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J. Patel, Poulasti Sahoo

SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION TO A CLASS OF ANALYTIC FUNCTIONS

Abstract. We introduce (and investigate various properties and characteristics of) certain class $\mathcal{H}_n^\lambda(A, B)$ of analytic functions in the open unit disc by using the techniques of Briot-Bouquet differential subordination. Relevant connections of the results obtained here with the earlier works are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. Let $\mathcal{S}, \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$ be the subclasses of functions in $\mathcal{A}$ which are respectively, univalent, starlike of order α and convex of order α in E. We denote $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$.

An analytic function f on E is said to be subordinate to an analytic function g on E (written $f \prec g$) if $f(z) = g(\phi(z))$, $z \in E$, for some analytic function ϕ with $\phi(0) = 0$ and $|\phi(z)| < 1, z \in E$. The Hadamard product (or convolution) of two power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=0}^{\infty} b_k z^k \qquad (z \in E)$$

is defined as the power series

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \qquad (z \in E).$$

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For an analytic f given by (1.1) and for all integer values of n, Flett [4] defined the multiplier transformation $\mathcal{I}^n f$ by

$$\mathcal{I}^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \quad (z \in E). \tag{1.2}$$

Clearly, the function $\mathcal{I}^n f(z)$ is analytic in E. We note that

$$\mathcal{I}^{-n} f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k = D^n f(z),$$

where the operator $D^n f$ was introduced by Salagean [13]. We also have

$$\mathcal{I}^n(\mathcal{I}^m f(z)) = \mathcal{I}^{n+m} f(z) \quad (z \in E)$$

for all integers n and m. Further, the operator $\mathcal{I}^n$ can be seen as a convolution of two functions. That is,

$$\mathcal{I}^n f(z) = (h * h * \cdots * h * f)(z),$$

where the function $h(z) = \log \dfrac{1}{1-z} = z + \sum_{k=2}^{\infty} k^{-1} z^k$ occurs n times. It follows from (1.2) that

$$z(\mathcal{I}^n f(z))' = \mathcal{I}^{n-1} f(z) \tag{1.3}$$

and

$$\mathcal{I}^{-1} f(z) = z f'(z), \ \mathcal{I}^{-2} f(z) = z(f'(z) + z f''(z)).$$

Making use of the operator $\mathcal{I}^n$, we now introduce a subclass of $\mathcal{A}$ as follows:

Definition. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_n^\lambda(A, B)$ $(-1 \leq B < A \leq 1,\ 0 \leq \lambda \leq 1)$, if and only if

$$\frac{\lambda \mathcal{I}^{n-2} f(z) + (1-\lambda) \mathcal{I}^{n-1} f(z)}{\lambda \mathcal{I}^{n-1} f(z) + (1-\lambda) \mathcal{I}^n f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in E),$$

where the symbol $' \prec '$ stands for subordination. For convenience, we put

$$\mathcal{H}_n^\lambda(1-2\alpha, -1) \equiv \mathcal{H}_n^\lambda(\alpha),$$

where $\mathcal{H}_n^\lambda(\alpha)$ denote the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\mathrm{Re}\left\{ \frac{\lambda \mathcal{I}^{n-2} f(z) + (1-\lambda) \mathcal{I}^{n-1} f(z)}{\lambda \mathcal{I}^{n-1} f(z) + (1-\lambda) \mathcal{I}^n f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1, 0 \leq \lambda \leq 1, z \in E).$$

In the present paper, we derive certain properties and characteristics of the class $\mathcal{H}_n^\lambda(A, B)$ by using the techniques of Briot-Bouquet differential subordination. Our results presented here besides generalizing and improving the work of earlier authors yield a number of new results.

2. Preliminary lemmas

In our present investigation of the class $\mathcal{H}_n^{\lambda}(A,B)$, we shall require the following lemmas.

LEMMA 1. *Let h be a convex(univalent) function in E with $h(0)=1$, and let*
$p(z)=1+p_1z+p_2z^2+\cdots$ be analytic in E. If

$$p(z)+\frac{zp'(z)}{\gamma}\prec h(z)\quad(z\in E)$$

for $\gamma\neq 0$ and $\mathrm{Re}(\gamma)\geq 0$, then

$$p(z)\prec q(z)=\frac{\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}h(t)dt\prec h(z)\quad(z\in E)$$

and $q(z)$ is the best dominant.

