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F. Defever[1], R. Deszcz, M. Głogowska,
V. V. Goldberg, L. Verstraelen

# A CLASS OF FOUR-DIMENSIONAL WARPED PRODUCTS

*Dedicated to the memory of Professor Dr. Roger Holvoet*

**Abstract.** We investigate properties of 4-dimensional warped product manifolds satisfying a particular set of curvature conditions. As an application, we obtain a generalization of a pseudosymmetric property for Ricci-flat warped product spacetimes which was established previously in some special cases, including the Schwarzschild metric.

## 1. Introduction

Curvature properties of four-dimensional semi-Riemannian manifolds $(M, g)$, in relation to a semiintegrable almost Grassmann structure (see [1]), were investigated recently in [13]. Among other results, it was shown that the metric $g$ from Example 3.5 in [1] satisfies the following relations

$$(1) \qquad \text{(i) } \operatorname{rank} S \leq 2, \quad \text{(ii) } S^2 = 0, \quad \text{(iii) } \kappa = 0, \quad \text{(iv) } S \cdot C = 0,$$

$$(2) \qquad \omega(X)\mathcal{R}(Y,Z) + \omega(Y)\mathcal{R}(Z,X) + \omega(Z)\mathcal{R}(X,Y) = 0.$$

We note that if (2) is satisfied at a point $x \in M$ and the 1-form $\omega$ is nonzero at this point, then the relation

$$(3) \qquad R \cdot R = Q(S, R)$$

holds at $x$ (see [10]). Thus (3) is a necessary condition for (2) to hold; however this condition is not sufficient. For precise definitions of the symbols used, we refer to Section 2.

In the present paper we broaden the scope of the investigations in [13] and obtain generalizations of the results for all four-dimensional warped products $\overline{M} \times_F \widetilde{N}$ and for all possible dimensions of the base manifold. In doing so, the level of technical complexity increases considerably. More precisely, we study curvature properties of four-dimensional warped products, subject to the conditions (1)(ii)–(iv) and (3). It was shown in [13]

[1]Postdoctoral Researcher F.W.O.-Vlaanderen; Belgium.

that the Ricci tensor $S$ of a four-dimensional warped product $\overline{M} \times_F \tilde{N}$, with $\dim \overline{M} = 1$ or $\dim \overline{M} = 3$, satisfying (1)(ii) and (1)(iii), has the rank at most one at every point. We prove that this statement is also true when the base $(\overline{M}, \overline{g})$ is a two-dimensional manifold. Furthermore, it was also shown in [13] that if (1)(*ii*) and (1)(iii) are satisfied on a four-dimensional warped product $\overline{M} \times_F \tilde{N}$, $\dim \overline{M} = 1$, then $\overline{M} \times_F \tilde{N}$ is a semisymmetric manifold, which is a particular case of a pseudosymmetric manifold. We extend this result to the case when the base $(\overline{M}, \overline{g})$ is two- or three-dimensional. More precisely, we prove that if (1)(ii)–(1)(iv) are satisfied on a 4-dimensional warped product $\overline{M} \times_F \tilde{N}$, $\dim \overline{M} = 2$, then $\overline{M} \times_F \tilde{N}$ is a pseudosymmetric manifold. In the case when the base $(\overline{M}, \overline{g})$ is a three-dimensional manifold, the warped product $\overline{M} \times_F \tilde{N}$ is a semisymmetric manifold. However, in order to prove this, we must assume additionally that such warped product satisfies also (3).

In Section 2 we define the symbols and comment on the concepts we use. In Section 3 we present the main results. Finally, in Section 4 we give an example of a warped product manifold satisfying (1) and (3). However, this manifold does not satisfy (2).

## 2. Preliminaries

Let $(M, g)$ be a connected $n$-dimensional, semi-Riemannian manifold of class $C^\infty$ and let $\nabla$ be its Levi-Civita connection. We define on $M$ the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$, and $\mathcal{C}(X, Y)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$
$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right),$$

respectively, where $A$ is a $(0, 2)$-tensor on $M$, $X, Y, Z \in \Xi(M)$, and $\Xi(M)$ is the Lie algebra of vector fields of $M$. The Ricci operator $\mathcal{S}$ is defined by $S(X, Y) = g(X, \mathcal{S}Y)$, where $S$ is the Ricci tensor and $\kappa$ the scalar curvature of $(M, g)$, respectively. We define the tensor $G$, the Riemann-Christoffel curvature tensor $R$, and the Weyl conformal tensor $C$ of $(M, g)$ by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$
$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$
$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

respectively. For $(0, 2)$-tensors $A$ and $B$, we define their Kulkarni-Nomizu product $A \wedge B$ by

$$(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y) - A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).$$

