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ON AN INITIAL VALUE PROBLEM FOR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

Abstract. The purpose of this paper is to study the existence and asymptotic behaviour of solutions of a nonlinear singular integro-differential equation.

1. Introduction

In the past two decades, several papers have been devoted to the study of singular initial value problems for differential and integro-differential equations under various conditions on the nonlinearity and the kernel (see e.g. [1], [3], [4], [6], [7]). Integro-differential equations have different properties from ordinary differential equations even in the simplest cases (see [2]). Therefore known qualitative methods of investigation of ordinary differential equations, e.g. Wazewki's topological method, cannot be applied to integro-differential equations. The fundamental tools used in the existence proofs of all above mentioned works are essentially Schauder-Tychonoff's fixed point theorem and Banach contraction principle.

In this paper we deal with the following problem

$$(1)\quad g(t)y'(t) = a(t)y(t)\Big(1 + f\Big(t, y(t), \int_{0+}^{t} K(t,s,y(s))ds\Big)\Big), \quad y(0^+) = 0,$$

where $f \in C^0(J \times R \times R)$, $K \in C^0(J \times J \times R)$, $J = (0, t_0]$, $t_0 > 0$. Let us introduce the following notation:

$f(t) = o(g(t))$ for $t \to t_0^+$ if $\lim_{t \to t_0^+} \frac{f(t)}{g(t)} = 0$,

$f(t) \sim g(t)$ for $t \to t_0^+$ if there is valid $\lim_{t \to t_0^+} \frac{f(t)}{g(t)} = 1$.

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Consider the following assumptions:

I) $g(t) \in C^1(J)$, $g(t) > 0$, $g(0^+) = 0$, $g'(t) \sim \psi_1(t)g^{\lambda_1}(t)$ for $t \to t_0^+$, $\lambda_1 > 0$, $\psi_1(t)g^{\tau}(t) = o(1)$ for $t \to t_0^+$, for each $\tau > 0$.

II) $a(t) \in C^0(J)$, $a(t) > 0$, $a(t) \sim \psi_2(t)g^{\lambda_2}(t)$ for $t \to t_0^+$, $0 < \lambda_2 < \lambda_1$, $\psi_2(t)g^{\tau}(t) = o(1)$ *for* $t \to t_0^+$.

III) $|f(t,u,v)| \le a(t)(|u| + |v|)$, $|\int_{0+}^t K(t,s,y(s))ds| \le \phi(t,C)|y|$, where $\phi(t,C) = C\exp\{\int_{t_0}^t \frac{a(s)}{g(s)}ds\}$ is the general solution of the equation

$$g(t)y'(t) = a(t)y(t).$$

2. The main result

The technique used for the existence and asymptotic behaviour of solutions of (1) is based on the well-known Schauder's fixed point theorem and Wazewski's topological method for ordinary differential equations (see [5]).

SCHAUDER'S THEOREM. *Let E be Banach space and S its nonempty convex and closed subset. If P is a continuous mapping of S into itself and PS is relatively compact then tha mapping P has at least one fixed point.*

THEOREM 1. *Let assumptions* I)–III) *hold. Then for each $C \neq 0$ there exists one solution $y(t,C)$ of the equation (1) such that*

$$|y^{(i)}(t,C) - \phi^{(i)}(t,C)| \le \delta(\phi^2(t,C))^{(i)},\ i = 0,1, \tag{2}$$

for $t \in J$, $\delta > 1$ is a constant.

Proof. 1) Denote by E the Banach space of continuous functions $h(t)$ on the interval $[0,t_0]$ with the norm

$$\|h(t)\| = \max_{t\in[0,t_0]} |h(t)|.$$

Let S be the subset of E consisting of all functions $h(t)$ satisfying the inequality

$$|h(t) - \phi(t,C)| \le \delta\phi^2(t,C). \tag{3}$$

The set S is obviously nonempty, convex and closed.