A more general form of this lemma is contained in [7].

LEMMA 2 [8]. *If $-1\leq B<A\leq 1,\beta>0$ and the complex number γ satisfy $\mathrm{Re}(\gamma)\geq-\beta(1-A)/(1-B)$, then the differential equation*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz}$$

has a univalent solution in E given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt}-\dfrac{\gamma}{\beta}, & B\neq 0\\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)dt}-\dfrac{\gamma}{\beta}, & B=0.\end{cases}\tag{2.1}$$

If $p(z)$ is analytic in E and satisfies

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\frac{1+Az}{1+Bz}\quad(z\in E),$$

then

$$p(z)\prec q(z)\prec\frac{1+Az}{1+Bz}\quad(z\in E)$$

and $q(z)$ is the best dominant.

LEMMA 3 [15]. *Let μ be a positive measure on the unit interval $[0,1]$. Let $g(t,z)$ be an analytic function in E for each $t\in[0,1]$, and integrable in t for each $z\in E$ and for almost all $t\in[0,1]$, and suppose that $\mathrm{Re}\{g(t,z)\}>0$ on E, $g(t,-r)$ is real and $\mathrm{Re}\left\{1/g(t,z)\right\}\geq 1/g(t,-r)$ for $|z|\leq r$ and $t\in[0,1]$. If $g(z)=\int_0^z g(t,z)d\mu(t)$, then $\mathrm{Re}\left\{1/g(z)\right\}\geq 1/g(-r)$ for $|z|\leq r$.*

For real or complex numbers $a, b,$ and $c\,(c \neq 0, -1, -2, \cdots)$, the hypergeometric series

$$(2.2)\qquad {}_2F_1(a,b;c;z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1).b(b+1)}{2!c(c+1)}z^2 + \cdots$$

represents an analytic function in E $[14, p.\,281]$. The following identities are well-known $[14]$.

LEMMA 4. *For real or complex numbers $a, b,$ and $c\,(c \neq 0, -1, -2, \cdots)$, and* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, *we have*

$$(2.3)\qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z),$$

$$(2.4)\qquad {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z),$$

$$(2.5)\qquad {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\Big(a, c-b; c; \frac{z}{z-1}\Big),$$

and

$$(2.6)\qquad (b+1)\,{}_2F_1(1,b;b+1;z) = (b+1) + bz\,{}_2F_1(1,b+1;b+2;z).$$

3. Main results

THEOREM 1. *Let $-1 \leq B < A \leq 1$ and n be any integer. If $f \in \mathcal{H}_{n-1}^{\lambda}(A,B)$, then*

$$(3.1)\qquad \frac{\lambda\mathcal{I}^{n-2}f(z) + (1-\lambda)\mathcal{I}^{n-1}f(z)}{\lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^{n}f(z)} \prec Q(z)^{-1} = \tilde{q}(z),$$

where

$$(3.2)\qquad Q(z) = \begin{cases} \int_0^1 \Big(\dfrac{1+Bsz}{1+Bz}\Big)^{(A-B)/B} ds, & B \neq 0, \\ \int_0^1 \exp(A(s-1)z)\,ds, & B = 0, \end{cases}$$

and $\tilde{q}(z)$ is the best dominant of (3.1). Furthermore, if $1 + (A/B) > 0$ with $B < 0$, then

$$(3.3)\qquad \mathcal{H}_{n-1}^{\lambda}(A,B) \subset \mathcal{H}_{n}^{\lambda}(\rho_1(A,B)),$$

where $\rho_1(A,B) = \{{}_2F_1(1,(B-A)/B;2;B/(B-1))\}^{-1}$. The result is best possible.

