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ON THE SOLUTION SET OF A VECTOR DUFFING EQUATION

Abstract. In this paper we consider a vector Duffing equation with periodic boundary conditions. First we prove an existence result assuming on f(t,x) Caratheodory type conditions. Then by imposing also a monotonicity assumption we show that the solution set is acyclic.

1. Introduction

In this paper we study the following vector Duffing equation with periodic boundary conditions

$$\left\{ \begin{array}{l} x''(t) + cx'(t) = f(t, x(t)), \text{a.e. on } T = [0, b] \\ x(0) = x(b), x'(0) = x'(b) \end{array} \right\}. \tag{1}$$

This problem has been studied extensively in the scalar case. We refer to the works of J. Bebernes – M. Martelli [1], S. Fucik – V. Lovicar [2], C. Gupta [4], C. Gupta – J. Nieto – L. Sanchez [5], J. Mawhin - J. R. Ward [6], J. Nieto [7], J. Nieto - V.Rao [8], [9] and S. Tersian [10]. In almost all of these works f is assumed to be continuous and only C. Gupta – J. Nieto – L. Sanchez and J. Mawhin – J. R. Ward allow f to be a Caratheodory function.

In this paper we consider the vector valued version of (1). Assuming a one-sided growth restriction in x on the Caratheodory function $f(t, x)$, we prove the existence of a solution for (1) using the Leray-Schauder alternative theorem. Then by imposing an additional restriction on $f(t, .)$ we prove a structural result for the solution set of (1). Namely we show that is acyclic in the Sobolev space $W^{1,1}(T, \mathbb{R}^N)$.

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Moreover, in contrast to C. Gupta – J. Nieto – L. Sanchez who worked with the Hilbert space $L^2(T,\mathbb{R}^+)$, here we develop the L^1-existence theory for the problem. We should point out that on the structural properties of the solution set of the scalar version of (1), the most general results can be found in the works of C. Gupta - J. Nieto - L. Sanchez [5], J. Nieto [7], and J.Nieto- V. Rao [8], [9].

2. Existence result

First we prove an existence result for problem (1). We will need the following hypotheses on the function f.

$H(f)_1$: $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that

(i) for every $x \in \mathbb{R}^N, t \to f(t,x)$ is measurable;
(ii) for almost all $t \in T, x \to f(t,x)$ is continuous;
(iii) for every $r > 0$, there exists $\varphi_r \in L^1(T,\mathbb{R}^+)$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \leq r$, we have that $\|f(t,x)\| \leq \varphi_r(t)$;
(iv) there exists $M > 0$ such that if $\|x_0\| > M$, then we can find $\delta > 0$ and $\eta > 0$ such that for almost all $t \in T$ and all $\|x - x_0\| < \delta$, we have $(f(t,x),x)_N \geq \eta$.

THEOREM 1. *If hypotheses $H(f)_1$ hold, then problem (1) has a solution and the solution set is weakly compact in $W^{2,1}(T,\mathbb{R}^N)$.*

Proof. Let $D = \{x \in W^{2,1}(T,\mathbb{R}^N) : x(0) = x(b), x'(0) = x'(b)\}$ and let $\hat{L} : D \subset L^1(T,\mathbb{R}^N) \to L^1(T,\mathbb{R}^N)$ be defined by $\hat{L}(x) = -x'' - cx'$.

CLAIM 1. For every $x_1, x_2 \in D$ it follows that $\|x_1 + \hat{L}(x_1) - x_2 - \hat{L}(x_2)\|_1 \geq \mu\|x_1 - x_2\|_1$ for some $\mu > 0$.

In what follows by $\|.\|$ we denote the Euclidean norm in $\mathbb{R}^N$ and by $\|.\|_*$ the l^1-norm (i.e. if $z = (z_k)_{k=1}^N \in \mathbb{R}^N$ then $\|z\|_* = \sum_{k=1}^N |z_k|$). The two norms are equivalent, in other words there exist $\vartheta_1, \vartheta_2 > 0$ such that $\vartheta_1\|z\|_* \leq \|z\| \leq \vartheta_2\|z\|_*, \forall x \in \mathbb{R}^N$. Let $z_1 = x_1 - x_2$, we have

$$\text{(2.1)} \qquad \|z - z'' - cz'\|_1 = \int_0^b \|z(t) - z''(t) - cz'(t)\|dt \geq$$

$$\vartheta_1 \int_0^b \|z(t) - z''(t) - cz'(t)\|_* dt = \vartheta_1 \sum_{k=1}^N \int_0^b |z_k(t) - z_k''(t) - cz_k'(t)|dt.$$