2) Now we shall construct the mapping P. Let $h_0(t) \in S$ be an arbitrary function. Substituting $h_0(s)$ instead of y(s) into (1) we obtain the differential equation

$$g(t)y'(t) = a(t)y(t)\Big(1 + f\Big(t, y(t), \int_{0+}^{t} K(t,s,h_0(s))ds\Big)\Big). \tag{4}$$

Set

$$y(t) = \phi(t,C) + Y_0(t)\exp\left\{\int_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)}ds\right\}. \tag{5_1}$$

Then

$$(5_2)\qquad y'(t) = \phi'(t,C) + \frac{Y_1(t)}{g(t)} \exp\left\{ \int\limits_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\},$$

where $0 < \alpha < 1$ is a constant and functions $Y_0(t)$, $Y_1(t)$ satisfy the differential equation

$$(6)\qquad g(t)Y_0'(t) = (\alpha - 1)a(t)Y_0(t) + Y_1(t).$$

From (3) it follows that

$$(7)\qquad h_0(t) = \phi(t,C) + H_0(t), \quad |H_0(t)| \le \delta\phi^2(t,C).$$

Substituting (5_1), (5_2), (7) into (4) we get

$$(8)\quad Y_1(t) = a(t)Y_0(t) + \left(a(t) \exp\left\{ \int\limits_{t_0}^{t} \frac{\alpha a(s)}{g(s)} ds \right\} + a(t)Y_0(t) \right)$$

$$\times f\left(t, \phi(t,C) + Y_0(t) \exp\left\{ \int\limits_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\}, \int\limits_{0+}^{t} K(t,s,\phi(s,C) + H_0(s))ds \right).$$

In view of (8) the equation (6) can be written in the form

$$(9)\quad g(t)Y_0'(t) = \alpha a(t)Y_0(t) + \left(a(t) \exp\{ \int\limits_{t_0}^{t} \frac{\alpha a(s)}{g(s)} ds \} + a(t)Y_0(t) \right)$$

$$\times f\left(t, \phi(t,C) + \exp\left\{ \int\limits_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\} Y_0(t), \int\limits_{0+}^{t} K(t,s,\phi(s,C) + H_0(s))ds \right).$$

In view (5_1), (5_2) it is obvious that any solution of (9) determines a solution of (4).

In the sequel we shall use Wazewki's topological method. We consider the behaviour of integral curves of (9) with respect to the boundary of the set

$$\Omega_0 = \{(t, Y_0) : 0 < t < t_0,\ u_0(t, Y_0) < 0\},$$

where

$$u_0(t, Y_0) = Y_0^2 - \left(\delta \exp\{ \int\limits_{t_0}^{t} \frac{(1+\alpha)a(s)}{g(s)} ds \} \right)^2.$$

Calculating the derivative $\dot{u}_0(t, Y_0)$ along the trajectories of (9) on the set

$$\partial\Omega_0 = \{(t, Y_0) : 0 < t < t_0,\ u_0(t, Y_0) = 0\},$$

we obtain

$$\dot{u}_0(t,Y_0) = \frac{2a(t)}{g(t)}\Big[\alpha Y_0^2(t) + \Big(Y_0(t)\exp\{\int_{t_0}^{t}\frac{\alpha a(s)}{g(s)}ds\} + Y_0^2(t)\Big) \tag{10}$$

$$\times f\Big(t,\ \phi(t,C)+\exp\{\int_{t_0}^{t}\frac{(1-\alpha)a(s)}{g(s)}ds\}Y_0(t),\ \int_{0+}^{t}K(t,s,\phi(s,C)+H_0(s))ds\Big)$$

$$-\ \delta^2(1+\alpha)\exp\{\int_{t_0}^{t}\frac{2(1+\alpha)a(s)}{g(s)}ds\}\Big].$$

It is obvious that $\lim_{t\to 0^+}\phi(t,C)=0$ and by de L'Hospital's rule $\phi^\tau(t,C)g^\sigma(t)$ $= o(1)$ for $t\to 0^+$ and $\sigma\in R$, imply that the powers of $\phi(t,C)$ have effect, in decisive way, to the convergence to zero of the terms in (10).

Using the assumptions of Theorem 1. and the definition of $Y_0(t)$, $\phi(t,C)$, we obtain

$$\operatorname{sgn}\dot{u}_0(t,Y_0) = \operatorname{sgn}\Big(-\delta^2(1+\alpha)\exp\{\int_{t_0}^{t}\frac{2(1+\alpha)a(s)}{g(s)}ds\}\Big) = -1, \tag{11}$$

for sufficiently small t_0 depending on C,δ.