For a $(0,k)$-tensor $T$, $k \geq 1$, and a symmetric $(0,2)$-tensor $A$, we define the $(0,k)$-tensor $A \cdot T$ and the $(0,k+2)$-tensors $R \cdot T$ and $Q(A,T)$ by

$$(A \cdot T)(X_1, \ldots, X_k) = -T(\mathcal{A}X_1, X_2, \ldots, X_k) - \cdots - T(X_1, X_2, \ldots, \mathcal{A}X_k),$$
$$(R \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{R}(X,Y) \cdot T)(X_1, \ldots, X_k)$$
$$= -T(\mathcal{R}(X,Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X,Y)X_k),$$
$$Q(A,T)(X_1, \ldots, X_k; X, Y) = ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k)$$
$$= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),$$

where $\mathcal{A}$ is the endomorphism of the algebra $\Xi(M)$ corresponding to $A$, defined by $g(\mathcal{A}X, Y) = A(X,Y)$. In particular, taking in the above formulas $T = R$, $T = S$, $T = C$, $A = g$, and $A = S$, we obtain the tensors $R \cdot R$, $R \cdot S$, $R \cdot C$, $S \cdot C$, $Q(g,R)$, $Q(g,S)$, and $Q(S,R)$, respectively.

A semi-Riemannian manifold $(M,g)$, $n \geq 3$, is said to be *pseudosymmetric* (see [7], [15]) if at every point of $M$ the following condition is satisfied:

$(*)_1$ the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent.

Equivalently, the manifold $(M,g)$ is pseudosymmetric if and only if the following

$$R \cdot R = L_R Q(g,R) \tag{4}$$

holds on the set $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)}G \neq 0 \text{ at } x\}$, where $L_R$ is some function on $U_R$. It is clear that any semisymmetric manifold ($R \cdot R = 0$—see [14]) is pseudosymmetric. For more information on the geometric motivation for the introduction of the concept of pseudosymmetry and a survey of various applications, we refer to the papers [7], [9] and [15]; in particular, for connections with the general theory of relativity, see e.g. [12].

The curvature condition (3) is a particular case of the situation when

$(*)_2$ the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent.

According to [3] (see also [4]) a semi-Riemannian manifold $(M,g)$, $\dim M = n \geq 3$, where $(*)_2$ is satisfied, is said to be *Ricci-generalized pseudosymmetric.* As it was shown in [10], if at a point $x \in M$, (2) is satisfied and $\omega$ is nonzero at this point, then (3) holds at $x$. Note also that every hypersurface $M$ of an $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}^{n+1}_s$ with signature $(n+1-s,s)$, $n \geq 3$, satisfies (3) (see [11]). Examples of warped products satisfying (3) can be found in [4].

Let now $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, $\dim \overline{M} = p$, $\dim \widetilde{N} = n - p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U}; x^a\}$ and $\{\widetilde{V}; y^\alpha\}$, respectively. Let $F$ be a positive smooth function on $\overline{M}$. The *warped product* $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g}$ defined by

$$\overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1)\pi_2^* \widetilde{g},$$

where $\pi_1 : \overline{M} \times \widetilde{N} \longrightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \longrightarrow \widetilde{N}$ are the natural projections on $\overline{M}$ and $\widetilde{N}$, respectively. Let $\{\overline{U} \times \widetilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart are $g_{rs} = \overline{g}_{ab}$ if $r = a$ and $s = b$, $g_{rs} = F\widetilde{g}_{\alpha\beta}$ if $r = \alpha$ and $s = \beta$, and $g_{rs} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $r, s, t, \ldots \in \{1, 2, \ldots, n\}$. We denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$). The local components $\Gamma^r_{st}$ of the Levi-Civita connection $\nabla$ of $\overline{M} \times_F \widetilde{N}$ are

