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UTILITY FUNCTIONS ASSOCIATED TO RELATIVELY INVARIANT MEASURES ON PARTIALLY ORDERED LOCALLY COMPACT GROUPS

Abstract. Let G be a topological locally compact group (abelian or not) endowed with a left Haar measure and a left translation-invariant and strongly continuous strict partial ordering $\prec$. We consider a positive finite measure ν on G, such that this order is $\nu-$separable. Then, we associate to each positive relatively invariant measure λ on G a class of continuous numerical representations for the order $\prec$.

1. Introduction

In Economics, the notion of utility function is a useful tool in the theory of consumption. Suppose that a society of consumers (constituted by individulas, households, tribes, etc, ...) is given and that the members of this society have an order of preferences for the products they want to choose. Then an utility function is a numerical representation of this order. It transforms preferences into numerical scales. Therefore, numerical representations of preordered sets are tools for decision making.

In Mathematics, the problem of representability of complete ordering by means of a numerical function was posed long ago by Cantor (1895, 1897) (see [3] and [4]). Different studies and solutions of that problem can be found in the paper [20] of Milgram (1939) and the papers ([7, 8]) of Debreu (1954, 1959) or the paper of Fishburn (see [10]) in 1970.

A natural extension of these ideas is the study of this problem for preorders or partial orders (not necessarily completes) in topological spaces. This study has producted a rich literature: see the works by Eilenberg (1941), Debreu (1954), Fleischer (1961) and Jaffray (1975) on completely preordered

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spaces, and those by Mehta (1985, 1986) and Herden (1989) for partially ordered topological spaces. (see the papers [9], [7], [11], [15], [17], [18] and [19]). It appears from exposition of the results in the above references that it is not usual to describe the utility functions by explicite formulas.

The idea of using measure theory has appeared in several papers (see for instance the papers [24], [21], [5], and [6]). But all these papers used particular topologies on particular sets of preferences, with restrictive conditions on the consumption set.

In the paper [2], J.C. Candeal and E. Indurain analyzed the problem of representablity of a given preference from an abstract point of view based on the concept of measures and they construct, under some conditions, continous utility functions for separable orders in topological spaces. In the paper [1], the same authors were interested in constructing continuous utility functions on a locally compact abelian group endowed with a translation-invariant, strongly continuous, and separable strict partial ordering. They used the concept of Haar measure.

The aim of this paper is to generalize the results of the paper [1] to the case of locally compact groups that are not necessarily abelian. Our methods are more general and based on the concept of positive relatively (left) invariant measures. This paper is organized as follows: in Section two, we establish our main result (see Theorem 2.6). In Section three, we provide an illustrative example in the group of affine transformations of the real line.

2. Utility functions on a partially ordered locally compact group

2.1 In all of this paper, G is a topological locally compact group with a left Haar measure μ and Δ is its modulus function. (see for example [14], [16] and [23]). We suppose that G is endowed with a left translation-invariant strict partial ordering $\prec$. (i.e., $\prec$ is irreflexive and transitive such that for all $s, x, y \in G$, the inequality $x \prec y$ implies $ax \prec ay$). Without loss of generality we may suppose that $(G, \prec)$ has neither maximal nor minimal elements. For all $x, y \in G$ verifying $x \prec y$, we denote $]x, y[$ the set of elements $z \in G$ such that $x \prec z \prec y$. We suppose that the order $\prec$ is strongly continuous as regards the topology of G (i.e., for every $x \in G$ the sets $I(x) := \{y \in G : y \prec x\}$ and $J(x) := \{y \in G : x \prec y\}$ are open) (see [25]).

For every function f on G and for every $x, y \in G$, we set $\gamma(x)f(y) := f(xy)$, and $\tau(x)f(y) := f(yx)$,. We recall the following classical result (see [14], Theorem (20.4), p. 285).

2.2 THEOREM. *Let p be a number such that $1 \leq p < \infty$ and let f be a function in the Lebesgue space $L_p(G, \mu)$ endowed with its usual norm $\|\|_p$. For every $\epsilon > 0$, there exists a neighborhood U of e in G such that*

(i) $\|\gamma(x)f - \gamma(y)f\|_p < \epsilon$ *if* $x, y \in G$ *and* $xy^{-1} \in U$.
That is, the mapping $x \mapsto \gamma(x)f$ *of* G *into* $L_p(G,\mu)$ *is right uniformly continuous.*

For every fixed $x \in G$ *and every* $\epsilon > 0$, *there exists a neighborhood* V *of* e *in* G *such that*

(ii) $\|\tau(x)f - \tau(y)f\|_p < \epsilon$ *if* $y \in xV$.
That is, the mapping $x \mapsto \tau(x)f$ *of* G *into* $L_p(G,\mu)$ *is continuous.*

We want to construct continuous utility functions u on G (i.e., $u : G \to \mathbb{R}$ continuous such that for every $x, y \in G$, $\quad x \prec y \Rightarrow u(x) < u(y)$). To this end, we need the following concept by which we can find lower semicontinuous utility functions.

