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ON SMOOTHNESS AND APPROXIMATION PROPERTIES OF THE KANTOROVICH TYPE OPERATORS

Abstract. In this paper we present inequalities concerning the weighted moduli of continuity of Kantorovich type operators $L_n^* f$. Moreover, we give some estimates of the degree of approximation of f by $L_n^* f$ in the Hölder type norms.

1. Preliminaries

Let I be a finite or infinite interval and let $M(I)$ be the class of all measurable complex-valued functions bounded on I. In the case when I is an infinite interval, denote by $M_{\text{loc}}(I)$ the class of all functions bounded on every compact subinterval of I. Given any $n \in N := \{1, 2, \ldots\}$, let J_n be a set of indices contained in $Z := \{0, \pm 1, \pm 2, \ldots\}$ and let I be the union of non-overlapping intervals $I_{j,n} (j \in J_n)$ with increasing left (right) end points. Introduce, formally, for functions f belonging to $M(I)$ or $M_{\text{loc}}(I)$, the discrete operators L_n defined by

$$(1.1) \qquad L_n f(x) = \sum_{j \in J_n} f(\xi_{j,n}) p_{j,n}(x) \qquad (x \in I, n \in N),$$

where $\xi_{j,n} \in I_{j,n}$ and $p_{j,n}$ are non-negative functions continuous on I. Denote by L_n^* the Kantorovich type modification of operators (1.1) given by

$$(1.2) \qquad L_n^* f(x) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} f(t)dt \quad (x \in I, n \in N),$$

with $m_{j,n} = \operatorname{meas} I_{j,n}$. This type modification of the classical Bernstein polynomials are called the Kantorovich polynomials and are presented e.g. in [9, Chap. II]. The Kantorovich type modification of some other discrete operators can be found e.g. in [3], [5], [2].

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Throughout this paper we make the standing assumption that the functions $p_{j,n}$ ($j \in J_n$, $n \in N$) are absolutely continuous on every compact interval contained in I. We say that $f \in \mathrm{Dom}(L_n^*)$ if the series in the definition (1.2) converges absolutely at every $x \in I$. Assuming that $f \in \mathrm{Dom}(L_n^*)$ we give the Kratz-Stadtmüller [7] type inequalities concerning the weighted moduli of continuity of f and $L_n^* f$. Moreover, we present an application of these inequalities to approximation of f by the Kantorowich type operators in some Hölder type norms. Analogous results for the discrete operators (1.1) are given [11], for the Feller operators in [10] and for the Durrmeyer type operators in [12].

We adopt the following notation. Given any non-negative function w defined on the interval $I \subseteq R$ and any $x, y \in I$ we write

$$\widehat{w}(x,y) = \min\{w(x), w(y)\}.$$

For an arbitrary function g defined on I we introduce the quantity

$$\|g\|_w := \sup\{|g(x)|w(x) : x \in I\}.$$

We denote by $\Lambda(I)$ the set of all continuous functions w on I, positive in the interior of I, with values not greater than 1, which satisfy the inequality $\widehat{w}(x,y) \le w(s)$ for any three points $x, s, y \in I$ such that $x \le s \le y$. (Obviously, this inequality holds if, for example, w is non-decreasing or non-increasing or concave on I). Given two weights $w, \rho \in \Lambda(I)$ we define the general weighted modulus of continuity of g on I by

$$\Omega_{w,\rho}(g;\delta) := \sup\{|g(x) - g(y)|\widehat{w}(x,y)\widehat{\rho}(x,y) : x, y \in I, |x-y| \le \delta\} \quad (\delta > 0).$$

If $\rho(x) \equiv 1$ on I, we will write $\Omega_w(g;\delta)$ instead of $\Omega_{w,\rho}(g;\delta)$. Further, in the case $w(x) \equiv 1$ on I the weighted modulus $\Omega_w(g;\delta)$ becomes the ordinary modulus of continuity $\omega(g;\delta)$ of g on I. In this case, $w \equiv 1$, the symbol $\|g\|$ will be used instead of $\|g\|_w$. Let $C_w(I)$ denote the class of all functions g continuous on I, such that $\|g\|_w < \infty$.

Taking into account a positive non-decreasing function φ on the interval $(0,1]$, such that $\varphi(1) \le 1$, we write

$$\|g\|_{w,\rho}^{(\varphi)} := \|g\|_{w\rho} + \sup\left\{\frac{|g(x) - g(y)|\widehat{w}(x,y)\widehat{\rho}(x,y)}{\varphi(|x-y|)} : x, y \in I,\ 0 < |x-y| \le 1\right\}.$$

If this quantity is finite we call it the weighted Hölder type norm of g on I. In case $\rho \equiv 1$ and $w \equiv 1$ on I, we denote this expression by $\|g\|^{(\varphi)}$; if $\rho \not\equiv 1$, $w \equiv 1$, then we will write $\|g\|_\rho^{(\varphi)}$ instead of $\|g\|_{w,\rho}^{(\varphi)}$.

