DEMONSTRATIO MATHEMATICA
Vol. XXXV No 4 2002

Tiberiu Trif

ON THE SUPERSTABILITY OF CERTAIN FUNCTIONAL EQUATIONS

Abstract. We investigate the superstability of the functional equations

(1) $$f(x^y) = yf(x),$$

(2) $$f(x^y) = yx^{y-1}f(x),$$

(3) $$f(\alpha x) = |\alpha|^p f(x).$$

We prove new results concerning the superstability of the equation (2) (both in the conventional sense and in the sense of R. Ger) and of the equation (3) (in the conventional sense). Likewise, we provide new simple proofs for stronger versions of already known results on the superstability of the equation (1) (both in the conventional sense and in the sense of R. Ger) and of the equation (3) (in the sense of R. Ger).

1. Introduction

The starting point of the present paper is the article [5] by S.-M. Jung, investigating the functional equation

(1) $$f(x^y) = yf(x).$$

In [5, Theorem 1], it was proved that a differentiable function $f : \mathbf{R}_+ \to \mathbf{R}$, satisfying (1) for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$, must be of the form $f(x) = c \ln x$, where $c = f(e)$ is an arbitrary real constant (here and throughout the rest of the paper $\mathbf{R}_+$ denotes the set of all positive real numbers). It should be noted that the same conclusion can be achieved under milder assumptions on the function f.

THEOREM 1. *If a continuous function $f : \mathbf{R}_+ \to \mathbf{C}$ satisfies $f(x^n) = nf(x)$ for all $x \in \mathbf{R}_+$ and all integers n, then it must be of the form $f(x) = c \ln x$, where $c = f(e)$.*

1991 *Mathematics Subject Classification*: 39B72, 39B22.

Proof. We have $f(x) = nf(x^{1/n})$, hence $f(x^{1/n}) = \frac{1}{n}f(x)$. Consequently, $f(x^{m/n}) = \frac{m}{n}f(x)$, i.e., $f(x^r) = rf(x)$ for all $x \in \mathbf{R}_+$ and all rational numbers r. In particular $f(e^r) = rf(e) = f(e)\ln(e^r)$ for all rational numbers r.

Now let x be an arbitrary positive real number, and let (r_k) be a sequence of rational numbers converging to $\ln x$. Then (e^{r_k}) converges to x, so, by virtue of the continuity of f, we have

$$f(x) = \lim_{k\to\infty} f(e^{r_k}) = \lim_{k\to\infty} f(e)\ln(e^{r_k}) = f(e)\ln x. \ \blacksquare$$

REMARK 1. According to a result of J. Milkman [8, Theorem II], the above theorem remains true if the word 'continuous' is replaced by 'monotone' and the range of f is $\mathbf{R}$.

With some effort, in [5, Theorem 2 and Theorem 8], S.-M. Jung proved the superstability of the functional equation (1), in the conventional setting as well as in the sense of R. Ger. It seems that Jung's proofs are more complicated than necessary. This assertion is justified by the very short proofs presented in the next two sections. Their simple idea is similar to that one used by S. Czerwik in the proofs of some results stated in his paper [2]. This simple idea can be also applied to the superstability of other functional equations, such as

$$f(x^y) = yx^{y-1}f(x) \tag{2}$$

or the homogeneous functional equation

$$f(\alpha x) = |\alpha|^p f(x). \tag{3}$$

It should be mentioned that the superstability of the equation (3) has already been investigated, but in different settings, by other authors: J. Chudziak [1], S. Czerwik [2], S.-M. Jung [6, 7], J. Tabor and J. Tabor [9] (see also the monograph [4, pp. 70–77]). To our knowledge, the superstability of the equation (2) is proved here for the first time.

2. Superstability in the conventional sense of the functional equations (1), (2) and (3)

In [5, Theorem 2], S.-M. Jung gave a long and complicated proof for the superstability of the equation (1), in the conventional setting:

THEOREM 2. *Let $\delta \geq 0$. If the function $f : \mathbf{R}_+ \to \mathbf{C}$ satisfies*

$$|f(x^y) - yf(x)| \leq \delta \tag{4}$$

for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$, then (1) holds true for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$.