Fix an arbitrary $k \in \{1, 2, ..., N\}$ and set

$$T_k^+ = \{t \in T : z_k(t) > 0\} \text{ and } T_k^- = \{t \in T : z_k(t) < 0\}.$$

Then we have

$$(2.2)\quad \int_0^b |z_k(t) - z_k''(t) - cz_k'(t)|dt$$
$$\geq \int_{T_k^+} |z_k(t) - z_k''(t) - cz_k'(t)|dt + \int_{T_k^-} |z_k(t) - z_k''(t) - cz_k'(t)|dt$$
$$\geq \int_{T_k^+} (z_k(t) - z_k''(t) - cz_k'(t))dt - \int_{T_k^-} (z_k(t) - z_k''(t) - cz_k'(t))dt$$
$$= \int_0^b |z_k(t)|dt - \int_{T_k^+} (z_k''(t) + cz_k'(t))dt + \int_{T_k^-} (z_k''(t) + cz_k'(t))dt.$$

Let (α_k, β_k) be a connected component of T_k^+. Then $z_k(\alpha_k) = z_k(\beta_k) = 0$ and $z_k'(\alpha_k) \geq 0, z_k'(\beta_k) \leq 0$. Thus we have $\int_{\alpha_k}^{\beta_k} z_k''(t)dt = z_k'(\beta_k) - z_k'(\alpha_k) \leq 0$ and $\int_{\alpha_k}^{\beta_k} cz_k'(t)dt = c(z_k(\beta_k) - z_k(\alpha_k)) = 0$. Therefore we infer that $-\int_{\alpha_k}^{\beta_k}(z_k''(t) + cz_k'(t))dt \geq 0$ and so

$$(2.3)\quad - \int_{T_k^+} (z_k''(t) + cz_k'(t))dt \geq 0.$$

Similarly we obtain

$$(2.4)\quad \int_{T_k^-} (z_k''(t) + cz_k'(t))dt \geq 0, \text{ for every } k \in \{1, 2, \ldots, N\}.$$

Using the inequalities (2.3) and (2.4) in (2.2), we obtain

$$(2.5)\quad \int_0^b |z_k(t) - z_k''(t) - cz_k'(t)|dt \geq \int_0^b |z_k(t)|dt, \text{ for every } k \in \{1, 2, \ldots, N\}.$$

Now using (2.5) in (2.1), we finally have that

$$\|z - z'' - cz'\|_1 \geq \vartheta_1 \int_0^b \|z(t)\|_* dt \geq \frac{\vartheta_1}{\vartheta_2} \int_0^b \|z(t)\| dt = \frac{\vartheta_1}{\vartheta_2} \|z\|_1,$$

therefore, recalling the definition of $\hat{L}$ and that $z = x_1 - x_2$, we obtain

$$\|x_1 + \hat{L}(x_1) - x_2 - \hat{L}(x_2)\|_1 \geq \frac{\vartheta_1}{\vartheta_2} \|x_1 - x_2\|_1.$$

This proves the claim.

CLAIM 2. $R(I + \hat{L}) = L^1(T, \mathbb{R}^N)$, where I is the identity map of $L^1(T, \mathbb{R}^N)$.

We need to show that for every $h \in L^1(T, \mathbb{R}^N)$, the periodic problem

$$(2) \qquad \left\{ \begin{array}{l} -x''(t) - cx'(t) + x(t) = h(t), \text{a.e. on } T \\ x(0) = x(b), x'(0) = x'(b) \end{array} \right\}$$

has a solution $x \in W^{2,1}(T, \mathbb{R}^N)$. If $h \in C(T, \mathbb{R}^N)$, then the existence of a unique solution follows by applying Theorem V.7 of [3], to each equation of the system (2). Now give $h \in L^1(T, \mathbb{R}^N)$, let $\{h_n\}_n \subset C(T, \mathbb{R}^N)$ be such that $h_n \to h$ in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$. Let $x_n \in W^{2,1}(T, \mathbb{R}^N)$ be the unique solution of (2) with forcing term h_n. Take the inner product with $x_n(t)$ and then integrate over T. We have

$$(2.6) \qquad \int_0^b (-x_n''(t), x_n(t))_N dt - c\int_0^b (-x_n'(t), x_n(t))_N dt + \|x_n\|_2^2 \le \|h_n\|_1 \|x_n\|_\infty.$$