The symbols $c_\nu (\nu = 1, 2, \ldots)$ will mean some positive constants depending only on a given sequence $\{L_n\}$ and eventually on the considered weights w, ρ, η.

2. Inequalities for weighted moduli of continuity

Let L_n^* be the operators defined by (1.2). Consider a function $f \in M_{\text{loc}}(I) \cap \operatorname{Dom}(L_n^*)$ for all $n \geq n_0$, with some fixed positive integer n_0.

THEOREM 2.1. *Suppose that*

$$\sum_{j \in J_n} p_{j,n}(x) \equiv 1 + r_n(x) \leq c_1 \quad \textit{for all} \;\; x \in I, n \geq n_0 \tag{2.1}$$

and that

$$\sum_{j \in J_n} (m_{j,n})^{-1} |p'_{j,n}(x)| \int_{I_{j,n}} |t - x| dt \leq \frac{c_2}{w(x)} \tag{2.2}$$

for $n \geq n_0$ *and a.e.* $x \in \operatorname{Int} I$,

where $w \in \Lambda(I)$. *Suppose, moreover, that there exist the weights* $\rho, \lambda \in \Lambda(I)$ *such that* ρ, λ *are positive in* I, $\rho \leq \lambda$ *and*

$$L_n^* \left(\frac{1}{\lambda} \right) (x) \leq \frac{c_3}{\rho(x)} \quad \textit{for all } x \in I, n \geq n_0, \tag{2.3}$$

$$\sum_{j \in J_n} (m_{j,n})^{-1} |p'_{j,n}(x)| \int_{I_{j,n}} \frac{|t - x|}{\lambda(t)} dt \leq \frac{c_4}{w(x)\rho(x)} \tag{2.4}$$

for $n \geq n_0$ *and a.e.* $x \in \operatorname{Int} I$

hold. Then, for $\delta \geq 0$ *and* $n \geq n_0$,

$$\Omega_{w,\rho}(L_n^* f; \delta) \leq 2(c_1 \|w\| + c_2 + c_3 \|w\| + c_4) \Omega_\lambda(f; \delta) + \|f\|_{w\rho} \omega(r_n; \delta).$$

P r o o f. Let $x, y \in I, 0 < y - x \leq \delta$ and let $\xi = (x + y)/2$. Then

$$\begin{aligned} &L_n^* f(y) - L_n^* f(x) \\ &= \sum_{j \in J_n} (m_{j,n})^{-1} (p_{j,n}(y) - p_{j,n}(x)) \int_{I_{j,n}} f(t) dt \\ &= \sum_{j \in J_n} (m_{j,n})^{-1} (p_{j,n}(y) - p_{j,n}(x)) \int_{I_{j,n}} (f(t) - f(\xi)) dt + f(\xi)(r_n(y) - r_n(x)). \end{aligned}$$

It is easy to check that for all $t, \tau \in I$ there holds the inequality

$$|f(t) - f(\tau)| \widehat{\lambda}(t, \tau) \leq \left(1 + \left[\frac{1}{\delta} |t - \tau| \right] \right) \Omega_\lambda(f; \delta), \tag{2.5}$$

where $[a]$ denotes the integer part of the number a. Hence

$$|L_n^* f(y) - L_n^* f(x)| \leq A_n(x, y) \Omega_\lambda(f; \delta) + |f(\xi)| \omega(r_n; \delta),$$

where

$$A_n(x,y) = \sum_{j\in J_n} (m_{j,n})^{-1}(p_{j,n}(y) - p_{j,n}(x)) \int\limits_{I_{j,n}} \left(1 + \left[\frac{|t-\xi|}{\delta}\right]\right) \frac{1}{\widehat{\lambda}(t,\xi)} dt.$$

Observing that for every $t \in I$

$$\frac{\widehat{\rho}(x,y)}{\widehat{\lambda}(t,\xi)} \le 1 + \frac{\widehat{\rho}(x,y)}{\lambda(t)} \tag{2.6}$$

and applying (2.1), we obtain

$$\begin{aligned} &A_n(x,y)\widehat{\rho}(x,y) \\ &\quad\le 2c_1 + \sum_{j\in J_n} (m_{j,n})^{-1}|p_{j,n}(x) - p_{j,n}(y)| \int\limits_{I_{j,n}} \frac{\widehat{\rho}(x,y)}{\lambda(t)} dt \\ &\qquad + \sum_{j\in J_n} (m_{j,n})^{-1} \int\limits_x^y |p'_{j,n}(s)|ds \int\limits_{I_{j,n}} \left(1 + \frac{\widehat{\rho}(x,y)}{\lambda(t)}\right) \left[\frac{|t-\xi|}{\delta}\right] dt. \end{aligned}$$