Proof. Let $f \in \mathcal{H}_{n-1}^{\lambda}(A,B)$, $g(z) = \lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^{n}f(z)$ $(0 \leq \lambda \leq 1)$ and
$r_1 = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$. Using (1.3), it follows that

$$(3.4)\qquad p(z) = \frac{zg'(z)}{g(z)} = \frac{\lambda\mathcal{I}^{n-2}f(z) + (1-\lambda)\mathcal{I}^{n-1}f(z)}{\lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^{n}f(z)}$$

is analytic in $|z| < r_1$ and $p(0) = 1$. Making use the logarithmic differentiation on both the sides of (3.4) and simplifying the resulting equation, we deduce that

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{\lambda\mathcal{I}^{n-3}f(z) + (1-\lambda)\mathcal{I}^{n-2}f(z)}{\lambda\mathcal{I}^{n-2}f(z) + (1-\lambda)\mathcal{I}^{n-1}f(z)} \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1). \tag{3.5}$$

By Lemma 2, we obtain

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1), \tag{3.6}$$

where $q(z)$ is the best dominant of (3.5) and is given by (2.1) for $\beta = 1$ and $\gamma = 0$. Now, rewriting $q(z)$ by changing the variables, we get

$$p(z) \prec Q(z)^{-1} = \widetilde{q}(z) \quad (|z| < r_1),$$

where $Q(z)$ is given by (3.2).

By (3.6), we see from (3.4) that $g(z)$ is starlike(univalent) in $|z| < r_1$. Thus, it is not possible that $g(z)$ vanishes in $|z| < r_1$. So, we conclude that $r_1 = 1$. Therefore, $p(z)$ is analytic in E. This proves (3.1).

Next we show that

$$\inf_{|z|<1} \operatorname{Re}\{\widetilde{q}(z)\} = \widetilde{q}(-1). \tag{3.7}$$

If we set $a = (B-A)/B, b = 1$ and $c = b+1$, then $c > b > 0$. From (3.2), by using (2.3), (2.4) and (2.5) we see that for $B \neq 0$

$$\begin{aligned} Q(z) &= (1+Bz)^a \int_0^1 (1+Bsz)^{-a}\, ds = (1+Bz)^a \; {}_2F_1(a,1;2;-Bz) \\ &= {}_2F_1\left(1,a;2;\frac{Bz}{Bz+1}\right). \end{aligned} \tag{3.8}$$

To prove (3.7), we show that $\operatorname{Re}\{(Q(z))^{-1}\} \geq (Q(-1))^{-1}, z \in E$. Again (3.2), by (3.8) for $1 + (A/B) > 0$ with $B < 0$ (so that $c > a > 0$) can be written as

$$Q(z) = \int_0^1 g(s,z)d\mu(s),$$

where

$$g(s,z) = \frac{1+Bz}{1+(1-s)Bz} \quad \text{and} \quad d\mu(s) = \frac{s^{a-1}(1-s)^{1-a}\, ds}{\Gamma(a)\Gamma(2-a)}$$

which is a positive measure on $[0,1]$.

For $B < 0$, it may be noted that $\operatorname{Re}\{g(s,z)\} > 0, g(s,-r)$ is real for $0 \leq r < 1, s \in [0,1]$ and

$$\operatorname{Re}\left\{\frac{1}{g(s,z)}\right\} = \operatorname{Re}\left\{\frac{1+(1-s)Bz}{1+Bz}\right\} \geq \frac{1-(1-s)Br}{1-Br} = \frac{1}{g(s,-r)}$$

for $|z| \leq r < 1$ and $0 \leq s \leq 1$. Therefore by using Lemma 3, we deduce that $\mathrm{Re}\{1/Q(z)\} \geq 1/Q(-r)$ for $|z| \leq r < 1$ and by letting $r \to 1^-$, we obtain

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)} \quad (z \in E).$$

This by (3.1) leads to (3.3). Hence the theorem.

Putting $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ in Theorem 1, and noting that

$$(3.9) \quad \beta(\alpha) = \left\{{}_2F_1(1, 2(1-\alpha); 2; \frac{1}{2})\right\}^{-1} = \begin{cases} \dfrac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})}, & \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\log 2}, & \alpha = \dfrac{1}{2}, \end{cases}$$

we get

Corollary 1. *For $0 \leq \lambda \leq 1, 0 \leq \alpha < 1$ and all integer values of n, we have*

$$\mathcal{H}_{n-1}^{\lambda}(\alpha) \subset \mathcal{H}_n^{\lambda}(\beta(\alpha)),$$

where $\beta(\alpha)$ is given by (3.9). The result is best possible.

