2}{\log(n-j)}\right].$$

Applying again (2.4.8) p. 39 in [4], we have that for $n>1$, D_2 in (18) can be estimated as follows

$$(25)\quad D_2\le C(x)\frac{(\log\log n)^2}{\log n}.$$

(18), (24) and (25) imply that, if $h(x)<j<n-1$, then

$$(26)\quad A_4=|\widehat{g}_{jn}|\le C(x)\left[\frac{j}{n}+\frac{(\log\log n)^2}{\log n}+\frac{(\log\log(n-j))^2}{\log(n-j)}\right].$$

Finally from (9), (17) and (26), we obtain that, there exists constant $C(x)$, such that for $h(x)<j<n-1$,

$$(27)\quad |g_{jn}|\le C(x)\Big[j\frac{1}{j^{2\gamma}}(\log j)^{\gamma}+j\frac{1}{j^{\gamma}}(\log j)^{\frac{\gamma}{2}}\frac{1}{n^{\gamma}}(\log n)^{\frac{\gamma}{2}} + n\frac{1}{n^{2\gamma}}(\log n)^{\gamma}+\frac{j}{n}+\frac{(\log\log(n-j))^2}{\log(n-j)}+\frac{(\log\log n)^2}{\log n}\Big],$$

where γ is defined in (11).

Now, let us notice that, if $K(N)$ and $S_N(x)$ are defined such as in (4) and (5), respectively, then

$$[S_N(x) - \Lambda(x)]^2 = \left\{ \frac{1}{K(N)} \sum_{n=1}^{N} \frac{1}{n} \left[I_x\left(\frac{M_n - b_n}{a_n} \right) - \Lambda(x) \right] \right\}^2$$
$$\leq \frac{2}{K^2(N)} \sum_{n=1}^{N} \sum_{j=1}^{n} \frac{1}{jn} \left[I_x\left(\frac{M_j - b_j}{a_j} \right) - \Lambda(x) \right] \left[I_x\left(\frac{M_n - b_n}{a_n} \right) - \Lambda(x) \right].$$

This and definition of g_{jn} in (8) imply that

$$E[S_N(x) - \Lambda(x)]^2 \leq \frac{2}{K^2(N)} \sum_{n=1}^{N} \sum_{j=1}^{n} \frac{|g_{jn}|}{jn}$$
$$= \frac{2}{K^2(N)} \sum_{n=1}^{h(x)} \sum_{j=1}^{n} \frac{|g_{jn}|}{jn} + \frac{2}{K^2(N)} \sum_{n=1+h(x)}^{N} \sum_{j=1}^{h(x)} \frac{|g_{jn}|}{jn}$$
$$+ \frac{2}{K^2(N)} \sum_{n=1+h(x)}^{N} \sum_{j=1+h(x)}^{n} \frac{|g_{jn}|}{jn}.$$

Let us write it as follows

$$E[S_N(x) - \Lambda(x)]^2 \leq \frac{2}{K^2(N)} \sum_{n=1}^{h(x)} \sum_{j=1}^{n} \frac{|g_{jn}|}{jn} \tag{28}$$
$$+ \frac{2}{K^2(N)} \sum_{n=1+h(x)}^{N} \sum_{j=1}^{h(x)} \frac{|g_{jn}|}{jn}$$
$$+ \frac{2}{K^2(N)} \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\leq j\leq n-2} \frac{|g_{jn}|}{jn}$$
$$+ \frac{2}{K^2(N)} \sum_{n=1+h(x)}^{N} \sum_{j=n-1}^{n} \frac{|g_{jn}|}{jn}$$
$$= F_1 + F_2 + F_3 + F_4.$$

We now estimate all the components F_1, F_2, F_3, F_4. For abbreviation, we introduce the following notation

$$C_N(x) = \frac{C(x)}{K^2(N)}, \tag{29}$$

where, for recollection, $C(x)$ denotes any non-negative constant, depending on x. It is easily seen that

$$F_1 + F_2 \leq C_N(x) \log N. \tag{30}$$

Thus, we need to estimate F_3 and F_4. From (28) and (27), we have

$$F_3 \le C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\le j\le n-2} \frac{(\log j)^\gamma}{nj^{2\gamma}} \tag{31}$$

$$+ C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\le j\le n-2} \frac{(\log n)^{\frac{\gamma}{2}}(\log j)^{\frac{\gamma}{2}}}{n^{1+\gamma}j^\gamma}$$

$$+ C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\le j\le n-2} \frac{(\log n)^\gamma}{n^{2\gamma}j}$$

$$+ C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\le j\le n-2} \frac{1}{nj}\frac{j}{n}$$

$$+ C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\le j\le n-2} \frac{(\log\log(n-j))^2}{nj\log(n-j)}$$

$$+ C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+h(x)\le j\le n-2} \frac{(\log\log n)^2}{nj\log n}$$

$$= G_1 + G_2 + G_3 + G_4 + G_5 + G_6.$$

We now estimate all the components G_1, G_2, G_3, G_4, G_5, G_6.

