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THE CAUCHY PROBLEM FOR CERTAIN GENERALIZED DIFFERENTIAL EQUATIONS OF SECOND ORDER

Abstract. The present paper concerns some generalized differential equations of second order for mappings from a subset of Banach space into a Banach space. The subject matter refers to studies of generalized differential equations of the first order submitted in [4].

Let X, Y be Banach spaces over the field $\mathbb{R}$ and let U, V be open subsets of X and Y, respectively.

Let h_1, h_2 be mappings from U into X. In this paper we study the Cauchy problem

$$D^2 f(x)(h_2(x), h_1(x)) = F(x, Df(x)(h_1(x))), \quad x \in U$$

for mappings from a subset of a Banach space into a Banach space, which are defined in C^2 class, at a neighbourhood of nonsingular point (that is, at a neighbourhood of such point x_0 for which $h_1(x_0) \neq 0$ and $h_2(x_0) \neq 0$).

1. Introduction

Let $a_1 \neq 0$ and $a_2 \neq 0$ be linearly independent vectors from a Banach space X over a field $\mathbb{R}$ and let $L_{a_i} = \{ka_i\,;\, k \in \mathbb{R}\}$ (i=1, 2).
Any (fixed in further considerations) subspace complementary to $L_{a_1} \bigoplus L_{a_2}$ will be denoted by $X_{[a_1,a_2]}$.

Let $X^{x_0}_{[a_1,a_2]} = \{x + x_0\,;\, x \in X_{[a_1,a_2]}\}$ where x_0 is a fixed point of X. By $B(x, r)$ we shall denote the ball with radius $r > 0$ and centre $x \in X$; let $X^{x_0}_{[a_1,a_2]}(r) = X^{x_0}_{[a_1,a_2]} \cap B(x_0, r)$.

DEFINITION 1.1. The mappings $y_{[a_1,a_2]} : X \to X_{[a_1,a_2]}$, $s_{a_1} : X \to \mathbb{R}$, $t_{a_2} : X \to \mathbb{R}$ such that

$$(1) \qquad x = y_{[a_1,a_2]}(x) + s_{a_1}(x)a_1 + t_{a_2}(x)a_2 \quad for \quad x \in X$$

will be called the projection operators (compare [4], p.6).

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REMARK 1.1. If $X_{[a_1,a_2]}$ is a closed subspace of X, then the projection operators are continuous and the space $L_{a_1} \bigoplus L_{a_2} \bigoplus X_{[a_1,a_2]}$ is isomorphic to X (see e.g. [6], p.372).

REMARK 1.2. Let U be an open subset of X and f_0 a function from $X^{x_0}_{[a_1,a_2]} \cap U$ into a Banach space Y. Such a function will be called differentiable on $X^{x_0}_{[a_1,a_2]} \cap U$ if the function $\widetilde{f_0}$, where $\widetilde{f_0}(x) = f_0(x + x_0)$ for $x \in X_{[a_1,a_2]} \cap U$, is differentiable on $X_{[a_1,a_2]} \cap U$.

2. The Cauchy problem for generalized differential equations of second order, at a neighbourhood of a nonsingular point

Let U and V be open subsets of Banach spaces X and Y over a field $\mathbb{R}$, respectively, h_1 and h_2 mappings from U into X, and F a mapping from $U \times V$ into Y. Let $x_0 \neq 0$ be any point of U, $h_1(x_0) \neq 0$ and $h_2(x_0) \neq 0$ linearly independent vectors from X and $X_{[h_1(x_0),h_2(x_0)]}$ a certain (fixed in further considerations) closed space such that $X = L_{h_1(x_0)} \bigoplus L_{h_2(x_0)} \bigoplus X_{[h_1(x_0),h_2(x_0)]}$. Let moreover f_1 be a function from $[L_{h_1(x_0)} \bigoplus X^{x_0}_{[h_1(x_0),h_2(x_0)]}] \cap U$ into V and f_2 a function from $[L_{h_2(x_0)} \bigoplus X^{x_0}_{[h_1(x_0),h_2(x_0)]}] \cap U$ into V.