The relation (11) implies that every point of the set $\partial\Omega_0$ is a strict ingress point with respect to the equation (9). Change the orientation of the axis t into opposite. Now each point of the set $\partial\Omega_0$ is a strict egress point with respect to the new system of coordinates. By Wazewki's topological method, we state that there exists at least one integral curve of (9) lying in Ω_0. It is obvious that this assertion remains true for an arbitrary function $h_0(t)\in S$.

Now we shall prove the uniqueness of a solution of (9). Let $\overline{Y_0}(t)$ be also the solution of (9). Putting $Z_0 = Y_0-\overline{Y_0}$ and substituting into (9), we obtain

$$g(t)Z_0' = \alpha a(t)Z_0 + \Big(a(t)\exp\{\int_{t_0}^{t}\frac{\alpha a(s)}{g(s)}ds\} + a(t)Z_0(t)\Big) \tag{12}$$

$$\times\Big[f\Big(t,\ \phi(t,C)+\exp\{\int_{t_0}^{t}\frac{(1-\alpha)a(s)}{g(s)}ds\}(Z_0(t)+\overline{Y_0}(t)),$$

$$\int_{0+}^{t}K(t,s,\phi(s,C)+H_0(s))ds\Big)$$

$$-f\Big(t,\ \phi(t,C)+\exp\{\int_{t_0}^{t}\frac{1-\alpha)a(s)}{g(s)}ds\}\overline{Y_0}(t),\ \int_{0+}^{t}K(t,s,\phi(s,C)+H_0(s))ds\Big)\Big].$$

Let

$$\Omega_1 = \{(t, Z_0):\ 0 < t < t_0,\ u_1(t, Z_0) < 0\},$$

where

$$u_1(t, Z_0) = Z_0^2 - \left(\delta \exp\{\int_{t_0}^{t} \frac{(1+\alpha-\mu)a(s)}{g(s)} ds\}\right)^2, \quad 0 < \mu < \alpha.$$

Using the same method as above, we have

$$\operatorname{sgn} \dot{u}_1(t, Z_0) = -1 \tag{13}$$

for sufficiently small t_0. It is obvious that $\Omega_0 \subset \Omega_1$. Let $\overline{Z_0}(t)$ be any nonzero solution of (12) such that $(t_1, \overline{Z_0}(t_1)) \in \Omega_1$ for $0 < t_1 < t_0$. Let $\overline{\delta} \in (0, \delta)$ be such a constant that $(t_1, \overline{Z_0}(t_1)) \in \partial\Omega_1(\overline{\delta})$. If the curve $\overline{Z_0}(t)$ lies in $\Omega_1(\overline{\delta})$ for $0 < t < t_1$, it would have to be valid $(t_1, \overline{Z_0}(t))$ is a strict egress point of $\partial\Omega_1(\overline{\delta})$. This contradicts the relation (13). Hence in the set $\Omega_0 \subset \Omega_1$ there is only the trivial solution $Z_0(t) \equiv 0$ of (12), so $\overline{Y_0}(t)$ is the unique solution of (9).

From (5_1) we obtain

$$|y_0(t, C) - \phi(t, C)| \le \delta\phi^2(t, C), \tag{14}$$

where $y_0(t, C)$ is the solution of (4) for $t \in (0, t_0]$. Similarly, from (5_2), (8) we have

$$\begin{aligned} |y_0'(t, C) - \phi'(t, C)| &= \left|\frac{1}{g(t)} \exp\left\{\int_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds\right\} Y_1(t)\right| \\ &\le \frac{2\delta a(t)}{g(t)} \exp\left\{\int_{t_0}^{t} \frac{a(s)}{g(s)} ds\right\} = (\phi^2(t, C))'. \end{aligned} \tag{15}$$

It is obvious (after a suitable extension of $y_0(t)$ for $t = 0$) that the correspondence $P(h_0) = y_0$ maps S into itself and $PS \subset S$.

3) We shall prove that PS is relatively compact and P is a continuous mapping. It is easy to see, by (14), (15), that PS is the set of uniformly bounded and equicontinuous functions for $t \in [0, t_0]$. By Ascoli's theorem, PS is relatively compact.