In what follows, we present a different

Short proof of Theorem 2. Take $x = 1$ in (4) and send y to infinity in order to see that $f(1) = 0$. Therefore (1) holds true for $y = 0$ and all $x \in \mathbf{R}_+$.

Replacing y by ty in (4) and then dividing both sides of the obtained inequality by $t|y|$, we find that

$$\left|\frac{f(x^{ty})}{ty} - f(x)\right| \le \frac{\delta}{t|y|} \qquad \text{for all } x, t \in \mathbf{R}_+ \text{ and all } y \in \mathbf{R} \setminus \{0\}.$$

This inequality implies that

$$\lim_{t\to\infty} \frac{f(x^{ty})}{ty} = f(x) \qquad \text{for all } x \in \mathbf{R}_+ \text{ and all } y \in \mathbf{R} \setminus \{0\}.$$

Taking this into account, for each $x \in \mathbf{R}$ and each $y \in \mathbf{R} \setminus \{0\}$ we have

$$f(x^y) = \lim_{t\to\infty} \frac{f\left((x^y)^{ty}\right)}{ty} = y \lim_{t\to\infty} \frac{f\left(x^{ty^2}\right)}{ty^2} = yf(x). \ \blacksquare$$

The above argument can be used to establish the superstability of other functional equations. Indeed, let us consider the functional equation (2), inspired by the power derivation formula $(u^y)' = yu^{y-1}u'$. This functional equation is closely related to the equation (1).

THEOREM 3. *A function $f : \mathbf{R}_+ \to \mathbf{C}$ satisfies (2) for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$ if and only if there exists a function $g : \mathbf{R}_+ \to \mathbf{C}$, satisfying*

$$g(x^y) = yg(x) \qquad \text{and} \qquad f(x) = xg(x) \tag{5}$$

for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$.

Proof. Let $f : \mathbf{R}_+ \to \mathbf{C}$ be a function satisfying (2) for all $x \in \mathbf{R}_+$ and $y \in \mathbf{R}$; $g : \mathbf{R}_+ \to \mathbf{C}$ be the function defined by $g(x) := \frac{f(x)}{x}$. Dividing both sides of (2) by x^y we find that $g(x^y) = yg(x)$ for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$.

Conversely, let $g : \mathbf{R}_+ \to \mathbf{C}$ be a function satisfying (5) for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$. Then we have $f(x^y) = x^y g(x^y) = yx^{y-1}xg(x) = yx^{y-1}f(x)$ for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$. $\blacksquare$

In the next theorem, the superstability of the functional equation (2) is proved.

THEOREM 4. *Let $\delta \ge 0$. If the function $f : \mathbf{R}_+ \to \mathbf{C}$ satisfies*

$$\left|f(x^y) - yx^{y-1}f(x)\right| \le \delta \tag{6}$$

for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$, then (2) holds true for all $x \in \mathbf{R}_+$ and all $y \in \mathbf{R}$.

Proof. Taking $x = 1$ in (6) and then sending y to infinity we see that $f(1) = 0$. Hence (2) holds true for $y = 0$ and all $x \in \mathbf{R}_+$, as well as for $x = 1$ and all $y \in \mathbf{R}$.

Replacing y by ty in (6) and then dividing both sides of the obtained inequality by $|ty|x^{ty-1}$, we get

$$\left|\frac{f(x^{ty})}{tyx^{ty-1}} - f(x)\right| \leq \frac{\delta}{|ty|x^{ty-1}} \qquad \text{for all } x \in \mathbf{R}_+ \text{ and all } y, t \in \mathbf{R} \setminus \{0\}.$$