With the above notations and assumptions we can formulate the following

THEOREM 2.1. *Let $x_0 \in U$. If h_1, h_2, F, f_1, f_2 are continuously differentiable (wherever defined) and the condition $Dh_1(x)(h_2(x)) = Dh_2(x)(h_1(x)) = 0$ takes place for $x \in U$. Then there exists a neighbourhood $U_1(x_0)$ of the point x_0 such that the Cauchy problem*

$$(2) \qquad D^2 f(x)(h_2(x), h_1(x)) = F(x, Df(x)(h_1(x))) \quad \text{for} \quad x \in U_1(x_0)$$

$$(2') \quad Df(x)(h_1(x)) = f_1(x) \quad \text{for} \quad x \in [L_{h_1(x_0)} \bigoplus X^{x_0}_{[h_1(x_0),h_2(x_0)]}] \cap U_1(x_0)$$

$$(2'') \qquad f(x) = f_2(x) \quad \text{for} \quad x \in [L_{h_2(x_0)} \bigoplus X^{x_0}_{[h_1(x_0),h_2(x_0)]}] \cap U_1(x_0).$$

has exactly one solution $f : U_1(x_0) \to Y$ in the class C^2.

Proof. First, consider the system of equations

$$(3) \qquad \begin{cases} \dfrac{\partial}{\partial s} v(s,t,x) = h_1(v(s,t,x)) \\ \dfrac{\partial}{\partial t} v(s,t,x) = h_2(v(s,t,x)) \end{cases}$$

with the initial condition

$$(3') \qquad v(0,0,x) = x, \quad \text{where} \quad x \in X^{x_0}_{[h_1(x_0),h_2(x_0)]} \cap U,$$

for $v : \mathbb{R} \times \mathbb{R} \times X^{x_0}_{[h_1(x_0),h_2(x_0)]} \to U$.

From the assumptions concerning the mappings h_1 and h_2 and the Frobenius - Diéudonne theorem [3] it follows that for any $x_0 \in U$ there exist numbers $\varepsilon_1 > 0, \varepsilon_2 > 0$, $r > 0$ such that the problem (3) – (3′) has exactly one continuously differentiable solution on $(-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r)$.

Next, consider the function $\widetilde{v}$, where $\widetilde{v}(s,t,\widetilde{x}) = v(s,t,x_0 + \widetilde{x})$ for $(s,t,x_0+\widetilde{x}) \in (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r)$.

Notice that

$$\frac{\partial}{\partial \widetilde{x}} \widetilde{v}(0,0,x_0) = I,$$

where I is the identity operator on X_0 and

$$\frac{\partial}{\partial s} \widetilde{v}(0,0,x_0) = h_1(x_0) \quad \text{and} \quad \frac{\partial}{\partial t} \widetilde{v}(0,0,x_0) = h_2(x_0).$$

Since $h_1(x_0) \neq 0$ and $h_2(x_0) \neq 0$ are linearly independent vectors from X, $D\widetilde{v}(0,0,x_0)$ is linear homeomorphism from $\mathbb{R} \times \mathbb{R} \times X_0$ onto X. By the inverse function theorem (Theorem 10.2.5 of [2]), there exist $\varepsilon_0' > 0, \varepsilon_0'' > 0, r_0 > 0$ and a neighbourhood $\widetilde{U}_0 \subset U$ of x_0 such that $\widetilde{v}$ is a diffeomorphism of class C^1 from $(-\varepsilon_0', \varepsilon_0') \times (-\varepsilon_0'', \varepsilon_0'') \times X_{[h_1(x_0), h_2(x_0)]}(r_0)$ onto $\widetilde{U}_0$. Set $v^{-1}(x) = (T_1(x), T_2(x), \mathcal{Y}(x))$ for $x \in \widetilde{U}_0$. Then v^{-1} is a continuously differentiable function from $\widetilde{U}_0$ onto $(-\varepsilon_0', \varepsilon_0') \times (-\varepsilon_0'', \varepsilon_0'') \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r_0)$.

Now, set $\widetilde{F}(s,t,x,y) = F(v(s,t,x),y)$ for $y \in V$ and consider the Cauchy problem

$$(4) \qquad \begin{cases} \dfrac{\partial}{\partial t} \widehat{w}(s,t,x) = \widetilde{F}(s,t,x,\widehat{w}(s,t,x) + f_1(x + s h_1(x_0))) \\ \widehat{w}(s,0,x) = 0 \quad \text{for} \quad (s,x) \in (-\varepsilon_0', \varepsilon_0') \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r_0). \end{cases}$$

From Theorem 10.8.1 and Theorem 10.8.2 [2] it follows that there exist numbers $\varepsilon_3 > 0$ $(\varepsilon_3 < \min(\varepsilon_0', \varepsilon_0''))$ and $r_1 > 0$ $(r_1 < r_0)$ such that the problem (4) has exactly one solution $\widehat{w} = \widehat{w}(s,t,x)$ which is defined and continuously differentiable on $(-\varepsilon_3, \varepsilon_3) \times (-\varepsilon_3, \varepsilon_3) \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r_1)$ and satisfies $\widehat{w}(s,0,x) = 0$.

