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ON RICCATI EQUATIONS IN ORDERED BANACH ALGEBRAS

Abstract. We consider Riccati differential equations in ordered Banach algebras $\mathcal{A}$, and prove invariance and comparison theorems for the case that the right hand side of a Riccati equation is quasimonotone increasing on the set of quasipositive elements (which are the quasimonotone increasing linear mappings in case that $\mathcal{A}$ is the operator algebra of an ordered Banach space).

1. Introduction

Let $(\mathcal{A}, ||\cdot||)$ be a real Banach algebra with unit $\mathbf{1}$, let $T \in (0,\infty]$, let $a, b_1, b_2, c_1, c_2, d \in C([0,T), \mathcal{A})$, and let $f : [0,T) \times \mathcal{A} \to \mathcal{A}$ be defined as

$$(1) \qquad f(t,x) = xa(t)x + b_1(t)x + xb_2(t) + c_1(t)xc_2(t) + d(t).$$

Riccati equations of the form $u'(t) = f(t, u(t))$ play a prominent role in the theory of transport and scattering and in other areas of technology, and especially important are conditions for invariance of an order cone $K \subseteq \mathcal{A}$, see for example [5], [7] and the references given there. Starting with Reid [8] various authors have studied Riccati equations in ordered spaces, see for example [1], [4], [6], [7], [12].

2. Ordered spaces

First, let $(E, ||\cdot||)$ be a Banach space with topological dual space E^*. A wedge W is a nonempty closed convex subset of E with $\lambda W \subseteq W$ $(\lambda \geq 0)$. A wedge K is called a cone if in addition $K \cap (-K) = \{0\}$. Now let E be ordered by a cone K, that is we consider the ordering defined by $x \leq y :\iff y - x \in K$. For $x \leq y$ let $[x,y]$ denote the order interval of all z with $x \leq z \leq y$. Analogously we define $[x,\infty)$ for $x \in E$. A cone K is called normal if $0 \leq x \leq y \Rightarrow ||x|| \leq \alpha||y||$ for a constant $\alpha \geq 1$. If K is normal

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each order interval $[x, y]$ is bounded, and in a finite dimensional spaces each cone is normal.

Let K^* denote the dual wedge of K, that is the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ $(x \geq 0)$.

If D is a subset of E, a function $g : D \to E$ is called quasimonotone increasing (qmi for short) on D, in the sense of Volkmann [14], if

$$x, y \in D,\ x \leq y,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) \Longrightarrow \varphi(g(x)) \leq \varphi(g(y)).$$

A function $g : [0, T) \times D \to E$ is called qmi on D if $x \mapsto g(t, x)$ is qmi on D for each $t \in [0, T)$.

Now, if $\mathcal{A}$ is a Banach algebra with unit $\mathbf{1}$, a cone K is called an algebra cone if $\mathbf{1} \in K$, and $K \cdot K \subseteq K$. In the sequel let $\mathcal{A}$ be ordered by such a cone. We define

$$Q_+ := \{a \in \mathcal{A} : \exp(ta) \geq 0 \ (t \geq 0)\}, \quad Q_\pm := Q_+ \cap (-Q_+).$$

To connect this setting with operator algebras consider a Banach space E ordered by a total cone K_E, that is $\overline{K_E - K_E} = E$. Then the Banach algebra $L(E)$ (the continuous endomorphisms on E with $\mathbf{1} = I := id_E$) is ordered by the algebra cone

$$K = \{A \in L(E) : Ax \geq 0 \ (x \geq 0)\}.$$

Moreover, according to classical results on differential inequalities [15], in this case $A \in Q_+$ if and only if $x \mapsto Ax$ is qmi.

3. Results

Let f be as in (1). Since f is locally Lipschitz continuous in x, each initial value problem $u'(t) = f(t, u(t))$, $u(0) = u_0 \in \mathcal{A}$ is uniquely solvable on an interval which is not extendable to the right, that is we have a unique solution

$$u(\cdot, u_0) : [0, \omega(u_0)) \to \mathcal{A}.$$

Note that a closed and convex subset $C \subseteq \mathcal{A}$ is called invariant for the equation $u'(t) = f(t, u(t))$, if $u(t, u_0) \in C$ $(t \in [0, \omega(u_0)))$ for each $u_0 \in C$. We will prove the following invariance theorem.