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\alpha\beta} = -\frac{1}{2}\overline{g}^{ab} F_b \widetilde{g}_{\alpha\beta}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta,$$

$$\Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}.$$

The local components

$$R_{rstu} = g_{rw} R^w_{stu} = g_{rw}(\partial_u \Gamma^w_{st} - \partial_t \Gamma^w_{su} + \Gamma^v_{st}\Gamma^w_{vu} - \Gamma^v_{su}\Gamma^w_{vt}), \qquad \partial_u = \frac{\partial}{\partial x^u},$$

of the Riemann–Christoffel curvature tensor $R$ and the local components $S_{ts}$ of the Ricci tensor $S$ of the warped product $\overline{M} \times_F \widetilde{N}$ which may not vanish identically are

$$(5) \quad R_{abcd} = \overline{R}_{abcd}, \quad R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \widetilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\beta} - \frac{\Delta_1 F}{4} \widetilde{G}_{\alpha\beta\gamma\delta},$$

$$(6) \quad S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \frac{1}{2}\left(\operatorname{tr} T + \frac{n-p-1}{2F}\Delta_1 F\right)\widetilde{g}_{\alpha\beta},$$

where

$$(7) \quad T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \quad \operatorname{tr} T = \overline{g}^{ab} T_{ab}, \quad \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b,$$

and $T$ is the $(0, 2)$-tensor with the local components $T_{ab}$. The scalar curvature $\kappa$ of $\overline{M} \times_F \widetilde{N}$ satisfies the relation

$$(8) \qquad \kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{F} - \frac{n-p}{F}\left(\operatorname{tr} T + \frac{n-p-1}{4F}\Delta_1 F\right).$$

## 3. Main results

We now proceed to the proof of the main results. We will show that any four-dimensional warped product subject to the conditions (1)(ii)–(1)(iv) and (3) bears the same features: $\operatorname{rank}(S) \leq 1$ and the manifold is pseudosymmetric. The proof proceeds case by case and is established in Proposition 3.1 for $\dim \overline{M} = 1$, in Proposition 3.2 for $\dim \overline{M} = 2$, and in Proposition 3.3 for $\dim \overline{M} = 3$. However, before proving these results we collect in Lemma

3.1 a few usefull formulas, which will be invoked repeatedly. The proof of these formulas consists of a carefull calculation and comparison of the left- and right-hand sides by using definitions of Section 2.

LEMMA 3.1. *Let $(M, g)$ be a four-dimensional semi-Riemannian manifold satisfying* (1)(ii), (1)(iii), *and* (1)(iv). *Then the following relations are satisfied on $M$:*

(9) $(S \cdot R)_{rstu} = (S \wedge S)_{rstu}$,

(10) $g^{st}(R \cdot S)_{rstu} = S^{st} R_{rstu} = 0$,

(11)
$$S_{rs}R_{tuvw} + S_{ts}R_{urvw} + S_{us}R_{rtvw} + S_{rv}R_{tuws} + S_{tv}R_{urws} + S_{uv}R_{rtws} + S_{rw}R_{tusv} + S_{tw}R_{ursv} + S_{uw}R_{rtsv} = g_{rs}(S \wedge S)_{tuvw} + g_{ts}(S \wedge S)_{urvw} + g_{us}(S \wedge S)_{rtvw} + g_{rv}(S \wedge S)_{tuws} + g_{tv}(S \wedge S)_{urws} + g_{uv}(S \wedge S)_{rtws} + g_{rw}(S \wedge S)_{tusv} + g_{tw}(S \wedge S)_{ursv} + g_{uw}(S \wedge S)_{rtsv}.$$

PROPOSITION 3.1 ([13], Lemma 3.1). *If the conditions* (1)(ii) *and* (1)(iii) *are satisfied on a four-dimensional warped product $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, then $\overline{M} \times_F \widetilde{N}$ is a semisymmetric manifold and* $\operatorname{rank} S \leq 1$.

PROPOSITION 3.2. *Let $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 2$, be a four-dimensional warped product manifold. Then*

(i) *If the conditions* (1)(*ii*) *and* (1)(iii) *are satisfied on $\overline{M} \times_F \widetilde{N}$, then* $\operatorname{rank} S \leq 1$.