Now set

$$(5) \qquad \widetilde{w}(s,t,x) = \widehat{w}(s,t,x) + f_1(x + s h_1(x_0))$$

for $(s,t,x) \in (-\varepsilon_3, \varepsilon_3) \times (-\varepsilon_3, \varepsilon_3) \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r_1)$. Then $\widetilde{w} : (-\varepsilon_3, \varepsilon_3) \times (-\varepsilon_3, \varepsilon_3) \times X^{x_0}_{[h_1(x_0), h_2(x_0)]}(r_1) \to X$ is differentiable and satisfies the equation

$$(6) \qquad \frac{\partial}{\partial t} \widetilde{w}(s,t,x) = \widetilde{F}(s,t,x,\widetilde{w}(s,t,x))$$

with the condition

$$(6') \qquad \widetilde{w}(s,0,x) = f_1(x + s h_1(x_0)).$$

Next, consider the function

$$w(s,t,x) = \int_0^s \widetilde{w}(\tau,t,x)d\tau + f_2(x + th_2(x_0)). \tag{7}$$

The above function satisfies the equation

$$\frac{\partial}{\partial t}[\frac{\partial}{\partial s}w(s,t,x)] = F(v(s,t,x), \frac{\partial}{\partial s}w(s,t,x)) \tag{8}$$

with the conditions

$$\frac{\partial}{\partial s}w(s,0,x) = f_1(x + sh_1(x_0)) \tag{8'}$$

and

$$w(0,t,x) = f_2(x + th_2(x_0)). \tag{8''}$$

Now, let $U_1(x_0) = v((-\varepsilon_3,\varepsilon_3) \times (-\varepsilon_3,\varepsilon_3) \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r_1))$, where v is the solution of the problem (3) – (3′). Define $f : U_1(x_0) \to X$ by

$$f(x) = w(T_1(x), T_2(x), \mathcal{Y}(x)) \quad \text{for} \quad x \in U_1(x_0)\,. \tag{9}$$

We now prove that f fulfils the equation (2). Fix $x \in U_1(x_0)$. From the definitions of the mappings $T_1, T_2, \mathcal{Y}$ it follows that for $s \in T_1(U_1(x_0))$ and $t \in T_2(U_1(x_0))$ we have

$$\begin{cases} T_1(v(s,t,\mathcal{Y}(x))) = s \\ T_2(v(s,t,\mathcal{Y}(x))) = t \\ \mathcal{Y}(v(s,t,\mathcal{Y}(x))) = \mathcal{Y}(x) \quad \text{for} \quad x \in X. \end{cases} \tag{10}$$

Differentiating (10) with respect to s for $s = T_1(x)$, $t = T_2(x)$ and with respect to t for $s = T_1(x)$, $t = T_2(x)$ we obtain

$$\begin{cases} DT_1(x)(h_1(x)) = 1 \\ DT_1(x)(h_2(x)) = 0 \\ DT_2(x)(h_1(x)) = 0 \\ DT_2(x)(h_2(x)) = 1 \\ D\mathcal{Y}(x)(h_1(x)) = 0 \\ D\mathcal{Y}(x)(h_2(x)) = 0. \end{cases} \tag{11}$$

Therefore, due to the form (9) of the solution f we have

$$\begin{aligned} Df(x) = {} & \frac{\partial}{\partial s}w(T_1(x), T_2(x), \mathcal{Y}(x))DT_1(x) + \frac{\partial}{\partial t}w(T_1(x), T_2(x), \mathcal{Y}(x))DT_2(x) \\ & + D_3w(T_1(x), T_2(x), \mathcal{Y}(x))D\mathcal{Y}(x). \end{aligned} \tag{12}$$

By (11) the equation (12) takes the form

$$Df(x)(h_1(x)) = \frac{\partial}{\partial s}w(T_1(x), T_2(x), \mathcal{Y}(x)). \tag{13}$$