From Green's formula we have

$$(2.7) \qquad \int_0^b (-x_n''(t), x_n(t))_N dt = -(x_n'(b), x_n(b))_N + (x_n'(0), x_n(0))_N + \|x_n'\|_2^2 = \|x_n'\|_2^2.$$

Also we have

$$(2.8) \qquad c\int_0^b (-x_n'(t), x_n(t))_N dt = -\frac{c}{2}\int_0^b \frac{d}{dt}\|x_n(t)\|^2 dt = -\frac{c}{2}[\|x_n(b)\|^2 - \|x_n(0)\|^2] = 0.$$

Using (2.7) and (2.8) in (2.6), we obtain

$$\|x_n'\|_2^2 + \|x_n\|_2^2 \le \|h_n\|_1 \|x_n\|_\infty \Rightarrow \|x_n\|_{1,2}^2 \le \|h_n\|_1 \|x_n\|_\infty.$$

But recall that $W^{1,2}(T, \mathbb{R}^N)$ embeds continuously (in fact compactly) in $C(T, \mathbb{R}^N)$, so there exists $\gamma > 0$ such that $\|x_n\|_\infty \le \gamma \|x_n\|_{1,2}$. Hence, since $h_n \to h$ in$L^1(T, \mathbb{R}^N)$ as $n \to \infty$, we have that there exists $M_1 > 0$ such that

$$\|x_n\|_{1,2} \le M_1.$$

Therefore $\{x_n\}_n$ is bounded in $W^{1,2}(T, \mathbb{R}^N)$ and moreover directly from the definition of $\{x_n\}_n$ we see that $\{x_n''\}_n$ is uniformly integrable. Hence $\{x_n\}_n$ is bounded in $W^{2,1}(T, \mathbb{R}^N)$; so, recalling that $W^{2,1}(T, \mathbb{R}^N)$ embeds compactly in $W^{1,1}(T, \mathbb{R}^N)$, by passing to a subsequence, if necessary, we may assume that $x_n \to x$ in $W^{1,1}(T, \mathbb{R}^N)$ and $x_n'' \to g$ weakly in $L^1(T, \mathbb{R}^N)$ as $n \to \infty$. Also for every $n \ge 1$ and every $0 \le s \le t \le b$, we have $\|x_n'(t) - x_n'(s)\| \le \int_s^t \|x_n''(\tau)\| d\tau$, which by virtue of the uniform integrability of $\{x_n''\}_n$, implies that $\{x_n'\}_n \subset C(T, \mathbb{R}^N)$ is equicontinuous. Moreover since $W^{2,1}(T, \mathbb{R}^N)$ embeds continuously in $C^1(T, \mathbb{R}^N)$, we see that $\{x_n'\}_n$ is bounded in $C(T, \mathbb{R}^N)$.

Thus by Arzela-Ascoli theorem $\{x'_n\}_n$ is relatively compact in $C(T, \mathbb{R}^N)$. Hence $x \in C^1(T, \mathbb{R}^N)$ and $x'_n(t) \to x'(t)$ in $\mathbb{R}^N$ as $n \to \infty$. Therefore for all $0 \le s \le t \le b$, we have $x'(t) - x'(s) = \int_s^t g(\tau)d\tau$, from which we obtain that $x''(t) = g(t)$, a.e. on T. Hence $x_n \to x$ weakly in $W^{2,1}(T, \mathbb{R}^N)$ and $-x''(t) - cx'(t) + x(t) = h(t)$, a.e. on T, $x(0) = x(b), x'(0) = x'(b)$. This proves the claim.

Let now $L = I + \hat{L} : D \subset L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$. From claims 1 and 2 we obtain that $L^{-1} : L^1(T, \mathbb{R}^N) \to D \subset L^1(T, \mathbb{R}^N)$ is well-defined and continuous.

CLAIM 3. $L^{-1} : L^1(T, \mathbb{R}^N) \to D \subset L^1(T, \mathbb{R}^N)$ is compact (i.e. continuous and maps bounded sets into relatively compact sets).

In fact, since $L : D \subset L^1(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ is linear and continuous, we have that also $L^{-1} : L^1(T, \mathbb{R}^N) \to D \subset L^1(T, \mathbb{R}^N)$ is linear and continuous and maps bounded sets of $L^1(T, \mathbb{R}^N)$ into bounded sets of $W^{2,1}(T, \mathbb{R}^N)$. But $W^{2,1}(T, \mathbb{R}^N)$ embeds compactly in $W^{1,1}(T, \mathbb{R}^N)$ and so the claim is proved.

