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MEASURABLE LINEAR OPERATORS INDUCED BY STOCHASTIC PROCESSES

Abstract. The purpose of this paper is to describe the structure of some class of measurable linear operators on the L^2–space with the probability measure induced by a stochastic process with independent increments. Our results extends the similar fact that has been considered in [3] and [5] for measurable linear functionals.

1. Introduction

Let X and Y be real separable complete locally convex linear metric spaces and let μ be a Borel probability measure on X. Denote by $\mathcal{B}(X)$ the Borel σ–algebra on X and by $\mathcal{B}_\mu(X)$ the completion in measure μ of $\mathcal{B}(X)$.

An operator A defined on a linear subset $D_A \subset X$ with values in Y is called a *μ—measurable linear operator* if:

(a) $D_A \in \mathcal{B}_\mu(X)$ and $\mu(D_A) = 1$
(b) A is a measurable mapping with respect to $(\mathcal{B}_\mu(X), \mathcal{B}(Y))$
(c) A is linear on D_A.

If $Y = R$ then we will say simply functional instead operator.

For the theory of measurable linear functionals and operators we refer to papers [5], [8] and [9].

Urbanik in [5] considered so-called Lusin measurable linear functionals as limits of sequences of continuous linear functionals with respect to the convergence μ—almost everywhere. Therefore μ-measurable linear functional f on X is a Lusin functional if there is a sequence $\{f_n\}$ of continuous linear functionals on X such that $f_n \to f$ μ - a.e. (or equivalently in the measure μ).

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Analogously we can define Lusin measurable linear operators (see [8]). Namely, if A is a μ–measurable linear operator from X into Y then we say that A is a *Lusin operator* if there exists a sequence $\{A_n\}$ of continuous linear operators from X into Y such that $A_n \to A$ μ – a.e..

A particular meaning in the theory of measurable linear operators possess so–called strongly measurable linear operators. Let $X = Y$ and let A be a μ–measurable linear operator on X (i.e. from $D_A \subset X$) into X. Denote by μ_A an image of μ under the operator A, i.e. the measure on X given by the formula $\mu_A(B) = \mu(A^{-1}(B))$ for $B \in \mathcal{B}(X)$. We will say that A is a *strongly measurable linear operator* if A is an one-to-one mapping and the measures $\mu$and μ_A are equivalent, (i.e $\mu(B) = 0$ iff $\mu_A(B) = 0$).

The aim of the present paper is to investigate the structure of strongly measurable linear Lusin operators for probability measures on the L^2–space over the unit interval, induced by symmetric, homogeneous, separable and continuous in probability stochastic processes with independent increments. It is well known that the probability measure induced by such process is in fact concentrated on the subset of L^2 consisting of bounded functions having no discontinuities of the second kind (see [1, Theorem 7.2]).

Let therefore $\{x(t) : 0 \leq t \leq 1\}$ be a symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments satisfying the initial condition $x(0) = 0$. This process defines a random measure M with independent values by means of the formula $M((a,b]) = x(b) - x(a)$.

For the definition of an integral with respect to the random measure M we refer to paper [6]. In the sequel we shall use the notation

$$\int_0^1 \varphi(t)dx(t) = \int_0^1 \varphi(t)M(dt).$$

The algebraic and topological structure of the space of all M–integrable functions was determined in [6]. Namely, this space is homeomorphic and linearly isomorphic to the Orlicz space $L(\psi)$ of all Borel functions φ on [0,1] satisfying the condition

$$\int_0^1 \psi(|\varphi(t)|)dt < \infty$$

where

$$\psi(t) = \int_{1/t}^{\infty} \frac{G(u)}{u^3} du$$

and G is the Lévy–Khinchine function of the process in question determined by means of the formula for the characteristic function γ of the increment

$x(b) - x(a)$:

$$\gamma(t) = \exp(b-a) \int_0^\infty (\cos tu - 1) \frac{1+u^2}{u^2} dG(u)$$

and the condition $G(0) = 0$.

Urbanik in [5] and Nguen Chi Bao in [3] showed the following theorem about the structure of Lusin functionals in the considered case.

THEOREM 1. ([3], [5]). *If μ is a probability measure on the L^2-space over the unit interval, induced by a symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments, then each Lusin functional f on L^2 is of the form*

$$f(x) = \int_0^1 \varphi(t) dx(t)$$

where φ belongs to the Orlicz space $L(\psi)$.

It is remarkable that the assumption in the above theorem, that f is a Lusin functional is essential, i.e. this theorem is not true for any μ–measurable linear functional on L^2 (see [2]).

2. The main results

In this section we will extend the Theorem 1 to the case of measurable linear operators. We prove the theorem which describes the structure of strongly measurable linear Lusin operators on L^2.

In the first place we must nevertheless define a quantity, which for almost all functions $x \in L^2$ and for any $t \in [0,1]$ can be regarded as the value of a function X at the point t.

Let thus, just in Sect.1, μ be a probability measure on the L^2–space over the unit interval induced by the the stochastic process in question. Let D be the subset of L^2 consisting of bounded functions having no discontinuities of the second kind, that is

$$D = \{x \in L^2 : x(t-0), x(t+0) \text{ exist for every } t \in [0,1]\}$$

where $x(t-0)$ and $x(t+0)$ denote the left–hand and the right–hand limit of a function x at the point t.

As mentioned in Sect.1 $\mu(D) = 1$. For $x \in D$ we put

$$x^*(t) = \frac{1}{2}[x(t-0) + x(t+0)] \quad \text{for } t \in [0,1].$$

For almost all trajectories of the process in question the set of point of their discontinuities is countable (see [4, Sect.11]). Hence for μ–almost all functions $x \in L^2$ $x^*(t) = x(t)$ almost everywhere with respect to the Lebesgue measure on [0,1]. Therefore for μ–almost all functions $x \in L^2$ the quantity $x^*(t)$ we can in fact consider as a value of a function x at the point t, for any $t \in [0,1]$.

