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Victor Pambuccian

A UNIVERSAL AXIOM SYSTEM FOR SALOW'S GENERALIZED HALBDREHUNGSEBENEN

Abstract. We present a universal axiom system for E. Salow's [1] *Halbdrehungsebenen.*

1. E. Salow [1] introduced a wide class of planes to include most classes of ring (Hjelmslev) geometries, by providing a characterization of them in the following terms: Given three pairwise disjoint nonempty sets $\mathcal{P}, \mathcal{L}, \mathcal{D}$ with $|\mathcal{P}| \geq 2$, the elements of the first two sets being called points and lines respectively, and $\mathcal{D}$ being an Abelian group, a binary relation I between points and lines (standing for 'incidence'), and a mapping $M : \mathcal{L} \times \mathcal{L} \to \mathcal{D}$, we call the structures $\langle \mathcal{P}, \mathcal{L}, \mathrm{I}, \mathcal{D}, M \rangle$ a generalized *Halbdrehungsebene* if
(i) for all $g, h, j \in \mathcal{L}$ we have $M(g, h) \cdot M(h, j) = M(g, j)$ ($\cdot$ being the group operation of the Abelian group $\mathcal{D}$;
(ii) for all $P \in \mathcal{P}$, $g \in \mathcal{L}, \alpha \in \mathcal{D}$ there is exactly one $h \in \mathcal{L}$ with $h\mathrm{I}P$ and $M(g, h) = \alpha$;
(iii) there is an $\alpha \in \mathcal{D}$, such that for all $g, h \in \mathcal{L}$ with $M(g, h) = \alpha$ there is a unique point P incident with both g and h, and such that for all $\beta \in \mathcal{D}$ and all $P \in \mathcal{P}$, if the lines a, b, c have a point in common, then $H_{P,\alpha,\beta}(a)$, $H_{P,\alpha,\beta}(b)$, $H_{P,\alpha,\beta}(c)$, have a point in common as well. Here $H_{P,\alpha,\beta}$ stands for a 'half rotation', a map transforming lines into lines, defined as follows: let $g \in \mathcal{L}$, let h be the uniquely determined line with $h\mathrm{I}P$ and $M(h, g) = \alpha$, and F the unique common point of g and h, then one defines $H_{P,\alpha,\beta}(g)$ as being the line g' which passes through F and satisfies $M(g, g') = \beta$.

This obviously does not represent an axiom system in the logical sense of the word, but rather a description of a class of planes. The purpose of this short note is to present a universal axiom system for this class of planes.

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2. The language in which the axiom system will be expressed consists of two sorts of individual variables, for points (upper case) and lines (lower case), as well as four individual constants, two points A_0, A_1, and two lines a_0, a_1, a binary relation between points and lines $\in$ standing for incidence, a quaternary relation between lines $\simeq$, with $(a,b) \simeq (c,d)$ to be interpreted as 'angles $\angle(a,b)$ and $\angle(c,d)$ have the same measure', a binary operation on lines ι, with $\iota(g,h)$ to be interpreted as 'the intersection point of the lines g and h, provided that it exists and is unique, an arbitrary point, otherwise', and two quaternary operation symbols, α_1 and α_2, with a point variable as the first argument, line variables as the next three arguments and a line variable as value, where $\alpha_1(P,a,b,g)$ is to be read as 'the line through P for which $(\alpha_1(P,a,b,g),g) \simeq (a,b)$', and $\alpha_2(P,a,b,g)$ is to be read as 'the line through P for which $(g,\alpha_2(P,a,b,g)) \simeq (a,b)$'.

We now define the following two abbreviations:
$\epsilon(P,g,h,u) := \alpha_2(\iota(u,\alpha_1(P,a_0,a_1,u)),g,h,u)$ and
$\chi(a,b) :\leftrightarrow \iota(a,b) \in a \wedge \iota(a,b) \in b \wedge (P \in a \wedge P \in b \rightarrow P = \iota(a,b))$.
The latter stands for 'the lines a and b intersect in one point" and the former stands for the half-rotation operation H, in a sense to be made precise shortly.

The axioms are:

A 1. $A_0 \neq A_1$,

A 2. $(g,g) \simeq (h,h)$,

A 3. $(g,h) \simeq (g,h)$,

A 4. $(a,b) \simeq (c,d) \wedge (e,f) \simeq (c,d) \rightarrow (a,b) \simeq (e,f)$,

A 5. $(g,h) \simeq (g',h') \rightarrow (h,g) \simeq (h',g')$,

A 6. $(g,h) \simeq (g',h') \wedge (h,j) \simeq (h',j') \rightarrow (g,j) \simeq (g',j')$,

A 7. $(g',h') \simeq (h,j) \wedge (h',j') \simeq (g,h) \rightarrow (g',j') \simeq (g,j)$,

A 8. $P \in \alpha_1(P,a,b,g) \wedge (\alpha_1(P,a,b,g),g) \simeq (a,b)$,

A 9. $P \in h \wedge (h,g) \simeq (a,b) \rightarrow h = \alpha_1(P,a,b,g)$,

A 10. $P \in \alpha_2(P,a,b,g) \wedge (g,\alpha_2(P,a,b,g)) \simeq (a,b)$,

A 11. $P \in h \wedge (g,h) \simeq (a,b) \rightarrow h = \alpha_2(P,a,b,g)$,

A 12. $(g,h) \simeq (a_0,a_1) \rightarrow \chi(g,h)$,

A 13. $\chi(x_1,x_2) \wedge \iota(x_1,x_2) \in x_3 \rightarrow \bigwedge_{i=1}^{3} \iota(\epsilon(P,g,h,x_1),\epsilon(P,g,h,x_2)) \in \epsilon(P,g,h,x_i)$.

To see that the above axiom system Σ is an axiom system for generalized *Halbdrehungsebenen*, we notice that a structure $\langle \mathcal{P}, \mathcal{L}, \mathrm{I}, \mathcal{D}, M \rangle$ satisfying (i), (ii), (iii) can be associated to every model $\mathfrak{M}$ of the axiom system A1–A13. Given such a model, let $\mathcal{P}$ and $\mathcal{L}$ denote the universes of point and lines of $\mathfrak{M}$, with I to be defined as the interpretation of $\in$ in $\mathfrak{M}$. We define $\mathcal{D}$ to be the set of equivalence classes $\mathcal{L}\times\mathcal{L}/\simeq$, with multiplication defined by $[(a,b)]\cdot[(c,d)] := [(a,\alpha_2(A_0,c,d,b))]$, the unit being $[(a_0,a_0)]$, the inverse of $[(a,b)]$ being $[(b,a)]$. That multiplication is well-defined (does not depend on the representative) is easily seen to follow from Σ, and one readily checks that (i),(ii), (iii) hold, with $\alpha = [(a_0,a_1)]$ in (iii), and $H_{P,[(a_0,a_1)],[(g,h)]}(u) := \epsilon(P,g,h,u)$.

From the additional axioms considered in [1] axiom (Δ) requires the enlarging of the language with two more line constants b_0, b_1, and axiom (E) requires the use of existential quantifiers.

References

[1] E. Salow, *Verallgemeinerte Halbdrehungsebenen*, Geom. Dedicata 13 (1982), 67–85.

DEPARTMENT OF INTEGRATIVE STUDIES
ARIZONA STATE UNIVERSITY WEST
P. O. BOX 37100
PHOENIX AZ 85069-7100, USA
E-mail: pamb@math.west.asu.edu

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