THEOREM 1. *1. Let f be as in* (1) *with*

$$a(t) \in K,\ b_1(t), b_2(t) \in Q_+,\ c_1(t), c_2(t) \in K,\ d(t) \in \mathcal{A},$$

for all $t \in [0, T)$. Then f is qmi on K. If in addition $d(t) \in K$ for all $t \in [0, T)$, then K is invariant.

2. Let f be as in (1) *with*

$$a(t) \in Q_\pm \cap K,\ b_1(t), b_2(t) \in Q_+,\ c_1(t), c_2(t) \in K,\ d(t) \in \mathcal{A},$$

for all $t \in [0,T)$. *Then* f *is qmi on* Q_+. *If in addition* $f(t,z) \geq 0$ $(t \in [0,T))$ *for some* $z \in Q_+$, *then* $[z,\infty)$ *is invariant.*

3. *Let* $\mu \in C([0,T),\mathbb{R})$ *with* $\mu(t) > 0$, *and let* f *be as in* (1) *with*

$$a(t) = \mu(t)\mathbf{1},\ b_1(t), b_2(t) \in Q_\pm,\ c_1(t), c_2(t) \in Q_\pm \cap K,$$

$$2\mu(t)d(t) - \left(b_1^2(t) + b_2^2(t)\right) \in Q_+, \tag{2}$$

for all $t \in [0,T)$. *Then* f *is qmi on* Q_+ *(by 2.), and* Q_+ *is invariant.*

By means of Theorem 1 we will prove:

THEOREM 2. *Let* f *be as in one of the cases in Theorem* 1, *hence* C *is invariant, with* $C = K$ *or* $C = [z,\infty)$ *or* $C = Q_+$, *respectively.*
If $u_0 \in C$ *and* $u_0 \leq v_0$, *then* $u(t,u_0) \leq u(t,v_0)$ *as long as both solutions exist. In particular, if* K *is normal* $\omega(u_1) \geq \min\{\omega(u_0),\omega(v_0)\}$ *for each* $u_1 \in [u_0,v_0]$.

We will give a discussion of our results together with some examples in the last section of this paper.

4. Preliminaries

For an ordered Banach algebra $\mathcal{A}$ we define

$$H_+ := \{a \in \mathcal{A} : \exists\lambda \in \mathbb{R} : a + \lambda\mathbf{1} \geq 0\}.$$

Note that $H_+ \subseteq Q_+$ (but i.g. $H_+ \neq Q_+$, [10]).

PROPOSITION 1. *In* $\mathcal{A}$ *the following assertions are valid:*

1. Q_+ *is a wedge.*
2. $Q_\pm$ *is a closed subspace of* $\mathcal{A}$ *(i.g. not a subalgebra).*
3. $Q_+ = \overline{H_+}$.
4. $a \in Q_+ \iff a \mapsto ax$ *is qmi on* $\mathcal{A} \iff x \mapsto xa$ *is qmi on* $\mathcal{A} \iff$ $\varphi(a) \geq 0$ *for each* $\varphi \in K^*$ *with* $\varphi(\mathbf{1}) = 0$.
5. $a \in Q_\pm \Rightarrow a^2 \in Q_+$ *(i.g. neither* $a^2 \in K$ *nor* $a^n \in Q_+$ *for* $n > 2$*).*
6. $a \in Q_+ \Rightarrow [a,\infty) \subseteq Q_+$.
7. $x \in Q_+$, $a \in Q_\pm \cap K \Rightarrow ax \in Q_+$, *and* $xa \in Q_+$.
8. $x \in Q_+$, $a \in Q_\pm$, $\varphi \in K^*$, $\varphi(x) = \varphi(\mathbf{1}) = 0 \Rightarrow \varphi(x^2/2+ax+a^2/2) \geq 0$, *and* $\varphi(x^2/2 + xa + a^2/2) \geq 0$.
9. $a, c_1, c_2 \geq 0 \Rightarrow x \mapsto xax$ *is increasing on* K *and* $x \mapsto c_1xc_2$ *is increasing on* $\mathcal{A}$.

REMARK. By 5. in Proposition 1, (2) implies $d(t) \in Q_+$ $(t \in [0,T))$, whereas $f(t,z) \geq 0$ as in part 2. of Theorem 1 is possible even if $d(t) \notin Q_+$.

Proof. For 1.-5. see [3], Theorem 1.

6. Follows from 4. since if $b \geq a$, then $\varphi(b) \geq \varphi(a) \geq 0$ for each $\varphi \in K^*$ with $\varphi(\mathbf{1}) = 0$.