2.3 DEFINITION. Let ν be a finite positive (Radon) measure on G. We say that the order is $\nu-$separable (or that ν is a separating measure for the order) if the following property is satisfied $\nu(]x,y[) \neq 0$ for all $x, y \in G$ such that $x \prec y$.

Let ν be a finite positive (Radon) measure on G for which the order $\prec$ is $\nu-$separable. We associate to ν the function ϕ_ν defined for all $x \in G$, by

$$\phi_\nu(x) := \sup\{\nu(I(y)) : y \prec x\}.$$

Then we have the following proposition.

2.4 PROPOSITION: *Let* ν *be a finite positive* (*Radon*) *measure on* G *for which the order* $\prec$ *is* $\nu-$*separable. Then the function* ϕ_ν *is an utility function which is lower semicontinuous.*

Proof. Let $x_0 \in G$ and let $a < \nu(G)$ be a real number such that $a < \phi_\nu(x_0)$. By definition of $\phi_\nu(x_0)$, there exists $y \in G$ such that $y \prec x$ and $a < \nu(I(y)) \leq \phi_\nu(x_0)$. Since $J(y)$ is open and nonvoid ($x \in J(y)$), we may find a neighbourhood V_x of x such that for every $z \in V_x$ the relation $y \prec z$ holds. Thus for all $z \in V_x$, we will have $a < \nu(I(y)) < \nu(I(z)) \leq \phi_\nu(z)$. We conclude that the set ${\phi_\nu}^{-1}(a, +\infty)$ is open and that ϕ_ν is lower semicontinuous. Now, let $x, y \in G$ such that $x \prec y$. Since the order $\prec$ is $\nu-$separable then the set $]x,y[$ is not empty. Let z be an element of $]x,y[$. Then we have

$$\phi_\nu(x) \leq \nu(I(x)) < \nu(I(z)) \leq \phi_\nu(y).$$

This completes the proof of the proposition.

2.5 Let λ be a (non trivial) positive (regular) and relatively left invariant measure on G (i.e., for each $x \in G$ there is a positive constant $\rho(x)$ such that $\rho(x)\lambda(xE) = \lambda(E)$ for every measurable set E). It is well known (see [16]) that ρ must be a real continuous character from G onto $(0, +\infty)$ and that λ must be equal to $C\rho d\mu$, where C is a positive constant. For every subset A of G, we denote by χ_A the characteristic function of A.

Let K be a compact subset of G having nonvoid interior. Since the function ϕ_ν is measurable and bounded, we may define the map U by setting for every $x \in G$,

$$R_\lambda(x) := \int_{Kx} \phi_\nu(y)\, d\lambda(y).$$

A change of variable will show that we have

$$R_\lambda(x) = C\rho(x)\Delta(x) \int_K \phi_\nu(kx)\rho(k)\, d\mu(k).$$

In the next theorem (which is the main result of this paper), we shall prove that $U_\lambda(x) := \rho(x^{-1})\Delta(x^{-1})R_\lambda(x)$ is a continuous utility function on G.

2.6 THEOREM. *Let G be a topological locally compact group with a left Haar measure μ. Let $\prec$ be a left translation-invariant, strongly continuous strict partial ordering. Let ν be a positive finite measure on G for which the order $\prec$ is separable. Then for all relatively (left) invariant positive measure λ on G and all compact subset K of G having nonvoid interior, the map*

$$x \mapsto U_\lambda(x) := C \int_K \phi_\nu(kx)\rho(k)\, d\mu(k)$$

is a continuous utility function.

Proof. Take $x, y \in G$ such that $x \prec y$. Then since $\prec$ is left translation-invariant, it follows that $kx \prec ky$ for every $k \in K$. Then according to proposition 2.4, we get $\phi_\nu(kx) < \phi_\nu(ky)$. Therefore $U_\lambda(x) < U_\lambda(y)$. Thus U_λ is a numerical representation of the order $\prec$. It remains to show that U_λ is continuous.