Let $(h_r(t))$ be an arbitrary sequence functions in S such that

$$\|h_r(t) - h_0(t)\| = \epsilon_r, \quad \lim_{r\to\infty} \epsilon_r = 0,\ h_0(t) \in S.$$

The solution $\overline{Y_k}(t)$ of the equation

$$g(t)Y_0'(t) = \alpha a(t)Y_0(t) + \left(a(t)\exp\{\int_{t_0}^{t} \frac{\alpha a(s)}{g(s)} ds\} + a(t)Y_0(t)\right) \tag{16}$$

$$\times f\Big(t,\ \phi(t,C)+\exp\Big\{\int_{t_0}^{t}\frac{(1-\alpha)a(s)}{g(s)}ds\Big\}Y_0(t),\ \int_{0+}^{t}K(t,s,\phi(s,C)+H_k(s))ds\Big),$$

corresponds to the function $h_k(t)$ and $\overline{Y_k}(t) \in \Omega_0$. Similarly, the solution $\overline{Y_0}(t)$ of (9) corresponds to the function $h_0(t)$. We shall show that $|\overline{Y_k}(t) - \overline{Y_0}(t)| \to 0$ uniformly on $[0, t_0]$. Consider the region

$$\Omega_{0k} = \{(t, Y_0):\ 0 < t < t_0,\ u_{0k}(t, Y_0) < 0\},$$

where

$$u_{0k}(t,Y_0) = (Y(t)-\overline{Y_0}(t))^2 - \Big(\epsilon_k \exp\Big\{\int_{t_0}^{t}\frac{(1+\alpha-\nu)a(s)}{g(s)}ds\Big\}\Big)^2,$$
$$0 < \nu < \alpha,\ k \geq 1.$$

Evidently, $\Omega_0 \subset \Omega_{0k}$ for any k and sufficiently small t_0. Investigate the behaviour of integral curves of (16) with respect to the boundary $\partial\Omega_{0k}$. Using the same method as above we obtain for trajectory derivatives

$$\operatorname{sgn} \dot{u}_{0k}(t, Y_0) = -1$$

for sufficiently small t_0 and any k. By Wazewki's topological method there exists at least one solution $\overline{Y_k}(t)$ lying in Ω_{0k}. Hence, it follows that

$$|\overline{Y_k}(t) - \overline{Y_0}(t)| \leq \epsilon_k \exp\Big\{\int_{t_0}^{t}\frac{1+\alpha-\nu)a(s)}{g(s)}ds\Big\} \leq M\epsilon_k,$$

$M > 0$ is a costant depending on t_0. From (5_1) we obtain

$$|y_k(t) - y_0(t)| = \exp\Big\{\int_{t_0}^{t}\frac{(1-\alpha)a(s)}{g(s)}ds\Big\}|\overline{Y_k}(t) - \overline{Y_0}(t)| \leq m\epsilon_k,$$

$m > 0$ is a constant depending on $t_0, M, t \in [0, t_0]$. This estimate implies that P is continuous.

We have thus proved that the mapping P satisfies the assumptions of Schauder's fixed point theorem and hence there exists a function $h(t) \in S$ with $h(t) = P(h(t))$. The proof of existence of a solution of (1) is complete.

Now we shall prove the uniqueness of a solution of (1). Substituting (5_1), (5_2) into (1), we get

$$(17)\qquad Y_1(t) = a(t)Y_0(t) + \Big(a(t)\exp\Big\{\int_{t_0}^{t}\frac{\alpha a(s)}{g(s)}ds\Big\} + a(t)Y_0(t)\Big)$$
$$\times f\Big(t,\ \phi(t,C) + Y_0(t)\exp\Big\{\int_{t_0}^{t}\frac{(1-\alpha)a(s)}{g(s)}ds\Big\},$$

$$\int_{0+}^{t} K\left(t, s, \phi(s, C) + Y_0(s) \exp\left\{ \int_{t_0}^{s} \frac{(1-\alpha)a(u)}{g(u)} du \right\})ds\right).$$