Applying (2.3) we get

$$\begin{aligned} &A_n(x,y)\widehat{\rho}(x,y) \\ &\le 2c_1 + 2c_3 + \frac{1}{\delta}\int\limits_x^y \sum_{j\in J_n} (m_{j,n})^{-1}|p'_{j,n}(s)| \int\limits_{I\setminus I_\delta} \left(1 + \frac{\widehat{\rho}(x,y)}{\lambda(t)}\right)|t-\xi|\chi_{j,n}(t)\,dt\,ds, \end{aligned}$$

where $I_\delta = I \cap (\xi-\delta, \xi+\delta)$, $\chi_{j,n}$ denotes the characteristic function of the interval $I_{j,n}$. The inequality $|t-\xi| \le 2|t-s|$ $(t \in I\setminus I_\delta, x \le s \le y)$ and the assumptions (2.2) and (2.4) lead to

$$A_n(x,y)\widehat{\rho}(x,y) \le 2c_1 + 2c_3 + \frac{2}{\delta}c_2\int\limits_x^y \frac{ds}{w(s)} + \frac{2}{\delta}c_4\int\limits_x^y \frac{\widehat{\rho}(x,y)}{w(s)\rho(s)} ds.$$

Hence

$$\begin{aligned} &A_n(x,y)\widehat{\rho}(x,y)\widehat{w}(x,y) \\ &\qquad\le 2(x_1 + c_2)\|w\| + \frac{2}{\delta}c_2\int\limits_x^y \frac{\widehat{w}(x,y)}{w(s)} ds + \frac{2}{\delta}c_4\int\limits_x^y \frac{\widehat{w}(x,y)}{w(s)} ds \\ &\qquad\le 2(c_1\|w\| + c_3\|w\| + c_2 + c_4). \end{aligned}$$

The assertion of Theorem 1.1 is now evident.

If $\lambda = 1, \rho = 1$ on I, then the constant c_3 in (2.3) becomes c_1 and the assumption (2.4) reduces to (2.2). Then from Theorem 2.1 we obtain

COROLLARY 2.1. *Under assumptions* (2.1) *and* (2.2) *we have*

$$\Omega_w(L_n^* f;\delta) \le 2(c_1\|w\| + c_2)\omega(f;\delta) + \|f\|_w\omega(r_n;\delta) \quad \textit{for } n \ge n_0 \textit{ and } \delta \ge 0.$$

If I is an unbounded interval and λ is a plynomial weight of class $\Lambda(I)$, then a slight modification of the proof of Theorem 2.1 (see e.g. [10], Th. 3 or [11], Th. 3) leads to the following

THEOREM 2.2. *Let conditions* (2.1), (2.2) *be satisfied and let* $\lambda(x) = (1+|x|)^{-p}$ *for* $x \in I$, *where* p *is a positive parameter. If condition (2.4) holds with* $\rho = \lambda$, *then for* $\delta \geq 0, n \geq n_0$

$$\Omega_{w,\lambda}(L_n^* f; \delta) \leq c_5 \Omega_\lambda(f; \delta) + \|f\|_{w\lambda}\omega(r_n; \delta),$$

where $c_5 = 2(c_1\|w\| + 2 \cdot 3^p\|w\| + c_2 + c_4)$.

3. Approximation in Hölder norms

In order to estimate the weighted norm of the difference $L_n^* f - f$, let us introduce the moments

$$\mu_{2,n}^*(x) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} (t-x)^2 dt \quad (x \in I,\ n \in N).$$

THEOREM 3.1. *Let condition* (2.1) *be satsfied and let*

$$\rho(x) L_n^* \left(\frac{1}{\lambda^2}\right)(x) \leq \frac{c_6}{\lambda(x)} \quad \text{for all } x \in I, n \geq n_0, \tag{3.1}$$

$$\rho(x)\mu_{2,n}^*(x) \leq c_7 \lambda(x) \delta_n^2 \quad \text{for all } x \in I, n \geq n_0, \tag{3.2}$$

where ρ, λ *are positive functions on* I, *such that* $\rho \leq \lambda$. *If* $f \in C_\rho(I)$, *then for* $n \geq n_0$,

$$\|L_n^* f - f\|_\rho \leq c_8 \Omega_\lambda(f; \delta_n) + \|f\|_\rho \|r_n\|,$$

where $c_8 = c_1 + \sqrt{c_1 c_6} + c_7 + \sqrt{c_1 c_7}$.