This inequality implies that

$$\lim_{t\to\infty} \frac{f(x^{ty})}{tyx^{ty-1}} = f(x) \qquad \text{for all } \begin{cases} x > 1, \ y > 0 \\ x < 1, y < 0 \end{cases} \tag{7}$$

and

$$\lim_{t\to-\infty} \frac{f(x^{ty})}{tyx^{ty-1}} = f(x) \qquad \text{for all } \begin{cases} x > 1, \ y < 0 \\ x < 1, y > 0. \end{cases} \tag{8}$$

Now let $x > 1$ and $y \in \mathbf{R} \setminus \{0\}$ be arbitrarily chosen. If $y > 0$, then $x^y > 1$, whilst if $y < 0$, then $x^y < 1$. So, by virtue of (7), we have

$$f(x^y) = \lim_{t\to\infty} \frac{f((x^y)^{ty})}{ty(x^y)^{ty-1}} = \lim_{t\to\infty} yx^{y-1} \frac{f(x^{ty^2})}{ty^2x^{ty^2-1}} = yx^{y-1}f(x).$$

Consequently, (2) holds true for all $x > 1$ and all $y \in \mathbf{R} \setminus \{0\}$. Analogously, but using (8) instead of (7), it can be proved that (2) holds also for all $x < 1$ and all $y \in \mathbf{R} \setminus \{0\}$, completing the proof. ∎

This approach can be also applied to the superstability of the homogeneous functional equation (3).

THEOREM 5. *Let $p > 0$ and $p_1 \geq 0$ with $p \neq p_1$, let X be a linear space over the field $\mathbf{K}$ of real or complex numbers, let Y be a normed space over $\mathbf{K}$, and let $k : X \to [0, \infty[$ be given. If a function $f : X \to Y$ satisfies*

$$\|f(\alpha x) - |\alpha|^p f(x)\| \leq |\alpha|^{p_1} k(x) \tag{9}$$

for all $\alpha \in \mathbf{K}$ and all $x \in X$, then (3) *holds true for all $\alpha \in \mathbf{K}$ and all $x \in X$ (by 0^0 we mean* 1).

Proof. Taking $x = 0$ in (9) and then sending α to zero for $p_1 > 0$, or to infinity for $p_1 = 0$, we get $f(0) = 0$. Hence (3) holds true for $\alpha = 0$ and all $x \in X$.

Replacing α by $t\alpha$ in (9) and then dividing both sides of the obtained inequality by $t^p|\alpha|^p$, we find that

$$\left\|\frac{f(t\alpha x)}{t^p|\alpha|^p} - f(x)\right\| \leq t^{p_1-p}|\alpha|^{p_1-p}k(x)$$

for all $x \in X$, all $t \in \mathbf{R}_+$, and all $\alpha \in \mathbf{K} \setminus \{0\}$. From this inequality we deduce that if $p > p_1$, then

$$\lim_{t\to\infty} \frac{f(t\alpha x)}{t^p|\alpha|^p} = f(x) \qquad \text{for all } x \in X \text{ and all } \alpha \in \mathbf{K} \setminus \{0\},$$

whilst if $p < p_1$, then

$$\lim_{t\searrow 0} \frac{f(t\alpha x)}{t^p|\alpha|^p} = f(x) \qquad \text{for all } x \in X \text{ and all } \alpha \in \mathbf{K}\setminus\{0\}.$$

Taking these into account, for each $x \in X$ and each $\alpha \in \mathbf{K}\setminus\{0\}$ we have

$$f(\alpha x) = \lim_{t\to\infty} \frac{f(t\alpha(\alpha x))}{t^p|\alpha|^p} = |\alpha|^p \lim_{t\to\infty} \frac{f(t\alpha^2 x)}{t^p|\alpha^2|^p} = |\alpha|^p f(x), \qquad \text{if } p > p_1,$$
$$f(\alpha x) = \lim_{t\searrow 0} \frac{f(t\alpha(\alpha x))}{t^p|\alpha|^p} = |\alpha|^p \lim_{t\searrow 0} \frac{f(t\alpha^2 x)}{t^p|\alpha^2|^p} = |\alpha|^p f(x), \qquad \text{if } p < p_1. \ \blacksquare$$