The equation (6) may be written in the form

$$(18) \quad g(t)Y_0'(t) = \alpha a(t) Y_0(t) + \left(a(t) \exp\left\{ \int_{t_0}^{t} \frac{\alpha a(s)}{g(s)} ds \right\} + a(t) Y_0(t)\right)$$
$$\times f\left(t,\ \phi(t, C) + Y_0(t) \exp\left\{ \int_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\},\right.$$
$$\left.\int_{0+}^{t} K\left(t, s, \phi(s, C) + Y_0(s) \exp\{ \int_{t_0}^{s} \frac{(1-\alpha)a(u)}{g(u)} du\})ds\right)\right).$$

Now we know that there exists the solution $y_0(t, C)$ of (1) satisfying (2) such that

$$(19) \qquad y_0(t, C) = \phi(t, C) + \exp\left\{ \int_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\} U_0(t),$$

where $U_0(t)$ is the solution of (18).

Denoting $W_0(t) = Y_0(t) - U_0(t)$ and substituting it into (18), we obtain

$$(20) \qquad g(t)W_0'(t) = \alpha a(t) W_0(t) + a(t)\left(\exp\left\{ \int_{t_0}^{t} \frac{\alpha a(s)}{g(s)} ds \right\} + W_0(t)\right)$$
$$\times \left[f\left(t, \phi(t, C) + (W_0(t) + U_0(t)) \exp\left\{ \int_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\},\right.\right.$$
$$\left.\int_{0+}^{t} K(t, s, \phi(s, C) + +(W_0(s) + U_0(s)) \exp\left\{ \int_{t_0}^{u} \frac{(1-\alpha)a(u)}{g(u)} du \right\})ds\right)$$
$$- f\left(t, \phi(t, C) + U_0(t) \exp\left\{ \int_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds \right\}\right.$$
$$\left.\left.\int_{0+}^{t} K(t, s, \phi(s, C) + U_0(s) \exp\left\{ \int_{t_0}^{u} \frac{(1-\alpha)a(u)}{g(u)} du \right\})ds\right)\right].$$

Let

$$\Omega_{00} = \{(t, W_0) :\ 0 < t < t_0,\ u_{00}(t, W_0) < 0\},$$

where

$$u_{00}(t, W_0) = W_0^2 - \left(\delta \exp\{\int\limits_{t_0}^{t} \frac{(1+\alpha-\mu)a(s)}{g(s)} ds\}\right)^2, \ 0 < \mu < \alpha.$$

If the equation (20) had only the trivial solution lying in Ω_{00} then $Y_0(t) = U_0(t)$ would be only solution of (20) and from here, by (19), $y_0(t, C)$ would be only solution of (1) satisfying (2) for $t \in J$.

We shall suppose that there a exists nontrivial solution $\overline{W_0}(t)$ of (20) lying in Ω_{00}. Substitute $\overline{W_0}(s)$ instead of $W_0(t)$ into (20), we obtain the differential equation:

$$(21) \qquad g(t)W_0'(t) = \alpha a(t)W_0(t) + a(t)\left(\exp\left\{\int\limits_{t_0}^{t} \frac{\alpha a(s)}{g(s)} ds\right\} + W_0(t)\right)$$
$$\times \left[f\left(t, \phi(t, C) + \exp\left\{\int\limits_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds\right\}(W_0(t) + U_0(t)),\right.\right.$$
$$\int\limits_{0+}^{t} K(t, s, \phi(s, C) + + \exp\left\{\int\limits_{t_0}^{u} \frac{(1-\alpha)a(u)}{g(u)} du\right\}(\overline{W_0}(s) + U_0(s))ds\right)$$
$$- f\left(t, \phi(t, C) + \exp\left\{\int\limits_{t_0}^{t} \frac{(1-\alpha)a(s)}{g(s)} ds\right\} \times\right.$$
$$\left.\left. \times U_0(t), \int\limits_{0+}^{t} K(t, s, \phi(s, C) + \exp\left\{\int\limits_{t_0}^{u} \frac{(1-\alpha)a(u)}{g(u)} du\right\} U_0(s))ds\right)\right].$$

Calculating the derivative $\dot{u}_{00}(t, W_0)$ along the trajectories of (21) on the set $\partial\Omega_{00}$, we get $\operatorname{sgn} \dot{u}_{00}(t, W_0) = -1$ for sufficiently small t_0.

By the same method as in the proof of the existence of a solution of (1), we obtain that in Ω_{00} there is only the trivial solution of (21). The proof is complete.

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