Since the proof is similar to that of the proof of Theorem 4 in [12] we do not write them explicitly.

REMARK 3.1. The Cauchy-Schwarz inequality ensures that the fulfilment of the assumption (3.1) implies (2.3) with $c_3 = \sqrt{c_6}$.

REMARK 3.2. It is easily seen that under the assumptions (2.1) and (3.1) with $\lambda(x) = 1$ and $\rho(x) \leq 1$,

$$\|L_n^* f - f\|_\rho \leq (c_1 + c_7)\omega(f; \delta_n) + \|f\|_\rho \|r_n\|.$$

Taking into account the Hölder type norm, we can state that for an arbitrary $\nu_n \in (0, 1]$,

$$(3.3)\qquad \|L_n^*f-f\|_{w,\lambda}^{(\varphi)} \le \left(1+\frac{2}{\varphi(\nu_n)}\right)\|L_n^*f-f\|_{w\lambda} + \sup\left\{\frac{1}{\varphi(\delta)}(\Omega_{w,\lambda}(f;\delta)+\Omega_{w,\lambda}(L_n^*f;\delta)) : 0<\delta\le\nu_n\right\}$$

(see [11] and [8]). This inequality, Corollary 2.1 and Theorem 3.1 allow us to state the following

THEOREM 3.2. *Let conditions* (2.1), (2.2) *be satisfied and let* $(\delta_n)_1^\infty$ *be a sequence of numbers from* $(0,1]$, *for which* (3.2) *holds with* $\rho=w$, $\lambda=1$ *on* I. *If* $f\in C_w(I)$, *then for* $n\ge n_0$,

$$\|L_n^*f-f\|_w^{(\varphi)} \le c_9\sup\left\{\frac{\omega(f;\delta)}{\varphi(\delta)} : 0<\delta\le\delta_n\right\} + \|f\|_w\Delta_n^{(\varphi)},$$

where $c_9=6c_1+3c_7+3\sqrt{c_1c_7}+2c_1\|w\|+2c_2+\|w\|$, *and*

$$\Delta_n^{(\varphi)} = \frac{3\|r_n\|}{\varphi(\delta_n)} + \sup\left\{\frac{\omega(r_n;\delta)}{\varphi(\delta)} : 0<\delta\le\delta_n\right\}.$$

4. The method of K-functionals

Let us note that Corollary 2.1 can be applied for every measurable locally bounded function f, for which the operators (1.2) are meaningful and $\omega(f;\delta)<\infty$ for $\delta>0$. Assuming that f is continuous and bounded on $I(f\in C_0(I))$ and applying the K-functional method we can get for certain operators better estimates. In [1], Anastassiou, Cottin, Gonska obtained the best estimate of the modulus of continuity of operators (1.1) in the case where I is a compact interval.

Let $f\in C_0(I)$. Then its modulus of continuity is equivalent to the K-functional

$$K(f;t)=\inf\{\,\|f-g\|+t\|g'\| : g\in C_0^1(I)\}\quad (t\ge 0),$$

where $C_0^1(I)$ denotes the class of all functions g possessing derivative g', such that $g'\in C_0(I)$. This means that there exists a constant $M>0$, independent of f and t, such that

$$(4.1)\qquad M^{-1}\omega(f;\delta)\le K(f;t)\le M\omega(f;t)$$

for $t\ge 0$ (see [3]).

THEOREM 4.1. *Let* $L: C_0(I)\to C_0(I), L\ne 0.$ *be bounded linear operator mapping* $C_0^1(I)$ *into* $C_0^1(I)$ *and let* w *be the weight function of class* $\Lambda(I)$. *If there exists a positive constant* c, *such that for every* $g\in C_0^1(I)$

$$(4.2)\qquad \|w(Lg)'\|\le c\|g'\|,$$

then there exists a constant $M > 0$, such that for all $f \in C_0(I)$ and all $\delta > 0$

$$\Omega_w(Lf;\delta) \le M\omega(f;\delta).$$

Proof. Let $f \in C_0(I), \delta > 0$. It is easy to check that $\Omega_w(Lf;\delta) \le 2\|L\|\,\|f-g\| + c\delta\|g'\|$ for $\delta > 0$, where $\|L\|$ means the norm of the operator L and $g \in C_0^1(I)$. Hence

$$\Omega_w(Lf;\delta) \le 2\|L\|K\left(f;\frac{c\delta}{2\|L\|}\right). \tag{4.3}$$

The desired estimate follows by (4.1).