For a function $f \in \mathcal{A}$, the generalized Bernardi-Libera-Livingston operator $\mathcal{F}_\mu$ is defined by

$$(3.10) \qquad \mathcal{F}_\mu(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt \ (\mu+1 > 0, z \in E).$$

It readily follows from (3.10) that

$$f \in \mathcal{A} \iff \mathcal{F}_\mu \in \mathcal{A}.$$

Theorem 2. *Let $-1 \leq B < A \leq 1$ and μ be any complex number satisfying*

$$(3.11) \qquad \mathrm{Re}(\mu) \geq -\frac{1-A}{1-B}.$$

(i) *If $f \in \mathcal{H}_n^{\lambda}(A, B)$, then the function $\mathcal{F}_\mu$ defined by (3.10) satisfies*

$$(3.12) \qquad \frac{\lambda \mathcal{I}^{n-2}\mathcal{F}_\mu(z) + (1-\lambda)\mathcal{I}^{n-1}\mathcal{F}_\mu(z)}{\lambda \mathcal{I}^{n-1}\mathcal{F}_\mu(z) + (1-\lambda)\mathcal{I}^{n}\mathcal{F}_\mu(z)} \prec \frac{1}{Q(z)} - \mu = \tilde{q}(z),$$

where

$$(3.13) \qquad Q(z) = \begin{cases} \int_0^1 s^\mu \left(\dfrac{1+Bsz}{1+Bz}\right)^{(A-B)/B} ds, & B \neq 0, \\ \int_0^1 s^\mu \exp(A(s-1)z)ds, & B = 0, \end{cases}$$

and $\tilde{q}(z)$ is the best dominant of (3.12).

(ii) *If in addition to (3.11), μ is real and $A/B > -(\mu+1)$ with $B < 0$, then for $f \in \mathcal{H}_n^\lambda(A,B)$, we have $\mathcal{F}_\mu \in \mathcal{H}_n^\lambda(\rho_2(A,B,\mu))$, where*

$$\rho_2(A,B,\mu) = \frac{\mu+1}{{}_2F_1\Big(1,(B-A)/B;\mu+2;B/(B-1)\Big)} - \mu.$$

The result is best possible.

Proof. Since

$$\mathcal{F}_\mu(z) = \left(z + \sum_{k=2}^{\infty} \frac{\mu+1}{\mu+k} z^k\right) * f(z),$$

it is easily seen from (1.3) that

$$z(\mathcal{I}^n\mathcal{F}_\mu(z))' = (\mu+1)\mathcal{I}^n f(z) - \mu\mathcal{I}^n F_\mu(z). \tag{3.14}$$

Let

$$g(z) = \lambda\mathcal{I}^{n-1}\mathcal{F}_\mu(z) + (1-\lambda)\mathcal{I}^n\mathcal{F}_\mu(z)$$

and

$$r_1 = \sup\{r : g(z) \neq 0,\ 0 < |z| < r < 1\}.$$

Then $g(z)$ is analytic in $|z| < r_1$ and

$$p(z) = \frac{zg'(z)}{g(z)} = \frac{\lambda\mathcal{I}^{n-2}\mathcal{F}_\mu(z) + (1-\lambda)\mathcal{I}^{n-1}\mathcal{F}_\mu(z)}{\lambda\mathcal{I}^{n-1}\mathcal{F}_\mu(z) + (1-\lambda)\mathcal{I}^n\mathcal{F}_\mu(z)} \tag{3.15}$$

is analytic in $|z| < r_1, p(0) = 1$. Using (1.3) and (3.14) in (3.15), we get

$$\frac{\lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^n\mathcal{F}_\mu(z)}{\lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^n f(z)} = \frac{\mu+1}{p(z)+\mu}. \tag{3.16}$$

Since $f \in \mathcal{H}_n^\lambda(A,B)$, we note that $\lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^n f(z) \neq 0$ in E. On differentiating (3.16) logarithmically and using (1.3) in the resulting equation, we have

(3.17)

$$\frac{\lambda\mathcal{I}^{n-2}f(z) + (1-\lambda)\mathcal{I}^{n-1}f(z)}{\lambda\mathcal{I}^{n-1}f(z) + (1-\lambda)\mathcal{I}^n f(z)} = p(z) + \frac{zp'(z)}{p(z)+\mu} \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1).$$

Using Lemma 1, we deduce that

$$p(z) \prec \tilde{q}(z) = \frac{1}{Q(z)} - \mu \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1),$$

where $Q(z)$ is given by (3.13) and $\tilde{q}(z)$ is the best dominant. Since $\mathrm{Re}\{p(z)\} > 0$ in $|z| < r_1$, it follows from (3.15) that $g(z)$ is univalent in $|z| < r_1$ and can not vanish on $|z| = r_1 < 1$. So, we conclude that $r_1 = 1$. Therefore, $p(z)$ is analytic in E and this proves the first part of the theorem.