(ii) *If the conditions* (1)(ii), (1)(iii), *and* (1)(iv) *are satisfied on $\overline{M} \times_F \widetilde{N}$, then it is a pseudosymmetric manifold.*

Proof. (i) From (6) we have

$$S_{\alpha\beta} = \frac{1}{2}\left(\widetilde{\kappa} - (\operatorname{tr} T + \frac{\Delta_1 F}{2F})\right)\widetilde{g}_{\alpha\beta}. \tag{12}$$

Further, (1)(ii) yields $S^2_{\alpha\beta} = 0$. Applying this to (12), we obtain

$$\widetilde{\kappa} = \operatorname{tr} T + \frac{\Delta_1 F}{2F}, \tag{13}$$

which reduces (12) to

$$S_{\alpha\beta} = 0. \tag{14}$$

Now (1)(iii) and (1)(ii) turn into $g^{ef}S_{ef} = 0$ and $g^{ef}S_{ae}S_{bf} = 0$, respectively. Set $T_{abcd} = S_{ad}S_{bc} - S_{ac}S_{bd}$. We can easily check that the identity

$$T_{abcd} = \frac{\kappa(T)}{2} G_{abcd}, \tag{15}$$

where $\kappa(T) = g^{ad}g^{bc}T_{abcd} = (g^{ef}S_{ef})^2 - g^{ad}g^{bc}S_{ac}S_{bd}$, is satisfied From the last relation, by making use of the definitions of $S^2$ and $\kappa$ and (14), (1)(ii) and (1)(iii), we obtain $\kappa(T) = \kappa^2 - \operatorname{tr} S^2 = 0$. But this reduces (15) to $S_{ad}S_{bc} - S_{ac}S_{bd} = 0$, whence $\operatorname{rank} S \leq 1$, completing the proof of (i).

(ii) Since $\operatorname{rank} S \leq 1$, (9) reduces to $S \cdot R = 0$. In particular, we have $(S \cdot R)_{\alpha ab\beta} = 0$, which by (5) and (14), leads to $g^{ef}(S_{ae}T_{bf} + S_{be}T_{af}) = 0$. By (6), this turns into

$$\overline{\kappa} T_{ab} = \frac{2}{F} T^2_{ab}. \tag{16}$$

On the other hand, (1)(*ii*) yields $\frac{\overline{\kappa}^2 F}{2}\overline{g}_{ab} - 2\overline{\kappa}T_{ab} + \frac{2}{F}T^2_{ab} = 0$ whence, by (16), we obtain

$$\overline{\kappa}\left(T_{ab} - \frac{\overline{\kappa}F}{2}\overline{g}_{ab}\right) = 0. \tag{17}$$

We assume that at a point $x \in \overline{M} \times_F \tilde{N}$ we have $\overline{\kappa} \neq 0$. Now (17) gives $T_{ab} = \frac{\overline{\kappa}F}{2}\overline{g}_{ab}$. We put $L = -\frac{\overline{\kappa}}{4}$ and $H_{ab} = \frac{1}{2}T_{ab} + FL\overline{g}_{ab}$. Applying this to (17), we obtain $H_{ab} = 0$. Now Corollary 2.1 of [8] implies $R \cdot R = LQ(g, R)$. We assume that at a point $x \in \overline{M} \times_F \tilde{N}$ we have $\overline{\kappa} = 0$. Next we note that by our assumptions, (8) turns into

$$\overline{\kappa} = \frac{1}{F} \operatorname{tr} T, \tag{18}$$

which gives $\operatorname{tr} T = 0$. Thus (13) reduces to $\tilde{\kappa} = \frac{1}{2}\frac{\Delta_1 F}{F}$. Now, in view of Corollary 2.1 of [8], we can state that $R \cdot R = 0$ is satisfied at $x$ if and only if

$$T_{ac}T_{bd} - T_{ab}T_{cd} = 0 \quad \text{and} \quad T^2_{da} = g^{ef}T_{ed}T_{fa} = 0 \tag{19}$$

hold at $x$. Since $\operatorname{rank} S = \operatorname{rank} T \leq 1$ and $0 = S^2_{da} = T^2_{ab}$ hold at $x$, we see that (19) is satisfied at $x$, i.e., $R \cdot R = 0$ holds at $x$. This completes the proof of (ii).

PROPOSITION 3.3. *Let $\overline{M} \times_F \tilde{N}$, $\dim \overline{M} = 3$, be a four-dimensional warped product manifold.*

(i) *If the conditions* (1)(ii) *and* (1)(iii) *are satisfied on* $\overline{M} \times_F \tilde{N}$, $\dim \overline{M} = 3$, *then* $\operatorname{rank} S \leq 1$.