Let $H_1 : W^{1,1}(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ be defined by

$$H_1(x)(.) = -f(., x(.)) + x(.).$$

Evidently this is a continuous, bounded map. Moreover our problem (1) is equivalent to the following abstract fixed point problem:

$$x = L^{-1}H_1(x).$$

So we consider the set $\Gamma = \{x \in W^{1,1}(T, \mathbb{R}^N) : x = \lambda L^{-1}H_1(x), 0 < \lambda < 1\}$.

CLAIM 4. The set Γ is bounded in $W^{1,1}(T, \mathbb{R}^N)$.

Let $x \in \Gamma$. Then for some $\lambda \in (0,1)$, we have $L\left(\frac{1}{\lambda}x\right) = H_1(x)$ which implies that $x \in W^{2,1}(T, \mathbb{R}^N)$ and

$$\left\{ \begin{array}{l} -x''(t) - cx'(t) = -\lambda f(t, x(t)) + (\lambda - 1)x(t), \text{a.e. on } T \\ x(0) = x(b), x'(0) = x'(b) \end{array} \right\}. \tag{2.9}$$

Take the inner product with $x(t)$ and then integrate over T. We obtain

$$\int_0^b (-x''(t), x(t))_N dt - c\int_0^b (-x'(t), x(t))_N dt \le -\lambda \int_0^b (f(t, x(t)), x(t))_N dt.$$

As before we have that $\int_0^b (-x''(t), x(t))_N dt = \|x'\|_2^2$, $\int_0^b (-x'(t), x(t))_N dt = 0$ and so

$$\|x'\|_2^2 \le -\lambda \int_0^b (f(t, x(t)), x(t))_N dt. \tag{2.10}$$

We will show that $\|x\|_\infty \leq M$, where M is the constant of hypothesis $H(f)_1$-(iv). Indeed let $t_0 \in T$ such that $\|x(t_0)\| = \max_{t\in T}\|x(t)\| > M$ and let $r(t) = \frac{1}{2}\|x(t)\|^2$. So $\max_{t\in T}\|r(t)\| = r(t_0)$ and assuming first that $0 < t_0 < b$, we have that $r'(t_0) = (x'(t_0), x(t_0))_N = 0$. Then by virtue of hypothesis $H(f)_1$-(iv), we can find $\delta_1 > 0$ and $\eta > 0$ such that

$$\lambda(f(t, x(t)), x(t))_N \geq \lambda\eta, \text{ a.e. on } [t_0, t_0 + \delta_1),$$

and so from (2.9) we obtain

$$(x''(t) + cx'(t) + (\lambda - 1)x(t), x(t))_N \geq \lambda\eta, \text{ a.e. on } [t_0, t_0 + \delta_1),$$

which implies that

$$(x''(t) + cx'(t), x(t))_N \geq \lambda\eta > 0, \text{ a.e. on } [t_0, t_0 + \delta_1).$$

By integrating we get

$$\int_{t_0}^{t}(x''(s), x(s))_N ds + c\int_{t_0}^{t}(x'(s), x(s))_N ds > 0, \text{ for all } t \in (t_0, t_0 + \delta_1). \tag{2.11}$$

As before by Green's formula, we have

$$\begin{aligned}\int_{t_0}^{t}(x''(s), x(s))_N ds &= (x'(t), x(t))_N - (x'(t_0), x(t_0))_N - \int_{t_0}^{t}\|x'(s)\|^2 ds\\ &= (x'(t), x(t))_N - \int_{t_0}^{t}\|x'(s)\|^2 ds, \text{ for all } t \in [t_0, t_0 + \delta_1),\end{aligned}$$

and also we have

$$c\int_{t_0}^{t}(x'(s), x(s))_N ds = \frac{c}{2}[\|x(t)\|^2 - \|x(t_0)\|^2], \text{ for all } t \in [t_0, t_0 + \delta_1),$$

therefore, from (2.11), we obtain

$$(x'(t), x(t))_N - \int_{t_0}^{t}\|x'(s)\|^2 ds + \frac{c}{2}[\|x(t)\|^2 - \|x(t_0)\|^2] > 0, \text{ for all } t \in (t_0, t_0 + \delta_1).$$

Now, since $\|x(t)\|^2 \leq \|x(t_0)\|^2$ in $[t_0, t_0 + \delta_1)$, we deduce that

$$r'(t) = 2(x'(t), x(t))_N > 0, \text{ for all } t \in (t_0, t_0 + \delta_1),$$

which implies that $r(t) > r(t_0)$, for all $t \in (t_0, t_0 + \delta_1)$, and this is a contradiction with the choice of t_0.