Proceeding as in Theorem 1, the second part follows.

Taking $A = 1-2\alpha, B = -1$ and $\lambda = 0$ in Theorem 2, we have

Corollary 2. *Let* $0 \le \alpha < 1$ *and* $\mu \ge -\alpha$. *If* $f \in \mathcal{H}_n^0(\alpha)$, *then* $\mathcal{F}_\mu \in \mathcal{H}_n^0(\rho_3(\alpha,\mu))$, *where*

$$\rho_3(\alpha,\mu) = \frac{\mu+1}{{}_2F_1(1, 2(1-\alpha), \mu+2; 1/2)} - \mu.$$

The result is best possible.

Remark. We observe that for $n = 0$, Corollary 2 improves a result due to Bajpai and Srivastava [1] and Bernardi [2] for $\mu = 1, 2, 3, \cdots$.

Theorem 3. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{\mathcal{I}^n f(z)}{z} \prec \frac{1+Az}{1+Bz} \quad (z \in E), \tag{3.18}$$

then

$$\frac{\mathcal{I}^n \mathcal{F}_\mu(z)}{z} \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (\mu+1>0, z \in E), \tag{3.19}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1,\mu+2;\frac{Bz}{Bz+1}), & B \neq 0, \\ 1 + \frac{\mu+1}{\mu+2}Az, & B = 0 \end{cases} \tag{3.20}$$

and is the best dominant of (3.19). *Furthermore,*

$$\operatorname{Re}\left\{\frac{\mathcal{I}^n \mathcal{F}_\mu(z)}{z}\right\} > \rho_4(A,B,\mu)$$

$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\mu+2;\frac{B}{B-1}\right), & B \neq 0, \\ 1 - \frac{\mu+1}{\mu+2}A, & B = 0. \end{cases}$$

The result is best possible.

Proof. From (1.2) and (3.10), it follows that

$$\mathcal{I}^n \mathcal{F}_\mu(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}\mathcal{I}^n f(t)\,dt. \tag{3.21}$$

Defining the function $p(z)$ in E by

$$p(z) = \frac{\mathcal{I}^n \mathcal{F}_\mu(z)}{z}, \tag{3.22}$$

we see that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in E. Differentiating both sides of (3.22), simplifying and using (3.18) in the resulting equation, we get

$$p(z) + \frac{zp'(z)}{\mu+1} \prec \frac{1+Az}{1+Bz} \quad (z \in E). \tag{3.23}$$

Using Lemma 1 in (3.23) followed by the use of the identities (2.3), (2.5) and (2.6), we get

$$p(z) \prec q(z) = (\mu+1)\int_0^1 s^{\mu}\frac{1+Asz}{1+Bsz}\,ds$$

$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\mu+2;\frac{Bz}{Bz+1}\right), & B \neq 0, \\ 1+\frac{\mu+1}{\mu+2}Az, & B = 0. \end{cases}$$

To prove the second part of the theorem, we proceed as in Theorem 1 [12]. The result is best possible as $q(z)$ is the best dominant. This completes the proof of the theorem.

Setting $A = 1-2\alpha, B = -1$ and $n = 0$ in Theorem 3, we get the following result which improves the corresponding work of Obradovic [10].

COROLLARY 3. *If* $f \in \mathcal{A}$ *satisfies* $\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \alpha$ $(0 \leq \alpha < 1, z \in E)$, *then for* $\mu+1>0$

$$\mathrm{Re}\left\{\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu-1}\,f(t)dt\right\} > \alpha+(1-\alpha)\Big\{{}_2F_1(1,1,\mu+2;1/2)-1\Big\} \quad (z\in E).$$

The result is best possible.