$$G_1 \le C_N(x) \sum_{n=1}^{N} \frac{1}{n} \sum_{j=1}^{n} \frac{(\log j)^\gamma}{j^{2\gamma}} \le C_N(x)(\log N)^\gamma \sum_{n=1}^{N} \frac{1}{n} \sum_{j=1}^{\infty} \frac{1}{j^{2\gamma}}.$$

By definition of γ in (11), $2\gamma > 1$. Hence $\sum_{j=1}^{\infty} \frac{1}{j^{2\gamma}} < \infty$ and

$$G_1 \le C_N(x)(\log N)^{\gamma+1}. \tag{32}$$

Similarly we can estimate G_2, G_3. We have

$$G_2 \le C_N(x) \sum_{n=1}^{N} \frac{(\log n)^{\frac{\gamma}{2}}}{n^{1+\gamma}} \sum_{j=1}^{n} \frac{(\log j)^{\frac{\gamma}{2}}}{j^\gamma}$$

$$\le C_N(x)(\log N)^\gamma \sum_{n=1}^{N} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j^{2\gamma}} \le C_N(x)(\log N)^\gamma \sum_{n=1}^{N} \frac{1}{n} \sum_{j=1}^{\infty} \frac{1}{j^{2\gamma}}$$

and hence

$$G_2 \le C_N(x)(\log N)^{\gamma+1}. \tag{33}$$

$$G_3 \le C_N(x) \sum_{n=1}^{N} \frac{(\log n)^\gamma}{n^{2\gamma}} \sum_{j=1}^{n} \frac{1}{j} \le C_N(x)(\log N)^\gamma \sum_{n=1}^{N} \frac{1}{n^{2\gamma}} \sum_{j=1}^{n} \frac{1}{j}$$

$$\le C_N(x)(\log N)^\gamma \sum_{n=1}^{\infty} \frac{1}{n^{2\gamma}} \sum_{j=1}^{N} \frac{1}{j}$$

and hence

$$G_3 \leq C_N(x)(\log N)^{\gamma+1}. \tag{34}$$

It is also easy to estimate G_4. We namely have

$$G_4 \leq C_N(x) \sum_{n=1}^{N} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j}\frac{j}{n} = C_N(x) \sum_{n=1}^{N} \frac{1}{n}$$

and

$$G_4 \leq C_N(x) \log N. \tag{35}$$

To get a bound for G_5, we use the fact that

$$(\log\log t)^2 \leq (\log t)^{\gamma}, \text{ for all sufficiently large } t. \tag{36}$$

Therefore

$$\begin{aligned} G_5 &\leq C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j=1}^{n-2} \frac{(\log(n-j))^{\gamma}}{nj\log(n-j)} \\ &= C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j=1}^{n-2} \frac{1}{nj(\log(n-j))^{1-\gamma}}. \end{aligned}$$

Hence, we can write that

$$\begin{aligned} G_5 &\leq C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1\leq j\leq [\frac{n}{2}]} \frac{1}{nj(\log(n-j))^{1-\gamma}} \\ &\quad + C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+[\frac{n}{2}]\leq j\leq n-2} \frac{1}{nj(\log(n-j))^{1-\gamma}} \\ &\leq C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1\leq j\leq [\frac{n}{2}]} \frac{1}{nj(\log\frac{n}{2})^{1-\gamma}} \\ &\quad + C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j:1+[\frac{n}{2}]\leq j\leq n-2} \frac{1}{nj(\log 2)^{1-\gamma}}, \end{aligned}$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

This implies that

$$\begin{aligned} G_5 \leq C_N(x) &\sum_{n=1+h(x)}^{N} \frac{1}{n}\frac{1}{(\log\frac{n}{2})^{1-\gamma}} \sum_{1\leq j\leq [\frac{n}{2}]} \frac{1}{j} \\ &+ C_N(x) \sum_{n=1+h(x)}^{N} \frac{1}{n}\frac{1}{(\log 2)^{1-\gamma}} \sum_{j:1+[\frac{n}{2}]\leq j\leq n-2} \frac{1}{j} \end{aligned}$$

$$\le C_N(x) \sum_{n=1+h(x)}^{N} \frac{1}{n} \frac{1}{(\log \frac{n}{2})^{1-\gamma}} \log \frac{n}{2} + C_N(x) \sum_{n=1+h(x)}^{N} \frac{1}{n} \frac{1}{(\log 2)^{1-\gamma}} \log 2$$