Taking into account the equality (7) we obtain

$$Df(x)(h_1(x)) = \widetilde{w}(T_1(x), T_2(x), \mathcal{Y}(x)), \tag{14}$$

hence

$$\begin{aligned} D^2f(x)(h_1(x)) &+ Df(x)(Dh_1(x)) \\ &= \frac{\partial}{\partial s}\widetilde{w}(T_1(x), T_2(x), \mathcal{Y}(x))DT_1(x) \\ &\quad + \frac{\partial}{\partial t}\widetilde{w}(T_1(x), T_2(x), \mathcal{Y}(x))DT_2(x) \\ &\quad + D_3\widetilde{w}(T_1(x), T_2(x), \mathcal{Y}(x))D\mathcal{Y}(x). \end{aligned} \tag{15}$$

By (11) the equation (15) takes the form

$$D^2f(x)(h_2(x), h_1(x)) = \frac{\partial}{\partial t}\widetilde{w}(T_1(x), T_2(x), \mathcal{Y}(x)). \tag{16}$$

Taking into consideration the equality (6) we have

$$D^2f(x)(h_2(x), h_1(x)) = \widetilde{F}(T_1(x), T_2(x), \mathcal{Y}(x), \widetilde{w}(T_1(x), T_2(x), \mathcal{Y}(x))). \tag{17}$$

Then, by (14) we obtain

$$D^2f(x)(h_2(x), h_1(x)) = \widetilde{F}(T_1(x), T_2(x), \mathcal{Y}(x), Df(x)(h_1(x)))).$$

Taking into consideration that $v(T_1(x), T_2(x), \mathcal{Y}(x)) = x$ for $x \in U_1(x_0)$ we have

$$D^2f(x)(h_2(x), h_1(x)) = F(x, Df(x)(h_1(x))) \quad \text{for} \quad x \in U_1(x_0).$$

Consequently, the function f given by the formula (7) fulfills the equation (2). Naturally, the conditions (2′) and (2″) are also fulfilled. The uniqueness of the solution of the equation (2) with the conditions (2′) and (2″) follows from the method of the construction of this solution (see the form (7) of the solution f) and from Theorem 10.8.1 and Theorem 10.8.2 [2].

3. Form of solution of certain generalized differential equations of second order

Let U and V be open subsets of Banach spaces X and Y over a field $\mathbb{R}$, respectively, h_1 and h_2 mappings from U into X. Let $x_0 \neq 0$ be any point of U, $h_1(x_0) \neq 0$ and $h_2(x_0) \neq 0$ linearly independent vectors from X and $X_{[h_1(x_0), h_2(x_0)]}$ a certain (fixed in further considerations) closed space such that $X = L_{h_1(x_0)} \bigoplus L_{h_2(x_0)} \bigoplus X_{[h_1(x_0), h_2(x_0)]}$. Let moreover f_1 be a function from $[L_{h_1(x_0)} \bigoplus X^{x_0}_{[h_1(x_0), h_2(x_0)]}] \cap U$ into V and f_2 a function from $[L_{h_2(x_0)} \bigoplus X^{x_0}_{[h_1(x_0), h_2(x_0)]}] \cap U$ into V.

Consider the system of equations (3)

$$\begin{cases} \dfrac{\partial}{\partial s} v(s,t,x) = h_1(v(s,t,x)) \\ \dfrac{\partial}{\partial t} v(s,t,x) = h_2(v(s,t,x)) \end{cases}$$

with the initial condition (3′)

$$v(0,0,x) = x, \quad \text{where} \quad x \in X^{x_0}_{[h_1(x_0),h_2(x_0)]} \cap U,$$

for $v : \mathbb{R} \times \mathbb{R} \times X^{x_0}_{[h_1(x_0),h_2(x_0)]} \to U$ (compare the proof of Theorem 2.1).