REMARK 2. The above theorem remains true for $p = p_1$ if, in addition, k satisfies the subhomogeneity condition

$$k(\alpha x) \le |\alpha|^{p_2} k(x) \qquad \text{for all } \alpha \in \mathbf{K} \text{ and all } x \in X,$$

with $p_2 \ge 0$, $p_2 \ne p$ (see [9, Corollary 2]). Moreover, the condition $p \ne p_2$ is indispensable, as it is shown by the following example: let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) := |x|^p e^{-x^2}$. Then for all $\alpha, x \in \mathbf{R}$ it holds that

$$|f(\alpha x) - |\alpha|^p f(x)| = |\alpha|^p |x|^p \left| e^{-\alpha^2 x^2} - e^{-x^2} \right| \le |\alpha|^p |x|^p,$$

but f is not homogeneous.

REMARK 3. Let $p, \varepsilon \in \mathbf{R}_+$ be fixed. Then there exist functions $f : \mathbf{R} \to \mathbf{R}$ satisfying

$$|f(\alpha x) - |\alpha|^p f(x)| \le \varepsilon |\alpha|^p |x|^p \qquad \text{for all } \alpha, x \in \mathbf{R} \tag{10}$$

and

$$\sup\{|f(x) - h(x)| \mid x \in \mathbf{R}\} = \infty \tag{11}$$

for every function $h : \mathbf{R} \to \mathbf{R}$ such that

$$h(\alpha x) = |\alpha|^p h(x) \qquad \text{for all } \alpha, x \in \mathbf{R}. \tag{12}$$

Indeed, let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) := \frac{\varepsilon}{\pi}|x|^p \arctan x$. Then for all $\alpha, x \in \mathbf{R}$ it holds that

$$|f(\alpha x) - |\alpha|^p f(x)| = \varepsilon |\alpha|^p |x|^p \frac{|\arctan(\alpha x) - \arctan x|}{\pi} \le \varepsilon |\alpha|^p |x|^p.$$

On the other hand, let $h : \mathbf{R} \to \mathbf{R}$ be an arbitrary function satisfying (12). Then $h(\alpha) = c|\alpha|^p$ for all $\alpha \in \mathbf{R}$, where $c = h(1)$. If $c \ne \frac{\varepsilon}{2}$, then

$$\lim_{x\to\infty} |f(x) - h(x)| = \lim_{x\to\infty} |x|^p \left| \frac{\varepsilon}{\pi} \arctan x - c \right| = \infty,$$

whilst if $c \ne -\frac{\varepsilon}{2}$, then $\lim_{x\to-\infty} |f(x) - h(x)| = \infty$. Consequently, (11) holds true.

REMARK 4. However, if $p, \varepsilon \in \mathbf{R}_+$ and $f : \mathbf{R} \to \mathbf{R}$ satisfies (10), then there exists a function $h : \mathbf{R} \to \mathbf{R}$, satisfying (12) and

$$|f(x) - h(x)| \le \varepsilon |x|^p \qquad \text{for all } x \in \mathbf{R}. \tag{13}$$

Indeed, the function $h : \mathbf{R} \to \mathbf{R}$, defined by $h(x) := f(1)|x|^p$, satisfies (12). On the other hand, letting $x = 1$ in (10), we see that h satisfies also (13).

3. Superstability in the sense of R. Ger of the functional equations (1), (2) and (3)

In [5, Theorem 8], S.-M. Jung established (also with a long and complicated proof) the superstability of the equation (1) in the sense of R. Ger. More precisely, he proved that if $\delta \ge 0$ and the function $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies

$$\left| \frac{f(x^y)}{y f(x)} - 1 \right| \le \frac{\delta}{x^y} \tag{14}$$

for all $x, y \in \mathbf{R}_+$, then (1) holds true for all $x > 1$ and all $y \in \mathbf{R}_+$. First of all, it should be noted that in this theorem it must be assumed that (14) holds only for all $x \in \mathbf{R}_+ \setminus \{1\}$ and all $\in \mathbf{R}_+$, because there is no function satisfying (14) for all $x, y \in \mathbf{R}_+$. Indeed, it suffices to take $x = 1$ in (14) and then to let $y \searrow 0$ in order to obtain a contradiction. In what follows we prove a stronger version of Jung's theorem.