7. According to 3. there is a sequence (x_n) in H_+ with limit x as $n \to \infty$. Let $\lambda_n > 0$ be such that $x_n + \lambda_n \mathbf{1} \geq 0$. Then, $0 \leq (x_n + \lambda_n \mathbf{1})a$, hence $x_n a \geq -\lambda_n a \in Q_+$. Therefore $x_n a \in Q_+$ by means of 6., and $xa \in Q_+$ since Q_+ is closed. Analogously $ax \in Q_+$

8. We have $\exp(ta)\exp(tx) \geq 0$ $(t \geq 0)$. Since $\varphi(a) = 0$ we have

$$0 \leq \lim_{t \to 0+} \frac{\varphi(\exp(ta)\exp(tx))}{t^2} = \varphi(x^2/2 + ax + a^2/2).$$

For the second inequality consider $\exp(ta)\exp(tx)$.

9. follows immediately from the properties of an algebra cone. ■

For some further investigations on Q_+ and $Q_\pm$ in matrix algebras see [2].

Next, let $a, b_1, b_2, c_1, c_2, d \in \mathcal{A}$, and let $g : \mathcal{A} \to \mathcal{A}$ be defined by

$$g(x) = xax + b_1 x + x b_2 + c_1 x c_2 + d. \tag{3}$$

PROPOSITION 2. *1. Let $a \in K$, $b_1, b_2 \in Q_+$, $c_1, c_2 \in K$, and $d \in \mathcal{A}$, then g defined by (3) is qmi on K.*

2. Let $a \in Q_\pm \cap K$, $b_1, b_2 \in Q_+$, $c_1, c_2 \in K$, and $d \in \mathcal{A}$, then g defined by (3) is qmi on Q_+.

Proof. 1. is an immediate consequence of 4. and 9. in Proposition 1. To prove 2. it is sufficient to prove that $x \mapsto xax$ is qmi on Q_+. Let $y \geq x \in Q_+$ and $\varphi \in K^*$ with $\varphi(x) = \varphi(y)$. Since $a \geq 0$ we have $ax \leq ay$ and $xa \leq ya$. Since $a \in Q_\pm$ part 4. in Proposition 1 gives $\varphi(ax) = \varphi(ay)$ and $\varphi(xa) = \varphi(ya)$. Hence $\varphi(xax) \leq \varphi(xay)$ since $x \in Q_+$, and $\varphi(xay) \leq \varphi(yay)$ since $y \in Q_+$, again by means of 4. in Proposition 1. ■

Now, let E be a Banach space, and let $g : [0, T) \times E \to E$ be continuous and locally Lipschitz continuous in x in the following sense: To each $(t_0, x_0) \in [0, T) \times E$ there exist real constants $\tau, r, L > 0$ such that

$$||g(t, x) - g(t, y)|| \leq L||x - y|| \quad (||x - x_0||, ||y - x_0|| < r,\ t \in [t_0, \tau)).$$

For such functions the following invariance theorem is valid [15], Satz 1:

PROPOSITION 3. *If $C \subseteq E$ is a closed convex set, and*

$$\varphi \in E^*,\ \varphi \neq 0,\ (t, x) \in [0, T) \times C,\ \varphi(x) = \inf\{\varphi(y) : y \in C\}$$
$$\Longrightarrow \varphi(g(t, x)) \geq 0,$$

then C is invariant for $u'(t) = g(t, u(t))$.

5. Proofs

Proof of Theorem 1: The quasimonotonicity of f on K or Q_+ follows from Proposition 2 in all three cases.

1. We apply Proposition 3. Let

$$\varphi \in E^*,\ \varphi \neq 0,\ (t,x) \in [0,T) \times K,\ \varphi(x) = \inf\{\varphi(y) : y \in K\}.$$

We have $\varphi(x) \leq \varphi(\lambda k)$ for each $k \in K$ and each real $\lambda \geq 0$. Hence $\varphi(k) \geq 0$, that is $\varphi \in K^*$. In particular $0 \leq \varphi(x) \leq \varphi(0)$, that is $\varphi(x) = 0$. Now $x \mapsto f(t,x)$ is qmi on K, hence

$$\varphi(f(t,x)) \geq \varphi(f(t,0)) = \varphi(d(t)) \geq 0.$$

2. Let

$$\varphi \in E^*,\ \varphi \neq 0,\ (t,x) \in [0,T) \times [z,\infty),\ \varphi(x) = \inf\{\varphi(y) : y \in [z,\infty)\}.$$