(ii) *If the conditions* (1)(ii), (1)(iii), (1)(iv), *and* (3) *are satisfied on* $\overline{M} \times_F \tilde{N}$, *then it is a semisymmetric manifold.*

Proof. (i) The proof of this subcase was covered by Lemma 3.2 of [13].

(ii) Let $x$ be a point of $\overline{M} \times_F \tilde{N}$. From (1)(ii) we get $S^2_{44} = g^{44}S_{44}S_{44} = 0$, and

$$S_{44} = 0. \tag{20}$$

Further, by (20), equations (6) yields

$$\operatorname{tr} T = 0. \tag{21}$$

Applying (21) to (8), we obtain

$$\overline{\kappa} = 0. \tag{22}$$

It is clear that if $S$ vanishes at $x$, then (3) reduces to $R \cdot R = 0$. Therefore, we assume that $\operatorname{rank} S = 1$ holds at $x$. Thus we have

$$S_{ad} = \rho \phi_a \phi_d, \quad \rho \in \mathbb{R}, \tag{23}$$

whence

$$\phi^f \phi_f = 0, \quad \phi^f = \overline{g}^{ef} \phi_e, \tag{24}$$

where $\phi_a$ are the local components of a covector $\phi$ at $x$. Applying (23) to (6), we obtain

$$\overline{S}_{ad} = \rho \phi_a \phi_d + \frac{1}{2F} T_{ad}. \tag{25}$$

Further, taking $r = s = 4$ and $t = a$, $u = b$, $v = c$, $w = d$ in (11), we find that

$$-S_{ac} C_{b44d} + S_{bc} C_{a44d} + S_{ad} C_{b44c} - S_{bd} C_{a44c} = 0,$$

which, by **(i)**, reduces to

$$-S_{ac} R_{b44d} + S_{bc} R_{a44d} + S_{ad} R_{b44c} - S_{bd} R_{a44c} = 0.$$

Applying this to (5) and (25), we obtain

$$\phi_a \phi_d T_{bc} - \phi_a \phi_c T_{bd} + \phi_b \phi_c T_{ad} - \phi_b \phi_d T_{ac} = 0. \tag{26}$$

Contracting (26) with $\overline{g}^{ac}$ and making use of (21) and (24), we find that

$$\phi_d \phi^f T_{fb} + \phi_b \phi^f T_{fd} = 0,$$

whence it follows that

$$\phi^f T_{fb} = 0. \tag{27}$$

Contracting (26) with $T^a_{\ e} = \overline{g}^{af} T_{fe}$ and using (27), we find that

$$T^2_{ec} = \rho_1 \phi_e \phi_c, \quad \rho_1 \in \mathbb{R}. \tag{28}$$

Using (5), (23) and (25), we get from (9) and (10) that

$$\phi_a \phi^f \overline{R}_{fbcd} - \phi_b \phi^f \overline{R}_{facd} + \phi_c \phi^f \overline{R}_{fdab} - \phi_d \phi^f \overline{R}_{fcab} = 0 \tag{29}$$

and $\phi^e \phi^f \overline{R}_{ebcf} = 0$. Contracting now (29) with $g^{bc}$ and using the last equality we obtain

$$\phi_d \phi^f \overline{S}_{fb} + \phi_b \phi^f \overline{S}_{fd} = 0,$$

whence

$$\phi^f \overline{S}_{fb} = 0. \tag{30}$$

Applying (30) to (29), we find that

$$\phi_a \phi_d \overline{S}_{bc} - \phi_a \phi_c \overline{S}_{bd} + \phi_b \phi_c \overline{S}_{ad} - \phi_b \phi_d \overline{S}_{ac} = 0. \tag{31}$$

Contracting (31) with $\overline{S}^a{}_e = g^{af}\overline{S}_{fe}$ and using (30), we obtain

$$\overline{S}^2_{ec} = \rho_2\phi_e\phi_c, \quad \rho_2 \in \mathbb{R}. \tag{32}$$