Now, we take into account operators L_n^* and assume that (2.1) and (2.2) are satisfied with $r_n(x)$ independent of x. For every $f \in C_0(I)$ we have $\sup_{x\in I}|L_n^* f(x)| \le c_1\|f\|$ $(n \in N, n \ge n_0)$. Moreover assuming that the series inthe definition (1.1) can be differentiated term by term in I, we get for $g \in C_0^1(I)$

$$(L_n^* g)'(x) = \sum_{j\in J_n}(m_{j,n})^{-1}p'_{j,n}(x)\int_{I_{j,n}} g(t)dt.$$

Using the Taylor expansion with the integral remainder, applying (2.2) and observing that $\sum_{j\in J_n} p'_{j,n}(x) = 0$ we get

$$\begin{aligned}|(L_n^* g)'(x)| &= \left|g(x)\sum_{j\in J_n} p'_{j,n}(x) + \sum_{j\in J_n}(m_{j,n})^{-1}p_{j,n}(x)\int_{I_{j,n}}\left(\int_x^t g'(s)\,ds\right)dt\right| \\ &\le \frac{c_2}{w(x)}\|g'\|.\end{aligned}$$

Finally by Theorem 4.1 we obtain

$$\Omega_w(L_n^* f;\delta) \le c_{10}\omega(f;\delta) \quad (\delta > 0), \tag{4.4}$$

for every $f \in C_0(I)$ and $n > n_0$. Hence Theorem 4.1 implies Corollary 2.1.

REMARK 4.1. Let us observe that for certain operators L_n^* it is convenient to verify condition (4.2) directly. Sometimes, this conditon is true with the weight $w(x) = 1$. Then the inequality (4.4) takes the form

$$\omega(L_n^* f;\delta) \le c_{11}\omega(f;\delta).$$

In particular, such estimate can be obtained for the Kantorovich polynomials, the Szász–Kantorvich operators and for the Baskakov–Kantorovich ones.

5. Examples

For many known operators defined by (1.2) the functions $r_n(x) = 0$ for all $x \in I, n \in N$ and the quantities

$$\mu_{2,n}(x) = \sum_{j\in J_n} (\xi_{j,n} - x)^2 p_{j,n}(x)$$

are finite at every $x \in I$ and positive in $\operatorname{Int} I$; moreover

$$p'_{j,n}(x)\mu_{2,\mu}(x) = p_{j,n}(x)(\xi_{j,n} - x) \tag{5.1}$$

for all $x \in I, n \in N$. In this case the left-hand side of the inequality (2.2) can be estimated from above by

$$\sum_{j\in J_n} (m_{j,n})^{-1}|p'_{j,n}(x)| \int\limits_{I_{j,n}} |t - \xi_{j,n}|dt + \frac{1}{\mu_{2,n}(x)} \sum_{j\in J_n} (\xi_{j,n} - x)^2 p_{j,n}(x)$$
$$\leq \frac{1}{\mu_{2,n}(x)} \sum_{j\in J_n} m_{j,n}|\xi_{j,n} - x|p_{j,n}(x) + 1,$$

whenever $x \in \operatorname{Int} I$. Further, by the Cauchy–Schwarz inequality, we get

$$\sum_{j\in J_n} (m_{j,n})^{-1}|p'_{j,n}(x)| \int\limits_{I_{j,n}} |t - x|\, dt \leq \frac{1}{\sqrt{\mu_{2,n}(x)}} \sqrt{\sum_{j\in J_n} m^2_{j,n} p_{j,n}(x)} + 1. \tag{5.2}$$

If, moreover $m_{j,n} = d_n$ for all $j \in J_n$, then condition (2.2) can be replaced by

$$\sum_{j\in J_n} (m_{j,n})^{-1}|p'_{j,n}(x)| \int\limits_{I_{j,n}} |t - x|dt \leq \frac{d_n\sqrt{c_1}}{\sqrt{\mu_{2,n}(x)}} + 1.$$

In particular, for certain discrete operators we have $\mu_{2,n}(x) = \sigma^2(x)/n$, $c_1 = 1$ and $d_n = 1/(n + \gamma)$ ($\gamma = 0$ or $\gamma = 1$).

1. The Kantorovich polynomials are defined as

$$B^*_n f(x) = (n+1) \sum_{j=0}^{n} p_{j,n}(x) \int\limits_{j/(n+1)}^{(j+1)/(n+1)} f(t)dt,$$

where $p_{j,n}(x) = \binom{n}{j} x^j (1-x)^{n-j}, x \in I = [0,1]$.

In this case $c_1 = 1, d_n = 1/(n+1), r_n(x) = 0, \mu_{2,n}(x) = x(1-x)/n$ and the equality (5.1) is fulfilled with $\xi_{j,n} = j/n$.