THEOREM 6. *Let $\delta \ge 0$ and let $f : \mathbf{R}_+ \to \mathbf{C}$ be a function satisfying the following conditions:*

(i) *$f(x) \ne 0$ for all $x \in \mathbf{R}_+ \setminus \{1\}$;*
(ii) *the inequality (14) holds true for all $x \in \mathbf{R}_+ \setminus \{1\}$ and all $y \in \mathbf{R} \setminus \{0\}$.*

Then (1) holds true for all $x \in \mathbf{R}_+ \setminus \{1\}$ and all $y \in \mathbf{R} \setminus \{0\}$.

Proof. Replacing y by ty in (14), we obtain

$$\left| \frac{f(x^{ty})}{ty f(x)} - 1 \right| \le \frac{\delta}{x^{ty}} \qquad \text{for all } x \in \mathbf{R}_+ \setminus \{1\} \text{ and all } y, t \in \mathbf{R} \setminus \{0\}.$$

This inequality ensures that

$$\lim_{t \to \infty} \frac{f(x^{ty})}{ty} = f(x) \qquad \text{for } \begin{cases} x > 1, \ y > 0 \\ x < 1, \ y < 0 \end{cases} \tag{15}$$

and

$$\lim_{t \to -\infty} \frac{f(x^{ty})}{ty} = f(x) \qquad \text{for } \begin{cases} x > 1, \ y < 0 \\ x < 1, \ y > 0. \end{cases} \tag{16}$$

Now, let $x > 1$ and $y \in \mathbf{R} \setminus \{0\}$ be arbitrarily chosen. If $y > 0$, then $x^y > 1$, whilst if $y < 0$, then $x^y < 1$. So, by virtue of (15), we have

$$f(x^y) = \lim_{t\to\infty} \frac{f((x^y)^{ty})}{ty} = y \lim_{t\to\infty} \frac{f(x^{ty^2})}{ty^2} = yf(x).$$

Consequently, (1) holds true for all $x > 1$ and all $y \in \mathbf{R} \setminus \{0\}$. Analogously, but using (16) instead of (15), it can be proved that (1) holds also for all $x \in]0,1[$ and all $y \in \mathbf{R} \setminus \{0\}$. ∎

The superstability of the equation (2) in the sense of R. Ger reduces to that of equation (1).

THEOREM 7. *Let $\delta \geq 0$ and let $f : \mathbf{R}_+ \to \mathbf{C}$ be a function satisfying the following conditions:*

(i) *$f(x) \neq 0$ for all $x \in \mathbf{R}_+ \setminus \{1\}$;*
(ii) *for all $x \in \mathbf{R}_+ \setminus \{1\}$ and all $y \in \mathbf{R} \setminus \{0\}$ it holds that*

$$\left| \frac{f(x^y)}{yx^{y-1}f(x)} - 1 \right| \leq \frac{\delta}{x^y}.$$

Then (2) holds true for all $x \in \mathbf{R}_+ \setminus \{1\}$ and all $y \in \mathbf{R} \setminus \{0\}$.

Proof. The function $g : \mathbf{R}_+ \to \mathbf{C}$, defined by $g(x) := \frac{f(x)}{x}$, satisfies $g(x) \neq 0$ for all $x \in \mathbf{R}_+ \setminus \{1\}$ and

$$\left| \frac{g(x^y)}{yg(x)} - 1 \right| \leq \frac{\delta}{x^y} \qquad \text{for all } x \in \mathbf{R}_+ \setminus \{1\} \text{ and all } y \in \mathbf{R} \setminus \{0\}.$$

By virtue of Theorem 6, we conclude that

$$g(x^y) = yg(x) \qquad \text{for all } x \in \mathbf{R}_+ \setminus \{1\} \text{ and all } y \in \mathbf{R} \setminus \{0\}.$$

This implies that (2) holds true for all $x \in \mathbf{R}_+ \setminus \{1\}$ and all $y \in \mathbf{R} \setminus \{0\}$. ∎

We finish this section with a result on the superstability of the equation (3). We point out that it is stronger than S.-M. Jung's theorem [7, Theorem 6], as well as that our proof is shorter and less complicated than that given in [7].