Taking $n=-1$, $A = 1-2\alpha$ and $B=-1$ in Theorem 3, we get

COROLLARY 4. *If* $f \in \mathcal{A}$ *satisfies* $\mathrm{Re}\{f'(z)\} > \alpha$ $(0 \leq \alpha < 1)$ *in* E, *then for* $\mu+1>0$

$$\mathrm{Re}\left\{\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu}\,f'(t)dt\right\} > \alpha+(1-\alpha)\{{}_2F_1(1,1;\mu+2;1/2)-1\} \quad (z\in E).$$

The result is best possible.

THEOREM 4. *Let* $\mathcal{F}_{\mu}$ *be defined by (3.10). If*

$$\mathrm{Re}\left\{\frac{\mathcal{I}^n\mathcal{F}_{\mu}(z)}{z}\right\} > \alpha \quad (0\leq\alpha<1,\ \mu+1>0,\ z\in E) \tag{3.24}$$

then

$$\mathrm{Re}\{\frac{\mathcal{I}^n f(z)}{z}\} > \alpha \quad (\,|z| < R_1\,),$$

where $R_1 = \{\sqrt{(\mu+1)^2+1}-1\}/(\mu+1)$. *The result is best possible.*

Proof. From (3.24), we have

$$\frac{\mathcal{I}^n\mathcal{F}_{\mu}(z)}{z} = \alpha+(1-\alpha)\omega(z), \tag{3.25}$$

where $\omega(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic and has a positive real part in E. Differentiating both sides of (3.25) and using (3.14) in the resulting equation, we get

$$(3.26) \quad \operatorname{Re}\left\{\frac{\mathcal{I}^n f(z)}{z} - \alpha\right\}$$
$$= (1-\alpha)\operatorname{Re}\left\{\omega(z) + \frac{z\omega'(z)}{\mu+1}\right\} \geq (1-\alpha)\left\{\operatorname{Re}(\omega(z)) - \frac{|z\omega'(z)|}{\mu+1}\right\}.$$

Using the well-known estimate

$$(3.27) \quad \frac{|z\omega'(z)|}{\operatorname{Re}\{\omega(z)\}} \leq \frac{2r}{1-r^2} \quad (|z| = r < 1)$$

in (3.26), we deduce that

$$\operatorname{Re}\left\{\frac{\mathcal{I}^n f(z)}{z} - \alpha\right\} \geq (1-\alpha)\operatorname{Re}\{\omega(z)\}\left\{1 - \frac{2r}{(\mu+1)(1-r^2)}\right\}$$

which is certainly positive if $r < R_1$ for R_1 given as in Theorem 4.

To show that the bound R_1 is sharp, we consider the function $\mathcal{F}_\mu$ defined by

$$\frac{\mathcal{I}^n \mathcal{F}_\mu(z)}{z} = \alpha + (1-\alpha)\frac{1+z}{1-z} \quad (\mu + 1 > 0, z \in E).$$

Noting that

$$\frac{\mathcal{I}^n f(z)}{z} - \alpha = (1-\alpha)\left\{\frac{(\mu+1)(1-z^2) + 2z}{(\mu+1)(1-z)^2}\right\} = 0$$

for $z = -R_1$, we complete the proof of the theorem.

Putting $n = 0$ in Theorem 4, we get the following result which inturn yields the result due to Bernardi [3] for $\alpha = 0$ and the result by Padmanabhan [11] for $\mu = 1$.

COROLLARY 5. *Let $\mathcal{F}_\mu$ be defined by (3.10) where $f \in \mathcal{A}$. If* $\operatorname{Re}\{\mathcal{F}_\mu'(z)\} > \alpha$ $(0 \leq \alpha < 1; \mu+1 > 0)$ *in E, then* $\operatorname{Re}\{f'(z)\} > \alpha$ *in* $|z| < \{\sqrt{(\mu+1)^2+1} - 1\}/(\mu+1)$. *The result is best possible.*

THEOREM 5. *Let $-1 \leq B < A \leq 1$ and μ be any complex number satisfying* (3.11).