Let us fix $x \in G$ and let W be a fixed compact neighbourhood of x in G. We put $Z := KW$. Then for every $y \in W$ we have

$$U_\lambda(y) = \frac{C}{\rho(y)\Delta(y)} \int_G \chi_K(zy^{-1})\phi_\nu(z)\rho(z)\chi_Z(z)\, d\mu(z).$$

Since the maps ρ and Δ are continuous it is sufficient to see that the mapping

$$F(y) = \int_G \chi_K(zy^{-1})\phi_\nu(z)\rho(z)\chi_Z(z)\, d\mu(z)$$

is continuous on W. Now, for all $y \in W$, we obtain the following

$$\begin{aligned}|F(x) - F(y)| &\le \int_Z |\chi_K(zx^{-1}) - \chi_K(zy^{-1})|\phi_\nu(z)\rho(z)\, d\mu(z)\\ &\le \nu(G) \sup_{s \in Z} \rho(s) \int_G |\chi_K(zx^{-1}) - \chi_K(zy^{-1})|\, d\mu(z).\end{aligned}$$

We put $M := \nu(G)\sup_{z\in Z}\rho(z)$ and take arbitrary $\epsilon > 0$. Then, according to Theorem 2.2, there exists a neighbourhood V of e (that can be supposed compact and symmetric, i.e., $V = V^{-1}$) such that

$$\int_G |\chi_K(zx^{-1}) - \chi_K(zy^{-1})|\,d\mu(z) < \frac{\epsilon}{M}$$

holds true whenever $y^{-1} \in x^{-1}V$ or equivalently $y \in Vx$. Now, the set $O := W \cap Vx$ is a neighbourhood of x satisfying $|F(x) - F(y)| < \epsilon$ for all $y \in O$. This proves the continuity of U_λ on G and completes the proof of this theorem.

2.7 Remarks:

1) Let f be a positive (measurable) $\mu-$ integrable function with compact support such that $\int_G f(x)\,d\mu(x) > 0$. Let λ be a relatively (left) invariant positive measure on G. Then the correspondence

$$x \mapsto U(x) := \int_G f(y)\phi_\nu(yx)\,d\lambda(y)$$

is a continuous utility function.

2) The above results can be extended to the case of complete preorderings for which the indifference class are $\mu-$negligible.

3) Similar results could be obtained for right translation-invariant strict partial ordering on the group G. We can also use right Haar measures to obtain other utility functions.

3. Illustrative example

Let G be the group of affine transformations of the real line. That is, $G = \{(x_1, x_2) : x_1 > 0, x_2 \in \mathbb{R}\}$ endowed with the law given by: $(x_1, x_2)(y_1, y_2) := (x_1y_1, x_2 + x_1y_2)$. It is the semidirect product of the usual groups $]0, \infty[$ with its usual multiplication and the real additive group. G is not abelian, not unimodular, a left Haar measure is given by

$$d\mu((x_1, x_2)) = \frac{dx_1\,dx_2}{x_1^2},$$

and its modulus function Δ is given by

$$\Delta((x_1, x_2)) = \frac{1}{x_1}.$$

The group G may be considered as the subgroup of real invertible matrices given by

$$G = \left\{ \begin{pmatrix} x_1 & x_2 \\ 0 & 1 \end{pmatrix} : x_1 > 0,\, x_2 \in \mathbb{R} \right\}.$$

Consider the following partial order $\prec$ defined on G by

$$(x_1, x_2) \prec (y_1, y_2) \Leftrightarrow x_1 > y_1 \quad \text{and} \quad x_2 < y_2,$$

where $<$ has the usual meaning on the real line. Then it is easy to see that this partial order is irreflexive, transitive, left trnaslation-invariant and strongly continuous as regards to the topology of G. The group G has neither maximal nor minimal elements. It is easy to show that $\prec$ is $\nu-$separable for every positive finite and regular Borel measure. So, let ν be a positive, finite and regular Borel measure on G. For every real number s the mapping $\rho((x_1, x_2)) := x_1^s$ is a continuous character of G. According to Theorem 2.6, for every compact subset K of G with nonvoid interior, the function

$$(x_1, x_2) \mapsto U((x_1, x_2)) := \int_K \phi_\nu(k_1 x_1, k_2 + k_1 x_2) k_1^{s-2}\, dk_1 dk_2$$

is a continuous utility function. If the measure ν is absolutely continuous with respect to the Haar measure having f as density, then it is easy to see that this mapping is given by:

$$U((x_1, x_2)) = \int_K \Big[\int_{k_1 x_1}^{+\infty} \int_{-\infty}^{k_2 + k_1 x_2} f(u_1, u_2) u_1^{-2}\, du_1 du_2 \Big] k_1^{s-2}\, dk_1 dk_2.$$

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