Observing that

$$p'_{j,n}(x) = n(p_{j-1,n-1}(x) - p_{j,n-1}(x)) \quad (j = 0, 1, \dots, n),$$

where $p_{k,n-1}(x) = 0$ if $k < 0$ and using the Lagrange mean value theorem we obtain

$$(B_n^* g)'(x) = n \sum_{j=0}^{n-1} p_{j,n-1}(x) \int_{j/(n+1)}^{(j+1)/(n+1)} g'(\Theta_{t,j,n}) dt$$

for $g \in C^1[0,1]$, where $\Theta_{t,j,n} \in \left[t + \frac{1}{n+1}, t\right]$.

Hence, the condition (4.2) takes the form

$$|(B_n^* g)'(x)| \leq \|g'\|.$$

In view of (4.3) and Corollary 6 of [1] we get

$$\omega(B_n^* f; \delta) \leq 2K\left(f; \frac{\delta}{2}\right) \leq 2\omega(f;\delta) \quad \text{for any } f \in C(I).$$

Theorem 3.1 applies with $\rho(x) = \lambda(x) = 1, c_6 = 1, c_7 = 7/12$ and $\delta_n = 1/\sqrt{n}$. Namely

$$\begin{aligned}\mu_{2,n}^*(x) &= \sum_{j=0}^{n} (m_{j,n})^{-1} p_{j,n}(x) \int_{j/(n+1)}^{(j+1)/(n+1)} (t-x)^2 dt \\ &= \frac{x(1-x)(n-1)}{(n+1)^2} + \frac{1}{3(n+1)^2} \leq \frac{7}{12n}.\end{aligned}$$

Thus Theorem 3.1 (via Remark 3.2) gives

$$\|B_n^* f - f\| \leq \left(1 + \frac{7}{12}\right) \omega\left(f; \frac{1}{\sqrt{n}}\right) \quad \text{for all } x \in N.$$

Consequently, from Theorem 3.2 it follows the estimate

$$\|B_n^* f - f\|^{(\varphi)} \leq \left(6 + \frac{7}{4}\right) \sup\left\{\frac{\omega(f;\delta)}{\varphi(\delta)} : 0 < \delta \leq \frac{1}{\sqrt{n}}\right\}. \tag{5.3}$$

2. The Szász–Kantorovich operators are defined as

$$S_n^* f(x) = n \sum_{j=0}^{\infty} p_{j,n}(x) \int_{j/n}^{(j+1)/n} f(t) dt,$$

where $p_{j,n}(x) = e^{-nx}(nx)^j/j!, x \in I = [0,\infty)$. Now $c_1 = 1, r_n(x) = 0, d_n = 1/n, \mu_{2,n}(x) = x/n$ and the identity (5.1) is fulfilled with $\xi_{j,n} = j/n$. In view of (5.2), condition (2.2) holds with the weight $w(x) = \sqrt{x/(1+x)}$ and with the constant $c_2 = 2$.

Let us choose the weight $\lambda(x) = (1+x)^{-p}$ for $x > 0$, where p is a positive parameter. In order to verify assumption (2.4) we first observe that

$$\int_{j/n}^{(j+1)/n} (1+t)^{2p} dt \leq 2^{2p}\left((1+x)^{2p}\frac{1}{n} + \left(\frac{j+1}{n} - x\right)^{2p} \int_{j/n-x}^{(j+1)/n-x} du\right)$$
$$\leq 2^{2p}\left((1+x)^{2p}\frac{1}{n} + 2^{2p}\left(\left(\frac{j}{n} - x\right)^{2p} + \left(\frac{1}{n}\right)^{2p}\right)\frac{1}{n}\right).$$

By the identity (5.1) and the Cauchy–Schwarz inequality we get, for $x \in \operatorname{Int} I$,

$$n\sum_{j=0}^{\infty} |p'_{j,n}(x)| \int_{j/n}^{(j+1)/n} |t-x|(1+t)^p dt$$
$$\leq \frac{\sqrt{\mu^*_{2,n}(x)}}{\mu_{2,n}(x)}\left(n\sum_{j=0}^{\infty}\left(\frac{j}{n} - x\right)^2 p_{j,n}(x) \int_{j/n}^{(j+1)/n} (1+t)^{2p}\, dt\right)^{1/2}.$$