From the assumptions concerning the mappings h_1 and h_2 and the Frobenius - Diéudonne theorem [3] it follows that for any $x_0 \in U$ there exist numbers $\varepsilon_1 > 0, \varepsilon_2 > 0$, $r > 0$ such that the problem (3) – (3′) has exactly one continuously differentiable solution on $(-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r)$. By the inverse function theorem (Theorem 10.2.5 of [2]), there exist $\varepsilon_0' > 0, \varepsilon_0'' > 0, r_0 > 0$ and a neighbourhood $\widetilde{U}_0 \subset U$ of x_0 such that $\widetilde{v}$ is a diffeomorphism of class C^1 from $(-\varepsilon_0', \varepsilon_0') \times (-\varepsilon_0'', \varepsilon_0'') \times X_{[h_1(x_0),h_2(x_0)]}(r_0)$ onto $\widetilde{U}_0$. Set $v^{-1}(x) = (T_1(x), T_2(x), \mathcal{Y}(x))$ for $x \in \widetilde{U}_0$. Then v^{-1} is a continuously differentiable function from $\widetilde{U}_0$ onto $(-\varepsilon_0', \varepsilon_0') \times (-\varepsilon_0'', \varepsilon_0'') \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r_0)$.

With the above notations and assumptions we can formulate the following

THEOREM 3.1. *Let $x_0 \in U$. If h_1, h_2, f_1, f_2 are continuously differentiable functions (wherever defined) and the condition*

$$Dh_1(x)(h_2(x)) = Dh_2(x)(h_1(x)) = 0$$

takes place for $x \in U$. Then there exists a neighbourhood $U_2(x_0)$ of the point x_0 such that the Cauchy problem

$$D^2 f(x)(h_2(x), h_1(x)) = f(x) \quad \text{for} \quad x \in U_2(x_0), \tag{18}$$

$$Df(x)(h_1(x)) = f_1(x) \quad \text{for} \quad x \in [L_{h_1(x_0)} \bigoplus X^{x_0}_{[h_1(x_0),h_2(x_0)]}] \cap U_2(x_0), \tag{18′}$$

$$f(x) = f_2(x) \quad \text{for} \quad x \in [L_{h_2(x_0)} \bigoplus X^{x_0}_{[h_1(x_0),h_2(x_0)]}] \cap U_2(x_0) \tag{18″}$$

has exactly one solution $f : U_2(x_0) \to Y$ in the class C^2 and it has the form

$$\begin{aligned} f(x) &= f_2(\mathcal{Y}(x)) J_0(2i\sqrt{T_1(x)T_2(x)}) \\ &+ \int_0^{T_1(x)} J_0(2i\sqrt{(T_1(x)-s)T_2(x)}) f_1(\mathcal{Y}(x) + s h_1(x_0)) ds \\ &+ \int_0^{T_2(x)} J_0(2i\sqrt{T_1(x)(T_2(x)-t)}) \frac{\partial}{\partial t} f_2(\mathcal{Y}(x) + t h_2(x_0)) dt \end{aligned} \tag{19}$$

for $x \in X$, where J_0 is the Bessel function of zero order.

P r o o f. First, we prove that if f is a solution of (18)–(18″), then it has the form (19). Consider the equation

$$\frac{\partial^2}{\partial t \partial s} w(s,t,x) = w(s,t,x) \tag{20}$$

with the conditions

$$\frac{\partial}{\partial s} w(s,0,x) = f_1(x + sh_1(x_0)) \quad \text{for } (s,x) \in (-\varepsilon_0', \varepsilon_0') \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r_0), \tag{20'}$$

$$w(0,t,x) = f_2(x + th_2(x_0)) \quad \text{for } (t,x) \in (-\varepsilon_0'', \varepsilon_0'') \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r_0). \tag{20''}$$

The Cauchy problem (20)–(20″) has exactly one solution and it has the form

$$w(s,t,x) = f_2(x) J_0(2i\sqrt{st}) + \int_0^s J_0(2i\sqrt{(s-\tilde{s})t}) \frac{\partial}{\partial \tilde{s}} w(\tilde{s},0,x) d\tilde{s} + \int_0^t J_0(2i\sqrt{s(t-\tau)}) \frac{\partial}{\partial \tau} w(0,\tau,x) d\tau \tag{21}$$

for $(s,t,x) \in (-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3) \times (-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3) \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r_2)$, where $\tilde{\varepsilon}_3 > 0 (\tilde{\varepsilon}_3 < min(\varepsilon_0', \varepsilon_0''))$ and $r_2 > 0$ $(r_2 < r_0)$ (see [1]). Therefore

$$w(s,t,x) = f_2(x) J_0(2i\sqrt{st}) + \int_0^s J_0(2i\sqrt{(s-\tilde{s})t}) f_1(x + \tilde{s}h_1(x_0)) d\tilde{s} + \int_0^t J_0(2i\sqrt{s(t-\tau)}) \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0))\, d\tau. \tag{22}$$