From Proposition 6.1 of [5] it follows that (3) is satisfied at $x$ if and only if $Q(T, \overline{R}) = 0$ and the relation

$$T^f{}_a\overline{S}_{fd} = \frac{1}{3}g^{bc}T^f{}_b\overline{S}_{fc}g_{ad} \tag{33}$$

holds at $x$. Applying (33) to (25) and (28), we find that

$$\rho_1\phi_a\phi_d = \frac{1}{3}g^{bc}T^f{}_b\overline{S}_{fc}g_{ad},$$

whence $\rho_1 = 0$ and

$$g^{bc}T^f{}_b\overline{S}_{fc} = 0. \tag{34}$$

Thus (28) and (33) are reduced to

$$T^f{}_d\overline{S}_{fa} = 0 \tag{35}$$

and

$$T^2_{ad} = 0, \tag{36}$$

respectively. Similarly, using (25), (32) and (34), from (33) we find that $\rho_2 = 0$ and

$$\overline{S}^2_{ad} = 0. \tag{37}$$

In view of Lemma 2 of [10], it follows that the relation

$$\overline{R}\cdot\overline{S} = 0 \tag{38}$$

holds at $x$. We note that by Lemma 2 of [10], (38) is equivalent at $x$ to $\overline{R}\cdot\overline{R} = 0$. Now from Theorem 2.1 of [8] it follows that the condition $R\cdot R = 0$ is satisfied at $x$ if and only if the relation

$$T_{cd}\overline{S}_{ab} - T_{bd}\overline{S}_{ac} = \frac{1}{2F}(T_{ac}T_{bd} - T_{ab}T_{cd}) \tag{39}$$

holds at $x$. Further, using (21) and (36) and applying Lemma 2.1$(ii)$ of [6], we deduce that $T_{ad}T_{bc} - T_{ac}T_{bd} = 0$, whence the relation

$$T_{ad} = \tau_1\psi_a\psi_d, \quad \tau_1 \in \mathbb{R} \tag{40}$$

holds at $x$, where $\psi_a$ are the local components of the covector $\psi$ at $x$. From Lemma 2.1$(ii)$ of [6] and from (22) and (37), we find that $\overline{S}_{ad}\overline{S}_{bc} - \overline{S}_{ac}\overline{S}_{bd} = 0$, whence the relation

$$\overline{S}_{ad} = \tau_2\omega_a\omega_d, \quad \tau_2 \in \mathbb{R} \tag{41}$$

holds at $x$, where $\omega_a$ are the local components of the covector $\omega$ at $x$. Now by (40) and (41), equations (39) reduce to

$$\tau_1\tau_2(\psi_c\psi_d\omega_a\omega_b - \psi_b\psi_d\omega_a\omega_c) = 0. \tag{42}$$

By Lemma 2.1(*ii*) of [6], we deduce from (34) and (35) that

$$\overline{S}_{ad}\overline{S}_{bc} - \overline{S}_{ac}\overline{S}_{bd} + \overline{S}_{bc}\overline{S}_{ad} - \overline{S}_{bd}\overline{S}_{ac} = 0,$$

which, by (40) and (41), turns into

$$\tau_1\tau_2(\psi_a\psi_d\omega_b\omega_c - \psi_a\psi_c\omega_b\omega_d + \psi_b\psi_c\omega_a\omega_d - \psi_b\psi_d\omega_a\omega_c) = 0. \tag{43}$$

Evidently, if $\psi$ or $\omega$ is a zero covector, then (42) is satisfied at $x$ and $R\cdot R = 0$ holds at $x$. Assume that $\psi$ is nonzero covector at $x$. We can choose a vector $V$ at $x$ such that $V^f\psi_f = 1$. Contracting now (43) with $V^a$ and $V^d$, we obtain

$$\tau_1\tau_2(\omega_b - \tau\psi_b)(\omega_c - \tau\psi_c) = 0, \quad \tau = V^f\omega_f,$$

whence $\tau_1\tau_2\omega_b = \tau_1\tau_2\psi_b$. Now we see that (42) holds at $x$. Thus Proposition 3.3 is proved.

From Proposition 3.1 and Proposition 3.2(ii) we have the following

COROLLARY 3.1. *Every Ricci flat warped product solution* $\overline{M}\times_F\widetilde{N}$, $\dim\overline{M} \leq 2$, *of Einstein's field equations is a pseudosymmetric manifold.*

Corollary 3.1 generalizes Proposition 2 of [12] where this was shown for the Schwarzschild metric. We finish this section with the following

REMARK 3.1. It is well known that the Kerr metric is also Ricci flat. However, it is not a pseudosymmetric metric [7]. It follows that the Kerr metric satisfies $(1)(i)-(1)(iv)$, but cannot satisfy (3). In addition, we mention that the Kerr metric is a nonwarped product metric [2].