Applying the estimate (13') given in [7] (p. 332) we have

$$\sum_{j=0}^{\infty}\left(\frac{j}{n} - x\right)^{2p+2} p_{j,n}(x) \leq c(p)(1+x)^{2p}|\mu_{2,n}(x)|,$$

where c is a positive constant depending only on p. Hence

$$n\sum_{j=0}^{\infty} |p'_{j,n}(x)| \int_{j/n}^{(j+1)/n} |t-x|(1+t)^p\, dt$$
$$\leq \sqrt{\mu^*_{2,n}(x)/\mu_{2,n}(x)}\sqrt{2^{2p} + 2^{4p}c + 2^{4p}}(1+x)^p$$
$$\leq \sqrt{\mu^*_{2,n}(x)/\mu_{2,n}(x)}c_{12}(1+x)^p,$$

where c_{12} is a constant depending only on p. We can easily verify that in this case,

$$\mu^*_{2,n}(x) = \frac{x}{n} + \frac{1}{3n^2} \leq \frac{1}{n}(1+x).$$

Hence the left-hand side of (2.4) is not greater than $c_{12}(1+x)^p\sqrt{(1+x)/x}$. Thus the assumption (2.4) is satisfied with $w(x) = \sqrt{x/(1+x)}$, $\rho(x) = \lambda(x) = (1+x)^{-p}$ and $c_4 = c_{12}$. In view of Theorem 2.2, given any $f \in C_\lambda(I)$ we have

$$\Omega_{w,\lambda}(S^*_n f;\delta) \leq c_{13}\Omega_\lambda(f;\delta), \quad \text{for } n \in N, \tag{5.4}$$

where $c_{13} \le 2(3 + 2 \cdot 3^p + 2c_{12})$. The assumptions (3.1), (3.2) remain valid for $\lambda(x) = (1+x)^{-p}, \rho(x) = (1+x)^{-(p+1)}$ and $\delta_n = 1/\sqrt{n}$. Indead,

$$S_n^*(1/\lambda^2)(x) = n \sum_{j=0}^{\infty} p_{j,n}(x) \int_{j/n}^{(j+1)/n} (1+t)^{2p} dt$$
$$\le \sum_{j=0}^{\infty} p_{j,n}(x) \left\{ 2^{2p}(1+x)^{2p} + 2^{4p}\left(\frac{j}{n} - x\right)^{2p} + 2^{4p}\frac{1}{n^{2p}} \right\}.$$

By (2.1) and the mentioned estimate (13') of [7] we obtain

$$S_n^*(1/\lambda^2)(x) \le 2^{2p}(1+x)^{2p} + 2^{4p}c(1+x)^{2(p-1)}\mu_{2,n}(x) + 2^{4p}\frac{1}{n^{2p}} \le c_{14}(1+x)^{2p}.$$

Consequently

$$\rho(x)S_n^*(1/\lambda^2)(x) \le c_{14}(1+x)^p.$$

The assumption (3.2) has the form

$$\rho(x)\mu_{2,n}^*(x) \le \frac{1}{n}(1+x)^{-p}.$$

Hence $c_7 = 1, \delta_n = 1/\sqrt{n}$. Applying Theorem 3.1 we get

$$\|S_n^* f - f\|_\rho \le 2(1 + \sqrt{c_{14}})\Omega_\lambda\left(f; \frac{1}{\sqrt{n}}\right) \quad \text{for } n \in N.$$

Combining this results and (5.4) with the inequality (3.3) we can easily get the corresponding estimate for $\|S_n^* f - f\|_{w,\rho}^{(\varphi)}$ with $w(x) = \sqrt{x/(1+x)}$, $\rho(x) = (1+x)^{-p+1}$, $\lambda(x) = (1+x)^{-p}$.

3. Analogous results can be obtained for the Baskakov–Kantorovich operators:

$$U_n^* f(x) = n \sum_{j=0}^{\infty} p_{j,n}(x) \int_{j/n}^{(j+1)/n} f(t)dt,$$

where $p_{j,n}(x) = \binom{n+j-1}{j} x^j (1+x)^{-n-j}$, $x \in I = [0, \infty)$. Since the computations are similar to the preceding example we omit the details.

4. The generalized Favard–Kantorovich operators are defined by

$$F_n^* f(x) = n \sum_{j=-\infty}^{\infty} p_{j,n}(x) \int_{j/n}^{(j+1)/n} f(t)dt,$$

where $p_{j,n}(x) = p_{j,n}(\gamma; x) = \frac{1}{\sqrt{2\pi n}\gamma_n} \exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{j}{n} - x\right)^2\right)$, $x \in I = R$, $\gamma = (\gamma_n)_1^\infty$ being a positive null sequence satisfying the assumption: $n\gamma_n^2 \ge \frac{1}{2}\pi^{-2}\log n$ for $n \ge 2, \gamma_1^2 \ge \frac{1}{2}\pi^{-2}\log 2$.