We have $\varphi(\lambda k) \geq \varphi(x) - \varphi(z)$ for $k \in K$ and $\lambda \geq 0$. Hence $\varphi \in K^*$, and $\varphi(x) = \varphi(z)$. Since $x \mapsto f(t,x)$ is qmi on Q_+ we have

$$\varphi(f(t,x)) \geq \varphi(f(t,z)) \geq 0.$$

3. Let

$$\varphi \in E^*,\ \varphi \neq 0,\ (t,x) \in [0,T) \times Q_+,\ \varphi(x) = \inf\{\varphi(y) : y \in Q_+\}.$$

Again $\varphi \in K^*$. Moreover $\varphi(\lambda \mathbf{1}) \geq \varphi(x)$ for all $\lambda \in \mathbb{R}$. Hence $\varphi(\mathbf{1}) = 0$ and $\varphi(x) \leq 0$. Since $\lambda x \in Q_+$ for all $\lambda \geq 0$ we have $\varphi(x) \leq 2\varphi(x)$, therefore $\varphi(x) = 0$. According to 8. in Proposition 1

$$0 \leq \varphi\left(x^2 + \frac{b_1(t)}{\mu(t)}x + x\frac{b_2(t)}{\mu(t)} + \frac{b_1^2(t) + b_2^2(t)}{2\mu^2(t)}\right).$$

Since $h(t) := d(t) - (b_1(t)^2 + b_2(t)^2)/(2\mu(t)) \in Q_+$ we get

$$\varphi\left(\mu(t)x^2 + b_1(t)x + xb_2(t) + \frac{b_1^2(t) + b_2^2(t)}{2\mu(t)} + h(t)\right) \geq 0.$$

Moreover by 7. in Proposition 1 we have $c_1(t)xc_2(t) \in Q_+$, therefore

$$\varphi(c_1(t)xc_2(t)) \geq 0.$$

Altogether $\varphi(f(t,x)) \geq 0$. ∎

Proof of Theorem 2: We adapt the method in [15], Satz 2, to our case. Let $u_0 \in C$, $u_0 \leq v_0$ (hence $v_0 \in C$), and let u, v be the solutions of the corresponding initial value problems. According to Theorem 1 $u(t), v(t) \in C$ as long as these solutions exist. We set $h(t) = v(t) - u(t)$ $(0 \leq t < \min\{\omega(u_0), \omega(v_0)\})$. Then $h(0) \geq 0$ and h solves $h'(t) = g(t, h(t))$ with

$$g(t,x) = f(t, u(t) + x) - f(t, u(t)).$$

Since $u(t) \in C$, $x \in K$ implies $u(t) + x \in C$ in all three cases, the function g is qmi on K. Let

$$\varphi \in E^*,\ \varphi \neq 0,\ (t,x) \in [0,T) \times K,\ \varphi(x) = \inf\{\varphi(y) : y \in K\}.$$

Then $\varphi \in K^*$, $\varphi(x) = 0$, and therefore

$$\varphi(g(t,x)) \geq \varphi(g(t,0)) = 0.$$

According to Proposition 3 we get $h(t) \geq 0$, that is $u(t) \leq v(t)$ for $0 \leq t < \min\{\omega(u_0), \omega(v_0)\}$. If K is normal each solution starting at $u_1 \in [u_0, v_0]$ is bounded on compact subintervals of $[0, \min\{\omega(u_0), \omega(v_0)\})$ and by standard reasoning $\omega(u_1) \geq \min\{\omega(u_0), \omega(v_0)\}$. ∎

6. Examples

1.) First we consider the classical case $E = \mathbb{R}^n$ ordered by the natural cone $K_E = \{x \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0\}$. Then $\mathcal{A} = L(\mathbb{R}^n)$ is ordered by the corresponding cone K with $A = (a_{ij}) \in K$ if and only if $a_{ij} \geq 0$. Moreover $A \in Q_+$ if and only if $a_{ij} \geq 0$ for $i \neq j$, and $A \in Q_\pm$ if and only if A is diagonal. In this case the invariance of K as in part 1. of Theorem 1 is due to Reid [8], Theorem 9.2.

Part 2. of Theorem 1 says that a restriction of the quadratic term lifts the area of quasimonotonicity form K to Q_+. In this example $Q_\pm \cap K$ is the set of diagonal matrices with nonnegative entries. We admit that in general $Q_\pm \cap K$ is rather small, in many cases $Q_\pm \cap K = \{\lambda \mathbf{1} : \lambda \geq 0\}$. On the other hand Q_+ is always bigger than K.