THEOREM 8. *Let $p \in \mathbf{R}_+$ and $p_1 \in \mathbf{R} \setminus \{0\}$, let $k : \mathbf{C} \setminus \{0\} \to [0, \infty[$ be a given function, and let $f : \mathbf{C} \to \mathbf{C}$ be a function satisfying the following conditions:*

(i) *$f(x) \neq 0$ for all $x \in \mathbf{C} \setminus \{0\}$;*
(ii) *for all $\alpha, x \in \mathbf{C} \setminus \{0\}$ it holds that*

$$\left| \frac{f(\alpha x)}{|\alpha|^p f(x)} - 1 \right| \leq |\alpha|^{p_1} k(x). \tag{17}$$

Then (3) holds true for all $\alpha, x \in \mathbf{C} \setminus \{0\}$.

Proof. Replacing α by $t\alpha$ in (17), we get

$$\left|\frac{f(t\alpha x)}{|t|^p|\alpha|^p f(x)} - 1\right| \leq |t|^{p_1}|\alpha|^{p_1}k(x) \qquad \text{for all } \alpha, x, t \in \mathbf{C}\setminus\{0\}.$$

This inequality ensures that if $p_1 < 0$, then

$$\lim_{|t|\to\infty} \frac{f(t\alpha x)}{|t|^p|\alpha|^p} = f(x) \qquad \text{for all } \alpha, x \in \mathbf{C}\setminus\{0\},$$

whilst if $p_1 > 0$, then

$$\lim_{t\to 0} \frac{f(t\alpha x)}{|t|^p|\alpha|^p} = f(x) \qquad \text{for all } \alpha, x \in \mathbf{C}\setminus\{0\}.$$

Taking these into account, as in the proof of Theorem 5 it is easily seen that (3) holds true for all $\alpha, x \in \mathbf{C}\setminus\{0\}$. ■

References

[1] J. Chudziak, *Stability of the homogeneous equation*, Demonstratio Math. 31 (1998), 765–772.

[2] S. Czerwik, *On the stability of homogeneous mappings*, C. R. Math. Rep. Acad. Sci. Canada 14 (1992), 268–272.

[3] R. Ger, *Superstability is not natural*, Rocznik Naukowo-Dydaktyczny WSP w Krakowie, Prace Mat. 159 (1993), 109–123.

[4] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston–Basel–Berlin, 1998.

[5] S.-M. Jung, *On the superstability of the functional equation* $f(x^y) = yf(x)$, Abh. Math. Sem. Univ. Hamburg 67 (1997), 315–322.

[6] S.-M. Jung, *On a modified Hyers–Ulam stability of homogeneous equation*, Int. J. Math. Math. Sci. 21 (1998), 475–478.

[7] S.-M. Jung, *Superstability of homogeneous functional equation*, Kyungpook Math. J. 38 (1998), 251–257.

[8] J. Milkman, *Note on the functional equations* $f(xy) = f(x) + f(y)$, $f(x^n) = nf(x)$, Proc. Amer. Math. Soc. 1 (1950), 505–508.

[9] J. Tabor and J. Tabor, *Homogeneity is superstable*, Publ. Math. Debrecen 45 (1994), 123–130.

UNIVERSITATEA BABEŞ-BOLYAI
FACULTATEA DE MATEMATICĂ ŞI INFORMATICĂ
Str. Kogălniceanu 1
3400 CLUJ-NAPOCA, ROMANIA
E-mail: ttrif@math.ubbcluj.ro

Received October 4, 2001.