As is known ([4], p. 388; [7], p. 336), for $x \in I, n \in N$,

$$|r_n(x)| \equiv |r_n(\gamma; x)| = \Big| \sum_{j=-\infty}^{\infty} p_{j,n}(x) - 1 \Big| \le 2$$

or

$$|r_n(\gamma; x)| \le 7\pi\gamma_n$$

and

$$\mu_{2,n}(x) \equiv \mu_{2,n}(\gamma; x) = \sum_{j=-\infty}^{\infty} \left(\frac{j}{n} - x\right)^2 p_{j,n}(x) \le 51\gamma_n^2.$$

Moreover $\omega(r_n; \delta) \le 16\pi\delta$ for every $\delta \ge 0$. In [11] we can found the estimate $\mu_{2,n}(2\gamma; x) \le 23\gamma_n^2$, where $2\gamma = (2\gamma_n)_{n=1}^{\infty}$. Observing that

$$p'_{j,n}(\gamma; x) = \frac{1}{\gamma_n^2}\left(\frac{j}{n} - x\right) p_{j,n}(\gamma; x)$$

we estimate the left hand side of (2.2) by

$$\frac{1}{\gamma_n^2 n}\sqrt{\mu_{2,n}(x)}\sqrt{1 + r_n(x)} + 51 \le 124 \quad \text{for } n \ge 2.$$

Thus Corollary 2.1 yields the estimate

$$\omega(F_n^* f; \delta) \le 2(3 + 124)\omega(f; \delta) + \|f\| 16\pi\delta \quad \text{for } n \ge 2 \text{ and } f \in C(I).$$

Now, let $f \in C_\lambda(I)$, where $\lambda(x) = \exp(-px^2), p > 0$. Then the condition (2.3) is fulfilled for $n \in N$ such that $32p\gamma_n \le 3$, with $\rho(x) = \exp(-4px^2), \lambda(x) = \exp(-px^2)$ and the constant $c_3 = e^{2p}34/15$. It is easy to see that

$$\begin{aligned}\mu_{2n}^*(2\gamma; x) &\le \frac{1}{3n^2}(1 + r_n(2\gamma; x)) \\ &\quad + \frac{1}{n}\sqrt{\mu_{2,n}(2\gamma; x)}\sqrt{\Sigma p_{j,n}(2\gamma; x)} + \mu_{2,n}(2\gamma; x) \\ &\le c_{15}\gamma_n^2.\end{aligned}$$

Hence the condition (2.4) takes the form

$$n \sum_{j=-\infty}^{\infty} |p'_{j,n}(x)| \int_{j/n}^{(j+1)/n} |t - x| e^{pt^2} dt \le \sqrt{102 c_{15}} e^{2p} e^{4px^2}$$

for $n \in N$ such that $64p\gamma_n^2 \le 3$. Thus Theorem 2.1 applies with $\lambda(x) = \exp(-px^2), \rho(x) = \exp(-4px^2), w(x) = 1$ and $c_3 = 34e^{2p}/15, c_4 = \sqrt{102c_{15}}$, $c_2 = 124$. Theorem 2.1 gives

$$\Omega_\rho(F_n^* f; \delta) \le c_{16}\Omega_\lambda(f; \delta) + \|f\|_\rho \cdot 16\pi\delta,$$

for all $n \in N$ such that $64p\gamma_n^2 \leq 3$. We can show that

$$F_n^*\left(\frac{1}{\lambda^2}\right)(x) \leq 34e^{4p}e^{8px^2}/15$$

and

$$\mu_{2,n}^*(\gamma; x) \leq 158\gamma_n^2.$$

Thus Theorem 3.1 is true with $\lambda(x) = \exp(-px^2), \rho(x) = \exp(-7px^2)$, $c_5 = 34e^{4p}/15, c_6 = c_{16}$ and $\delta_n = \gamma_n$ for $n \in N$ such that $64p\gamma_n^2 \leq 3$. Consequently, Theorem 3.1 gives

$$\|F_n^* f - f\|_\rho \leq c_{17}\Omega_\lambda(f; \gamma_n) + \|f\|_\rho 7\pi\gamma_n,$$

for n such that $64p\gamma_n^2 \leq 3$. From these results we see at once that

$$\|F_n^* f - f\|_\rho^{(\varphi)} \leq (1 + c_{16} + 3c_{17}) \sup\left\{\frac{\Omega_\lambda(f;\delta)}{\varphi(\delta)} : 0 < \delta \leq \gamma_n\right\} + \|f\|_\rho \left(\frac{21\pi\gamma_n}{\varphi(\gamma_n)} + \sup\left\{\frac{16\pi\delta}{\varphi(\delta)} : 0 < \delta \leq \gamma_n\right\}\right).$$

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