$$= C_N(x) \sum_{n=1+h(x)}^{N} \frac{1}{n} \Big(\log \frac{n}{2} \Big)^{\gamma} + C_N(x) \sum_{n=1+h(x)}^{N} \frac{1}{n} (\log 2)^{\gamma}$$

$$\le C_N(x) (\log N)^{\gamma} \sum_{n=1}^{N} \frac{1}{n}$$

and hence

$$G_5 \le C_N(x)(\log N)^{\gamma+1}. \tag{37}$$

Finally, applying (36) again, we obtain that

$$G_6 \le C_N(x) \sum_{n=1+h(x)}^{N} \sum_{j=1}^{n-2} \frac{(\log n)^{\gamma}}{nj \log n} \le C_N(x) \sum_{n=1+h(x)}^{N} \frac{(\log n)^{\gamma-1}}{n} \sum_{j=1}^{n} \frac{1}{j}$$

$$\le C_N(x) \sum_{n=1}^{N} \frac{(\log n)^{\gamma}}{n} \le C_N(x)(\log N)^{\gamma} \sum_{n=1}^{N} \frac{1}{n}$$

and hence

$$G_6 \le C_N(x)(\log N)^{\gamma+1}. \tag{38}$$

From (31)–(35), (37) and (38), we have that

$$F_3 \le C_N(x)(\log N)^{\gamma+1}. \tag{39}$$

Besides, it is very easy to check that F_4 in (28) satisfies the inequality

$$F_4 \le C_N(x) \log N. \tag{40}$$

By (28), (30), (39) and (40) $E[S_N(x) - \Lambda(x)]^2 \le C_N(x)(\log N)^{\gamma+1}$. Using notation on $C_N(x)$ in (29), $K(N)$ in (4) and γ in (11), we obtain that, there exists some non-negative constant $C(x)$, such that

$$E[S_N(x) - \Lambda(x)]^2 \le \frac{C(x)}{(\log N)^{1-\gamma}} = \frac{C(x)}{(\log N)^{\frac{\delta}{1+\delta}}}. \tag{41}$$

From (41) and Chebyshev's inequality, we have

$$P(|S_N(x) - \Lambda(x)| > \varepsilon) \le \frac{C(x)}{\varepsilon^2 (\log N)^{\frac{\delta}{1+\delta}}}. \tag{42}$$

Now, we put $N_k = [e^{k^m} + 1]$, for some natural number $m > \frac{1+\delta}{\delta}$, where δ is such as above and $[e^{k^m} + 1]$ denotes the integer part of $e^{k^m} + 1$. Then (42) implies that

$$\sum_{k=1}^{\infty} P(|S_{N_k}(x) - \Lambda(x)| > \varepsilon) \le \frac{C(x)}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{m\delta}{1+\delta}}} < \infty, \quad \text{as } \frac{m\delta}{1+\delta} > 1.$$

From the Borel-Cantelli lemma, we conclude that

$$S_{N_k}(x) - \Lambda(x) \underset{k\to\infty}{\longrightarrow} 0 \quad \text{a.s.}.$$

On the other hand, we have that for $N_k < N < N_{k+1}$,

$$|S_N(x) - S_{N_k}(x)| \le 2\frac{K(N_{k+1}-1) - K(N_k)}{K(N_k)} \sim 2\frac{1}{\log N_k}\log\frac{N_{k+1}-1}{N_k}.$$

Hence, we can write that

$$\begin{aligned}|S_N(x) - S_{N_k}(x)| &\le C\frac{1}{k^m}\log\frac{e^{(k+1)^m}}{e^{k^m}} = C\frac{1}{k^m}[(k+1)^m - k^m]\\ &= C\frac{[(k+1)^{m-1}+(k+1)^{m-2}k+\ldots+(k+1)k^{m-2}+k^{m-1}]}{k^m}\\ &\le C\frac{m(k+1)^{m-1}}{k^m} = Cm\Big(1+\frac{1}{k}\Big)^m\frac{1}{k+1} \to 0 \text{ as } k\to\infty.\end{aligned}$$

Thus, we have

$$\lim_{N\to\infty} S_N(x) = \Lambda(x) \quad \text{a.s.}.$$

Set $\Gamma_q = \{\omega \in \Omega : \lim_{N\to\infty} S_N(q) = \Lambda(q)\}$. Then $P(\Gamma_q) = 1$. Now, write $\Gamma = \bigcap_q \Gamma_q$, where the intersection takes over all rational numbers q. Then $P(\Gamma) = 1$. Noting that Λ is continous for all $-\infty < x < \infty$, the set of rational numbers is dense in $(-\infty,\infty)$ and $S_N(x)$ are monotonous functions, we have (7) and (6) and the proof is completed. ■

Acknowledgement. I wish to thank Professor W. Dziubdziela for his comments and support.

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Received January 10, 2002.