If we assume that $t_0 = 0$ (or $t = b$), then $r'(0) \leq 0$. Note that because of the periodic condition $r(0) = r(b)$ and $r'(0) = r'(b)$. So it must be $r'(0) = r'(b) = 0$ and proceeding as before we arrive to a contradiction.

Therefore we have $\|x\|_\infty \leq M$ and so (cf. (2.10) and $H(f)_1$-(iii))

$$\|x'\|_2^2 \leq \lambda\| - f(.,x(.))\|_1\|x\|_\infty \leq \lambda\|\varphi_M\|_1 M \leq \|\varphi_M\|_1 M,$$

which proves the boundedness of Γ in $W^{1,1}(T,\mathbb{R}^N)$.

Thus by virtue of the claims 3 and 4, we can apply the Leray-Schauder alternative theorem, and obtain that there exists $x \in W^{1,1}(T,\mathbb{R}^N)$ such that $x = L^{-1}H_1(x)$: evidently $x \in W^{2,1}(T,\mathbb{R}^N)$ and it is a solution of problem (1). ∎

Now let $S = \{x \in W^{2,1}(T,\mathbb{R}^N) : x \text{ is a solution of problem (1)}\}$. From the previous considerations, it is clear that S is relatively weakly compact in $W^{2,1}(T,\mathbb{R}^N)$. So to prove weak compactness, we need to show that it is sequentially weakly closed in $W^{2,1}(T,\mathbb{R}^N)$. To this end let $\{x_n\}_n \subset S$ and assume that $x_n \to x$ weakly in $W^{2,1}(T,\mathbb{R}^N)$ as $n \to \infty$. Then, since $W^{2,1}(T,\mathbb{R}^N)$ embeds continuously in $C^1(T,\mathbb{R}^N)$, we have that $x_n \to x$ weakly in $C^1(T,\mathbb{R}^N)$ and so $x_n(t) \to x(t)$ and $x_n'(t) \to x'(t)$ for all $t \in T$, as $n \to \infty$. From condition $H(f)_1$, by applying Lebesgue's dominated convergence theorem to the sequence $\{f(.,x_n(.))\}_n$, we obtain that $f(.,x_n(.)) \to f(.,(.))$ in $L^1(T,\mathbb{R}^N)$. Now, since $\{x_n''\}_n$ is uniformly integrable it is possible to show $\{x_n''\}_n$ weakly converges to x'' in $L^1(T,\mathbb{R}^N)$, so we have that $x \in S$ and this proves the weak compactness in $W^{2,1}(T,\mathbb{R}^N)$ of the solution set of problem (1). ∎

3. The structure of the solution set

In this section we will prove that the solution set of the problem (1) is acyclic. We recall that a nonempty topological space is acyclic if all its reduced Čech homology groups over rationals vanish.

To determine the structure of the solution set of (1) we will need the following hypotheses on f:

$H(f)_2$ $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that

(i) for every $x \in \mathbb{R}^N, t \to f(t,x)$ is measurable;
(ii) for almost all $t \in T, x \to f(t,x)$ is continuous and monotone;
(iii) for every $r > 0$, there exists $\varphi_r \in L^1(T,\mathbb{R}^+)$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \leq r$, we have that $\|f(t,x)\| \leq \varphi_r(t)$;
(iv) there exists $M > 0$ such that if $\|x_0\| > M$, then we can find $\delta > 0$ and $\eta > 0$ such that for almost all $t \in T$ and all $\|x - x_0\| < \delta$, we have $(f(t,x),x)_N \geq \eta$.

THEOREM 2. *If hypotheses $H(f)_2$ hold, then the solution set of problem (1) is nonempty, compact and acyclic in $W^{1,1}(T,\mathbb{R}^N)$.*