Now, let $U_2(x_0) = v((-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3) \times (-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3) \times X^{x_0}_{[h_1(x_0),h_2(x_0)]}(r_2))$, where v is the solution of the problem $(3) - (3')$. Define $f : U_2(x_0) \to X$ by

$$f(x) = w(T_1(x), T_2(x), \mathcal{Y}(x)) \quad \text{for} \quad x \in U_2(x_0). \tag{23}$$

We now prove that f fulfils the problem (18)–(18″). Fix $x \in U_2(x_0)$. From the definitions of the mappings $T_1, T_2, \mathcal{Y}$ it follows that for $s \in T_1(U_2(x_0))$ and $t \in T_2(U_2(x_0))$ the equalities (10) and (11) are true (see the proof of Theorem 2.1).

The function f given by the formula (19) is twice differentiable because the function $\Phi_{\mathcal{Y}(x)}$ defined by the following way

$$\Phi_{\mathcal{Y}(x)}(s,t) = f_2(\mathcal{Y}(v(s,t,\mathcal{Y}(x)))) J_0(2i\sqrt{T_1(v(s,t,\mathcal{Y}(x)))T_2(v(s,t,\mathcal{Y}(x)))}) + \int_0^{T_1(v(s,t,\mathcal{Y}(x)))} J_0(2i\sqrt{(T_1(v(s,t,\mathcal{Y}(x))) - \tilde{s})T_2(v(s,t,\mathcal{Y}(x)))}) \bullet$$

$$\bullet f_1(\mathcal{Y}(v(s,t,\mathcal{Y}(x))) + \tilde{s}h_1(x_0))d\tilde{s}$$

$$+ \int\limits_0^{T_2(v(s,t,\mathcal{Y}(x)))} J_0(2i\sqrt{T_1(v(s,t,\mathcal{Y}(x)))(T_2((v(s,t,\mathcal{Y}(x))) - \tau)}) \bullet$$

$$\bullet \frac{\partial}{\partial \tau} f_2(\mathcal{Y}(v(s,t,\mathcal{Y}(x))) + \tau h_2(x))d\tau,$$

it means the function $\Phi_{\mathcal{Y}(x)}(s,t) = f(v(s,t,\mathcal{Y}(x))$ for $s,t \in \mathbb{R}$, is twice differentiable with respect to s and t, successively, for $s,t \in \mathbb{R}$. By the equation (10) (see the proof of Theorem 2.1) we have

$$(24)\ \Phi_{\mathcal{Y}(x)}(s,t) = f_2(\mathcal{Y}(x))J_0(2i\sqrt{st}) + \int\limits_0^s J_0(2i\sqrt{(s-\tilde{s})t}) f_1(x + \tilde{s}h_1(x_0))d\tilde{s}$$
$$+ \int\limits_0^t J_0(2i\sqrt{s(t-\tau)}) \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0))d\tau \quad \text{for} \quad s,t \in \mathbb{R}.$$

From the definition of $\Phi_{\mathcal{Y}(x)}$ (taking into consideration the equalities (10) and (11) in the proof of Theorem 2.1 and the conditions concerning the mappings h_1 and h_2) it follows that

$$D^2 f(x)(h_2(x), h_1(x)) = \frac{\partial^2}{\partial t \partial s} \Phi_{\mathcal{Y}(x)}(T_1(x), T_2(x)).$$

Simultaneously, by the theorem on differetiating of an integral with respect to a parameter (see [5]) we obtain

$$\frac{\partial}{\partial s}\Phi_{\mathcal{Y}(x)}(s,t) = \frac{\partial}{\partial s} J_0(2i\sqrt{st}) f_2(\mathcal{Y}(x)) + J_0(0) f_1(x + sh_1(x_0))$$
$$+ \int\limits_0^s \frac{\partial}{\partial s}[J_0(2i\sqrt{(s-\tilde{s})t})] f_1(x + \tilde{s}h_1(x_0))d\tilde{s}$$
$$+ \int\limits_0^t \frac{\partial}{\partial s}[J_0(2i\sqrt{s(t-\tau)})] \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0))d\tau \quad \text{for } s,t \in \mathbb{R}.$$