(i) *If $f \in \mathcal{H}_n^0(A,B)$, then the function $\mathcal{F}_\mu(z)$ defined by* (3.10) *satisfies*

$$(3.28) \quad \frac{\mathcal{I}^n f(z)}{\mathcal{I}^n \mathcal{F}_\mu(z)} \prec \frac{1}{(\mu+1)Q(z)} = \tilde{q}(z) \prec \frac{1 + \frac{1+\mu B}{\mu+1} z}{1+Bz} \quad (z \in E),$$

where

$$(3.29)\qquad Q(z)=\begin{cases}\int\limits_0^1 s^{\mu}\left(\dfrac{1+Bsz}{1+Bz}\right)^{(A-B)/B} ds, & B\neq 0,\\ \int\limits_0^1 s^{\mu}\exp(A(s-1)z)\,ds, & B=0,\end{cases}$$

and $\tilde{q}(z)$ *is the best dominant of* (3.28).

(ii) *If in addition to* (3.11), μ *is real*, $A/B > -(\mu+1)$ *and* $B<0$, *then for* $f\in\mathcal{H}_n^0(A,B)$, *we have*

$$(3.30)\qquad \operatorname{Re}\left\{\frac{\mathcal{I}^n f(z)}{\mathcal{I}^n\mathcal{F}_\mu(z)}\right\} > \left\{{}_2F_1\left(1,\frac{B-A}{B};\mu+2;\frac{B}{B-1}\right)\right\}^{-1}\quad (z\in E).$$

The result is best possible.

Proof. Setting

$$(3.31)\qquad p(z)=\frac{\mathcal{I}^n f(z)}{\mathcal{I}^n\mathcal{F}_\mu(z)}\quad (z\in E),$$

we see that $p(z)$ is analytic in E with $p(0)=1$. Making use of the logarithmic differentiation in (3.31), using (3.14) in the resulting equation and simplifying, we deduce that

$$(3.32)\qquad \frac{\mathcal{I}^{n-1}f(z)}{\mathcal{I}^n f(z)}=\{(\mu+1)p(z)-\mu\}+\frac{zp'(z)}{p(z)}$$
$$=P(z)+\frac{zP'(z)}{P(z)+\mu}\prec\frac{1+Az}{1+Bz}\quad (z\in E),$$

where $P(z)=(\mu+1)p(z)-\mu$. Using Lemma 2, we deduce that

$$(3.33)\qquad P(z)\prec q(z)\prec\frac{1+Az}{1+Bz}\quad (z\in E),$$

where $q(z)$ is the best dominant of (3.33) and is given by (2.1) for $\beta=1$ and $\gamma=\mu$. On simplifying (3.33), we get (3.28).

Proceeding as in Theorem 1, the second part follows. This completes the proof of Theorem 5.

Taking $n=0$, $A=1-2\alpha$ $(0\le\alpha<1)$ and $B=-1$ in Theorem 5, we have

COROLLARY 6. *If* $f\in\mathcal{S}^*(\alpha)$,, *then for* $\mu\ge-\alpha$

$$\operatorname{Re}\left\{\frac{z^{\mu}f(z)}{\int_0^z t^{\mu-1}f(t)\,dt}\right\} > (\mu+1)\left\{{}_2F_1(1,2(1-\alpha);\mu+2;1/2)\right\}^{-1}\quad (z\in E).$$

The result is best possible.

Similarly for $n = -1, A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$, Theorem 5 yields

COROLLARY 7. *If $f \in \mathcal{K}(\alpha)$, then for $\mu \ge -\alpha$*

$$\operatorname{Re}\left\{\frac{z^{\mu} f'(z)}{f(z) - \frac{\mu}{z^{\mu}}\int_0^z t^{\mu-1} f(t)\, dt}\right\} > (\mu+1)\{{}_2F_1(1, 2(1-\alpha), \mu+2; 1/2)\}^{-1} \quad (z \in E).$$

The result is best possible.

Putting $\mu = 0$ in Corollary 7, we obtain the following result which was also obtained by Goel [5] and MacGregor [7].

COROLLARY 8. *For $0 \le \alpha < 1$, we have*

$$\mathcal{K}(\alpha) \subset \mathcal{S}^*(\beta(\alpha)),$$

where $\beta(\alpha)$ is given by (3.9). The result is best possible.