## 4. Examples

Let $(\widetilde{N},\widetilde{g})$, $\dim\widetilde{N} = n-p \geq 1$, $n \geq 4$, $p \geq 1$, be a semi-Riemannian space of constant curvature. Further, let $\overline{M}$ be a nonempty open connected subset of $\mathbb{R}^p$, equipped with the standard metric $\overline{g}$, $\overline{g}_{ab} = \varepsilon_a\delta_{ab}$, $\varepsilon_a = \pm 1$. We put $F = F(x^1,\ldots,x^p) = k\exp(\xi_a x^a)$, where $k,\xi_1,\ldots,\xi_p \in \mathbb{R}$, $\xi_1^2 + \ldots + \xi_p^2 > 0$ and $k > 0$. Now (6)–(8) turn into

$$S_{ab} = -\frac{n-p}{4}\xi_a\xi_b, \quad S_{\alpha\beta} = \left(\frac{\widetilde{\kappa}}{n-p} - \frac{\operatorname{tr}T}{2} - \frac{n-p-1}{4F}\Delta_1 F\right)\widetilde{g}_{\alpha\beta}, \tag{44}$$
$$T_{ab} = \frac{F}{2}\xi_a\xi_b, \quad \operatorname{tr}T = \frac{F}{2}\xi^f\xi_f,$$
$$\Delta_1 F = F^2\xi^f\xi_f, \quad \kappa = \frac{\widetilde{\kappa}}{F} - \frac{(n-p)(n-p+1)}{4}\xi^f\xi_f,$$

respectively, where $\xi^f = \overline{g}^{ef}\xi_e$. We note that

$$(45) \qquad FS_{ad} + \frac{1}{2}T_{ad} = -\frac{n-p-1}{4}F\xi_a\xi_d.$$

(i) The manifold $\overline{M} \times_F \widetilde{N}$ satisfies (3) (see [4]). We can easily check that if (2) is satisfied on $\overline{M} \times_F \widetilde{N}$, for some 1-form $\omega$, then it must vanish on $\overline{M} \times_F \widetilde{N}$. Indeed, let $x \in \overline{M} \times_F \widetilde{N}$. If in the formula

$$(46) \qquad \omega_r R_{stuv} + \omega_s R_{truv} + \omega_t R_{rsuv} = 0,$$

we take $r = a$, $s = \alpha$, $t = \beta$, $u = \gamma$, $v = \delta$, then we get $\omega_a R_{\alpha\beta\gamma\delta} = 0$, whence, by our assumptions and (5), it follows that at the point $x$ we have $\omega_a = 0$. Further, taking in (46) $r = \alpha$, $s = \beta$, $t = a$, $u = b$, $v = \gamma$, we find that $(\omega_\alpha \widetilde{g}_{\beta\gamma} - \omega_\beta \widetilde{g}_{\alpha\gamma})T_{ab} = 0$, which implies $\omega_\alpha = 0$.

(ii) If $p = 1$ then the warped product $\overline{M} \times_F \widetilde{N}$ is a conformally flat manifold. Therefore in the following we assume that $p \geq 2$.

From Corollary 2.1 of [8] it follows that if $\widetilde{\kappa} \neq 0$, then $\overline{M} \times_F \widetilde{N}$ is a nonpseudosymmetric manifold. However, if $\widetilde{\kappa} = 0$, then $\overline{M} \times_F \widetilde{N}$ is a semisymmetric manifold. In addition, if the constants $\varepsilon_a$ and $\xi_a$ satisfy $\xi^f\xi_f = 0$, then from (44) it follows that the warped product $\overline{M} \times_F \widetilde{N}$ satisfies the following relations: $\operatorname{rank} S = 1$, (1)($ii$) and (1)($iii$). Furthermore, by making use of (5) and (44), we can state that the local components of the Weyl tensor $C$, which may not vanish identically, are

$$C_{abcd} = \frac{n-p}{4(n-2)}(g_{ad}\xi_b\xi_c + g_{bc}\xi_a\xi_d - g_{ac}\xi_b\xi_d - g_{bd}\xi_a\xi_c),$$
$$C_{a\alpha\beta d} = -\frac{p-2}{4(n-2)}\xi_a\xi_d g_{\alpha\beta}.$$

Now, in virtue of (44), we get easily (1)($iv$). Furthermore, in view of Lemma 3.1 and (44), we obtain $S \cdot R = 0$.