Therefore, taking into account that $J_0(0) = 1$ and $J_0'(y) = -J_1(y)$ for $y \in \mathbb{C}$ (see [7]) we have

$$\frac{\partial}{\partial s}\Phi_{\mathcal{Y}(x)}(s,t) = -\frac{i\sqrt{t}}{\sqrt{s}} J_1(2i\sqrt{st}) f_2(\mathcal{Y}(x))$$
$$- i\sqrt{t} \int\limits_0^s J_1(2i\sqrt{(s-\tilde{s})t}) \frac{1}{\sqrt{s-\tilde{s}}} f_1(x + \tilde{s}h_1(x_0))d\tilde{s}$$
$$- \frac{i}{\sqrt{s}} \int\limits_0^t J_1(2i\sqrt{s(t-\tau)}) \sqrt{t-\tau} \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0))d\tau \quad \text{for } s,t \in \mathbb{R}.$$

Then, taking into consideration that $J_n'(y) = \frac{1}{2}[J_{n-1}(y) - J_{n+1}(y)]$ for $y \in \mathbb{C}, n \in N$ (see [7]) we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial t \partial s}\Phi_{\mathcal{Y}(x)}(s,t) \\
&\quad = \frac{1}{2i\sqrt{st}} J_1(2i\sqrt{st}) f_2(\mathcal{Y}(x)) + \frac{1}{2}[J_0(2i\sqrt{st}) - J_2(2i\sqrt{st})] f_2(\mathcal{Y}(x)) \\
&\qquad - \frac{i}{2\sqrt{t}} \int_0^s J_1(2i\sqrt{(s-\tilde{s})t}) \frac{1}{\sqrt{s-\tilde{s}}} f_1(x + \tilde{s}h_1(x_0)) d\tilde{s} \\
&\qquad + \frac{1}{2} \int_0^s [J_0(2i\sqrt{(s-\tilde{s})t}) - J_2(2i\sqrt{(s-\tilde{s})t})] f_1(x + \tilde{s}h_1(x_0)) d\tilde{s} \\
&\qquad - \frac{i}{\sqrt{s}} \int_0^t \frac{\partial}{\partial t}[J_1(2i\sqrt{s(t-\tau)})\sqrt{t-\tau}] \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0)) d\tau.
\end{aligned}$$

Consequently, taking into account that $\frac{J_1(y)}{y} = \frac{1}{2}[J_0(y) + J_2(y)]$ for $y \in \mathbb{C}\backslash\{0\}$ (see [7]) we have

$$\begin{aligned}
&\frac{\partial^2}{\partial t \partial s}\Phi_{\mathcal{Y}(x)}(s,t) \\
&\quad = J_0(2i\sqrt{st}) f_2(\mathcal{Y}(x)) \\
&\qquad + \int_0^s J_0(2i\sqrt{(s-\tilde{s})t}) f_1(x + \tilde{s}h_1(x_0)) d\tilde{s} \\
&\qquad + \frac{1}{2} \int_0^t [J_0(2i\sqrt{s(t-\tau)}) - J_2(2i\sqrt{s(t-\tau)})] \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0)) d\tau \\
&\qquad - \frac{i}{\sqrt{s}} \int_0^t J_1(2i\sqrt{s(t-\tau)}) \frac{1}{2\sqrt{t-\tau}} \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0)) d\tau.
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{(25)} \qquad \frac{\partial^2}{\partial t \partial s}\Phi_{\mathcal{Y}(x)}(s,t) = {} & J_0(2i\sqrt{st}) f_2(\mathcal{Y}(x)) \\
& + \int_0^s J_0(2i\sqrt{(s-\tilde{s})t}) f_1(x + \tilde{s}h_1(x_0)) d\tilde{s} \\
& + \int_0^t J_0(2i\sqrt{s(t-\tau)}) \frac{\partial}{\partial \tau} f_2(x + \tau h_2(x_0)) d\tau.
\end{aligned}$$

Hence from the definition of $\Phi_{\mathcal{Y}(x)}$ and from (24) and (25) it follows that f

fulfils the equation (18). The conditions (18′)–(18″) are also fulfilled because

$$\Phi_{\mathcal{Y}(x)}(0,t) = f_2(\mathcal{Y}(x)) + \int_0^t \frac{\partial}{\partial\tau} f_2(x+\tau h_2(x_0))d\tau = f_2(x+th_2(x_0)),$$

and

$$\left[\frac{\partial}{\partial s}\Phi_{\mathcal{Y}(x)}(x,t)\right]\bigg|_{t=0} = f_1(x+sh_1(x_0)).$$

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