THEOREM 6. *Let $f \in \mathcal{A}$ and $\mathcal{F}_\mu$ be defined by (3.10). If*

$$\operatorname{Re}\left\{\frac{\mathcal{I}^n f(z)}{\mathcal{I}^n \mathcal{F}_\mu(z)}\right\} > \alpha \quad (0 \le \alpha < 1,\ \mu + 1 > 0,\ z \in E),$$

then

$$\operatorname{Re}\left\{\frac{\mathcal{I}^{n-1} f(z)}{\mathcal{I}^n f(z)}\right\} > (\mu+1)\alpha - \mu \qquad (|z| < R_2),$$

where R_2 is the smallest positive root of the equation

$$(1 - 2\alpha)(\mu+1)r^2 - 2\{(\mu+1)(1-\alpha) + 1\}r + (\mu+1) = 0. \tag{3.34}$$

The result is best possible.

Proof. Defining the function $p(z)$ in E by

$$p(z) = \frac{1}{1-\alpha}\left\{\frac{\mathcal{I}^n f(z)}{\mathcal{I}^n \mathcal{F}_\mu(z)} - \alpha\right\}, \tag{3.35}$$

we see that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic and has a positive real part in E. Making use of the logarithmic differentiation on both sides of (3.35) and using (3.14) in the resulting equation, we get

$$\begin{aligned} \operatorname{Re}\left\{\frac{\mathcal{I}^{n-1} f(z)}{\mathcal{I}^n f(z)} - ((\mu+1)\alpha - \mu)\right\} &= (1-\alpha)\operatorname{Re}\left\{(\mu+1)p(z) + \frac{zp'(z)}{\alpha + (1-\alpha)p(z)}\right\} \\ &\ge (1-\alpha)\left\{(\mu+1)\operatorname{Re}\{p(z)\} - \frac{|zp'(z)|}{|\alpha + (1-\alpha)p(z)|}\right\}. \end{aligned} \tag{3.36}$$

Using (3.27) and the estimate

$$\operatorname{Re}\{p(z)\} \geq \frac{1-r}{1+r} \quad (|z| = r < 1)$$

in (3.36), we get

$$\begin{aligned}\operatorname{Re}&\left\{\frac{\mathcal{I}^{n-1}f(z)}{\mathcal{I}^n f(z)} - ((\mu+1)\alpha - \mu)\right\} \\ &\geq (1-\alpha)\operatorname{Re}\{p(z)\}\left\{(\mu+1) - \frac{2r}{(1-r^2)\alpha + (1-\alpha)(1-r)^2}\right\}\end{aligned}$$

which is certainly positive if $r < R_2$ for R_2 given by Theorem 6.

The bound R_2 is sharp for the function $f \in \mathcal{A}$ defined in E by

$$\frac{\mathcal{I}^n f(z)}{\mathcal{I}^n \mathcal{F}_\mu(z)} = \alpha + (1-\alpha)\frac{1+z}{1-z},$$

where $\mathcal{F}_\mu$ is given by (3.10).

For $n = 0$, Theorem 6 gives

COROLLARY 9. *Let $f \in \mathcal{A}$ and $\mu + 1 > 0$. If*

$$\operatorname{Re}\left\{\frac{z^\mu f(z)}{\int_0^z t^{\mu-1} f(t)\,dt}\right\} > (\mu+1)\alpha \quad (0 \leq \alpha < 1, z \in E),$$

then $f \in \mathcal{S}^*(\beta(\alpha,\mu))$ in $|z| < R_2$, where $\beta(\alpha,\mu) = (\mu+1)\alpha - \mu$ and R_2 as given in Theorem 6.

Setting $n = -1$ in Theorem 6, we get

COROLLARY 10. *Let $f \in \mathcal{A}$ and $\mu + 1 > 0$. If*

$$\operatorname{Re}\left\{\frac{z^\mu f'(z)}{f(z) - \frac{\mu}{z^\mu}\int_0^z t^{\mu-1} f(t)\,dt}\right\} > (\mu+1)\alpha \quad (0 \leq \alpha < 1, z \in E),$$

then $f \in \mathcal{K}^*(\beta(\alpha,\mu))$ in $|z| < R_2$, where $\beta(\alpha,\mu)$ is given as in Corollary 9 and R_2 as given in Theorem 6.

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J. Patel
DEPARTMENT OF MATHEMATICS
UTKAL UNIVERSITY, VANI VIHAR
BHUBANESWAR-751004, INDIA
E-mail: jpatelmath@sify.com

P. Sahoo
DEPARTMENT OF MATHEMATICS
SONEPUR COLLEGE
SONEPUR-767017, INDIA

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