(iii) The manifold $\overline{M} \times_F \widetilde{N}$ cannot be realized as a hypersurface of a semi-Riemannian space of nonzero constant curvature. This is a consequence of Proposition 3.1 of [11] and the fact that $\overline{M} \times_F \widetilde{N}$ satisfies (3). We prove now that $\overline{M} \times_F \widetilde{N}$, $n-p \geq 2$, can be realized as a hypersurface of a semi-Euclidean space. Let $\tau$ be a function on $\overline{M} \times_F \widetilde{N}$ such that the relation

$$\tau^2 = \varepsilon\left(\frac{\widetilde{\kappa}}{(n-p)(n-p-1)F} - \frac{1}{4}\xi^f\xi_f\right), \qquad \varepsilon = \pm 1,$$

holds on $\overline{M} \times_F \widetilde{N}$. It is clear that there exist constants $\varepsilon$, $\varepsilon_a$, and $\xi_a$ such that the function $\tau$ is nonzero at every point $x$ of $\overline{M} \times_F \widetilde{N}$ and the right-hand side of the last relation is positive at every point $x$. Further, let $H$ be the

(0, 2)-tensor on $\overline{M} \times_F \widetilde{N}$ with local components $H_{rs}$ defined by

$$H_{ab} = -\frac{1}{4\tau}\xi_a\xi_b, \quad H_{a\alpha} = 0, \quad H_{\alpha\beta} = \varepsilon\tau g_{\alpha\beta}.$$

We can check that the following relations are satisfied on $\overline{M} \times_F \widetilde{N}$:

$$R(X_1, X_2, X_3, X_4) = \varepsilon(H(X_1, X_4)H(X_2, X_3) - H(X_1, X_3)H(X_2, X_4)),$$
$$\nabla_X H(Y, Z) = \nabla_Y H(X, Z),$$

where $X, Y, Z, X_1, \ldots, X_4$ are vectors fields on $\overline{M} \times_F \widetilde{N}$. Thus we see that the manifold $\overline{M} \times_F \widetilde{N}$ can be realized as a hypersurface of a semi-Euclidean space $\mathbb{E}^{n+1}_s$ with signature $(n + 1 - s, s)$.

Remark 4.1. All known examples of manifolds satisfying (3) are either warped products, hypersurfaces of semi-Euclidean spaces, or manifolds satisfying (2) (see e.g. [10]). Example 3.5 of [1] is an example of a semi-Riemannian nonwarped product manifold satisfying (2), but which cannot be realized as a hypersurface of a semi-Euclidean space [13]. Example 4.1 gives an example of a manifold which is a warped product and a hypersurface of a semi-Euclidean space, but does not satisfy (2). Explicit examples realizing (3), which do not belong to one of the above mentioned classes, are hitherto unknown.

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Filip Defever
DEPARTEMENT INDUSTRIELE WETENSCHAPPEN EN TECHNOLOGIE
KATHOLIEKE HOGESCHOOL BRUGGE-OOSTENDE
Zeedijk 101
B-8400 OOSTENDE, BELGIUM
E-mail: filip.defever@kh.khbo.be

Ryszard Deszcz and Małgorzata Głogowska
DEPARTMENT OF MATHEMATICS
AGRICULTURAL UNIVERSITY OF WROCŁAW
Grunwaldzka 53
50-357 WROCŁAW, POLAND
E-mail: rysz@ozi.ar.wroc.pl and mglog@ozi.ar.wroc.pl

Vladislav V. Goldberg
DEPARTMENT OF MATHEMATICAL SCIENCES
NEW JERSEY INSTITUTE OF TECHNOLOGY
UNIVERSITY HEIGHTS
NEWARK, NJ 07102, U.S.A.
E-mail: vlgold@m.njit.edu

Leopold Verstraelen
DEPARTMENT WISKUNDE
KATHOLIEKE UNIVERSITEIT LEUVEN
Celestijnenlaan 200B
B-3001 LEUVEN, BELGIUM
E-mail: LeopoldVerstraelen@wis.kuleuven.ac.be

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