Proof. Using the notations of Theorem 1, from the proof of the previous theorem we know that, denoting by T the map: $T = L^{-1}H_1 : W^{1,1}(T,\mathbb{R}^N) \to D$, T is compact by considering on D the topology of $W^{1,1}(T,\mathbb{R}^N)$. Moreover it is evident that

$$S = \{x \in W^{1,1}(T,\mathbb{R}^N) : x = Tx\}.$$

Since, as we have seen, the set $\Gamma = \{x \in W^{1,1}(T,\mathbb{R}^N) : x = \lambda L^{-1}H_1(x), \lambda \in [0,1]\}$ is bounded let $M_2 > 0$ be such that $\|x\|_{1,1} \leq M_2$, $\forall x \in \Gamma$. Now we put $\Omega = \{x \in W^{1,1}(T,\mathbb{R}^N) : \|x\|_{1,1} < M_2\}$ and let $H_{1,n} : W^{1,1}(T,\mathbb{R}^N) \to L^1(T,\mathbb{R}^N)$ be defined by

$$H_{1,n}(x) = \frac{n-1}{n} H_1(x), x \in W^{1,1}(T,\mathbb{R}^N),$$

and set $T_n = L^{-1}H_{1,n} : W^{1,1}(T,\mathbb{R}^N) \to D$. Obviously, as T, also T_n is compact and

$$\begin{aligned} \|T - T_n\| &= \sup\{\|T(x) - T_n(x)\|_{1,1}, x \in \bar{\Omega}\} \\ &= \sup\{\|L^{-1}(H_1(x) - H_{1,n}(x))\|_{1,1}, x \in \bar{\Omega}\} \\ &\leq \|L^{-1}\|_L \frac{1}{n} \sup\{\|H_1(x)\|_1, x \in \bar{\Omega}\}. \end{aligned}$$

(We denote by $\|L^{-1}\|_L$ the norm of L^{-1} as a linear, bounded operator). Therefore, since H_1 is bounded we obtain that

$$\|T - T_n\| \to 0, \text{ as } n \to \infty.$$

Now we will prove that for every $n \in \mathbb{N}$, and all $y \in W^{1,1}(T,\mathbb{R}^N)$ such that $\|y\|_{1,1} \leq \|T - T_n\|$, the equation

$$x = T_n(x) + y \tag{3.1}$$

has at most one solution in Ω. Indeed suppose that $x_1, x_2 \in \Omega$ are two solutions of the equation (3.1), we get that $x_1 - x_2 = T_n(x_1) - T_n(x_2)$ and so

$$L(x_1 - x_2) = H_{1,n}(x_1) - H_{1,n}(x_2).$$

So we have:

$$\begin{aligned} &(x_1 - x_2)(t) - (x_1 - x_2)''(t) - c(x_1 - x_2)'(t) \\ &= \frac{n-1}{n}(-f(t, x_1(t)) + f(t, x_2(t))) + \frac{n-1}{n}(x_1(t) - x_2(t)), \text{ a.e. on } T \end{aligned}$$

and

$$\begin{aligned} (x_1 - x_2)(0) &= (x_1 - x_2)(b) \\ (x_1 - x_2)'(0) &= (x_1 - x_2)'(b). \end{aligned}$$

Take the inner product with $(x_1 - x_2)(t)$ and then integrate over T, we have

$$\int_0^b (-(x_1 - x_2)''(t), (x_1 - x_2)(t))_N dt - c\int_0^b ((x_1 - x_2)'(t), (x_1 - x_2)(t))_N dt + \|x_1 - x_2\|_2^2$$

$$= -\frac{n-1}{n}\int_0^b (f(t, x_1(t)) - f(t, x_2(t)), (x_1 - x_2)(t))_N dt + \frac{n-1}{n}\|x_1 - x_2\|_2^2.$$

From Green's formula, we have

$$\int_0^b (-(x_1 - x_2)''(t), (x_1 - x_2)(t))_N dt = \|(x_1 - x_2)'\|_2^2.$$

Also we have

$$c\int_0^b ((x_1 - x_2)'(t), (x_1 - x_2)(t))_N dt = \frac{c}{2}[\|x_1(b) - x_2(b)\|^2 - \|x_1(0) - x_2(0)\|^2] = 0.$$

Thus finally we have

$$\|x_1' - x_2'\|_2^2 + \frac{1}{n}\|x_1 - x_2\|_2^2 = -\frac{n-1}{n}\int_0^b ((f(t, x_1(t)) - f(t, x_2(t)), (x_1 - x_2)(t))_N dt$$

and so from the monotonicity of $f(t, .)$ we get

$$\|x_1' - x_2'\|_2^2 + \frac{1}{n}\|x_1 - x_2\|_2^2 \leq 0$$

which means that $\|x_1 - x_2\|_{1,1} = 0$ and so $x_1 = x_2$ in $W^{1,1}(T, \mathbb{R}^N)$.

So we are in position to apply Lemma 2 of [1] and we conclude that S is nonempty, compact and acyclic in $W^{1,1}(T, \mathbb{R}^N)$. ∎

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