Our main results reads as follows.

THEOREM 2. *Let μ be a probability measure on the L^2–space over the unit interval, induced by a symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments. If A is strongly measurable Lusin operator on L^2 then there exists a function $K(s,t) : [0,1] \times [0,1] \longrightarrow R$ such that*

$$Ax(t) = \int_0^1 K(s,t)dx(s)$$

where $K(\cdot,t) \in L(\psi)$ for every $t \in [0,1]$ ($L(\psi)$ is the Orlicz space defined in Sect.1).

Proof. Let a point $t \in [0,1]$ be fixed and define a functional $f_t : L^2 \to R$ by the formula

$$f_t(x) = Ax(t) = y^*(t) \tag{1}$$

where $y = Ax$.

Let us remark that the functional f_t is correctly defined on a set of the full measure. Indeed, let $D_0 = A^{-1}(D) \cap D_A$, where D_A is a domain of the operator A (From the definition, D_A is a linear subset of L^2 and $\mu(D_A) = 1$). Since D is a linear set then the set $A^{-1}(D)$ is also linear. Moreover, since A is a strongly measurable linear operator and $\mu(D) = 1$ then $\mu_A(D) = 1$, i.e. $\mu(A^{-1}(D)) = 1$. Therefore D_0 is a linear set and $\mu(D_0) = 1$. If now $x \in D_0$, then Ax is defined (since $x \in D_A$) and $Ax \in D$ (since $x \in A^{-1}(D)$) and consequently there exists a quantity $y^*(t)$, where $y = Ax$. Therefore the functional f_t, given by the formula (1), is in fact correctly defined and its domain is the set D_0.

Now we show that f_t is a μ–measurable linear functional . The linearity of this functional is obvious. We prove that f_t is a μ–measurable mapping.

Let $\{e_n : 1,2,\ldots\}$ be an orthonormal trigonometric basis in the L^2–space, and denote by (x,y) the scalar product in L^2. By Fejér's theorem ([7, Sect. 9.4]), it follows that if $x \in D$ then the Fourier series $\sum_{n=1}^{\infty}(e_n,x)e_n(t)$ of a function x is convergent in the sense of Cesàro (see [7, Sect. 8.43]) for every $t \in [0,1]$ and has the sum $\frac{1}{2}[x(t-0)+x(t+0)] = x^*(t)$, that is

$$x^*(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big[\sum_{j=1}^{k} (e_j,x)e_j(t) \Big]$$

for every $x \in D$ and every $t \in [0,1]$.

Therefore from Fejér's theorem we obtain if $x \in D_0$ then

$$f_t(x) = (Ax)^* = \lim_{n\to\infty} f_n(x), \tag{2}$$

where

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n}\Big[\sum_{j=1}^{k}(e_j, Ax)e_j(t)\Big]. \tag{3}$$

Since A is a μ–measurable linear operator then for any $j = 1, 2, \ldots$ (e_j, Ax) is a μ–measurable linear functional. Hence and from (3) it follows that for every $n = 1, 2, \ldots$ $f_n(x)$ is also a μ–measurable linear functional. Thus, by virtue of (2), the functional f_t as a limit (in the sense of the μ-a.e. convergence) of a sequence $\{f_n\}$ of μ–measurable mappings is also μ–measurable. This completes the proof of the fact that f_t is a μ–measurable linear functional.

Now we will show that f_t is a Lusin functional. To prove this fact it is enough to show that for every $\varepsilon > 0$ and every $\varrho > 0$ there exists a continuous linear functional f on L^2 such that

$$\mu\{x : |f_t(x) - f(x)| > \varepsilon\} < \varrho. \tag{4}$$

Indeed, if we show (4) then choosing the sequences $\varepsilon \to 0$ and $\varrho \to 0$ we can construct a sequence of continuous linear functional on L^2 which is convergent in the measure μ to f, and from this sequence we may choose a subsequence which is convergent to f μ-a.e..

To prove (4) let us remark that since A is a Lusin operator then for any $j = 1, 2, \ldots$ (e_j, Ax) is a Lusin functional. Hence for every $n = 1, 2, \ldots$ the functional f_n defined by the formula (3) is also a Lusin functional.

Since now $f_n \to f_t$ μ-a.e., we have that there is $n_0 \in N$ such that

$$\mu\{x : |f_t(x) - f_{n_0}(x)| > \varepsilon/2\} < \varrho/2. \tag{5}$$

But f_{n_0} is a Lusin functional. Thus there exists a continuous linear functional f on L^2 such that

$$\mu\{x : |f_{n_0}(x) - f(x)| > \varepsilon/2\} < \varrho/2. \tag{6}$$

From (5) and (6) we receive

$$\mu\{x : |f_t(x) - f(x)| > \varepsilon\} \le \mu\{x : |f_t(x) - f_{n_0}(x)| > \varepsilon/2\} + \mu\{x : |f_{n_0}(x) - f(x)| > \varepsilon/2\} < \varrho/2 + \varrho/2 = \varrho.$$

This completes the proof of the condition (4).

Recapitulating, we have proved that for every $t \in [0,1]$ the mapping f_t defined by the formula (2) is a Lusin functional on L^2. Therefore from the Theorem 1 we have that for any $t \in [0,1]$ there exists a function $\varphi_t \in L(\psi)$

such that

$$f_t(x) = \int_0^1 \varphi_t(s)dx(s).$$

Let $K(s,t) = \varphi_t(s)$. Then

$$Ax(t) = \int_0^1 K(s,t)dx(s).$$

The theorem is thus proved.

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