Consider for example $f(t, W) = W^2 - 2\sin(t)W$ and

$$Z = \begin{pmatrix} -1 & 4 & 4 \\ 4 & -1 & 4 \\ 4 & 4 & -1 \end{pmatrix}.$$

Then $f(t, Z) \geq 31I \geq 0$ $(t \geq 0)$. Hence, each solution $W : [0, \omega(W_0)) \to L(\mathbb{R}^3)$ of $W'(t) = W^2(t) - 2\sin(t)W(t)$, $W(0) = W_0 \geq Z$ stays in $[Z, \infty)$.

2.) Consider $\mathbb{R}^n$ ordered by the n-dimensional ice-cream cone

$$K = \{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \ldots + x_{n-1}^2}\},$$

and $\mathcal{A} = L(\mathbb{R}^n)$. Here, $A = (a_{ij}) \in Q_\pm$ if and only if $a_{ii} = a_{jj}$ $(1 \leq i, j \leq n)$, $a_{ij} = -a_{ji}$ $(1 \leq i, j \leq n-1,\ i \neq j)$, and $a_{nj} = a_{jn}$ $(1 \leq j \leq n-1)$, see [11]. See also [11] for a characterization of Q_+. Thus, if $n = 3$ we have $A \in Q_\pm$ if and only if it is of the form

$$\begin{pmatrix} \alpha & \beta & \gamma \\ -\beta & \alpha & \delta \\ \gamma & \delta & \alpha \end{pmatrix}.$$

For example, if $B_1(t), B_2(t)$ are of this form ($t \in [0,T)$), then Q_+ is invariant for

$$W'(t) = W^2(t) + B_1(t)W(t) + W(t)B_2(t) + \frac{B_1^2(t) + B_2^2(t)}{2} + r(t)I$$

for any function $r \in C([0,T),\mathbb{R})$, according to 3. in Theorem 1.

3.) Consider the (commutative) Banach algebra $\mathcal{A} = l^1(\mathbb{Z})$, with norm and multiplication

$$||x|| = \sum_{k\in\mathbb{Z}} |x_n|, \qquad (x * y)_n = \sum_{k\in\mathbb{Z}} x_{n-k}y_k,$$

and unit $\mathbf{1} = (\delta_{0,n})_{n=-\infty}^{\infty}$. Let $\mathcal{A}$ be ordered by the (normal) algebra cone $K = \{x \in \mathcal{A} : x_n \geq 0 \ (n \in \mathbb{Z})\}$. Then $H_+ = \{x \in \mathcal{A} : x_n \geq 0 \ (n \neq 0)\}$, and this set is closed, hence $Q_+ = H_+$ by 3. in Proposition 1 . Consider the Riccati equation

$$u'(t) = u(t) * u(t) + \beta(t)u(t) + \delta(t)\mathbf{1}.$$

with $\beta, \delta \in C([0,T),\mathbb{R})$. Here $a(t) = \mathbf{1}, b_1(t) = \beta(t)\mathbf{1}, b_2(t) = c_1(t) = c_2(t) = 0$, and $d(t) = \delta(t)\mathbf{1}$. According to 3. in Theorem 1 Q_+ is invariant, since

$$d(t) - \frac{b_1(t)^2 + b_2(t)^2}{2} = \Big(\delta(t) - \frac{\beta^2(t)}{2})\Big)\mathbf{1} \in Q_+ \quad (t \in [0,T)).$$

4.) A well studied case that is not covered by the Banach algebra frame is the following. Consider $E = S_n$, the Banach space of all real symmetric matrices (which is not a Banach algebra), ordered by the cone K of all positive semidefinite matrices. Then $f : [0,T) \times S_n \to S_n$ defined by

$$f(t,W) = WA(t)W + B_1(t)W + WB_2(t) + C_1(t)WC_2(t) + D(t)$$

is qmi on the whole space S_n if $A(t) \in S_n$, $B_1(t) = B_2^T(t) \in L(\mathbb{R}^n)$, $C_1(t) = C_2^T(t) \in L(\mathbb{R}^n)$, and $D(t) \in S_n$ for all $t \in [0,T)$. For this setting see for example [1], [4], [8], [9], [12], [13], and [7] for symmetric operators in Hilbert spaces.

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