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ON THE SUMS OF UNILATERALLY CONTINUOUS JUMP FUNCTIONS

Abstract. In this article we prove that each jump function $f : \mathcal{R} \to \mathcal{R}$ is the sum of two unilaterally continuous jump functions. Moreover we observe that there are jump functions $g : \mathcal{R} \to \mathcal{R}$ which are not the products of any finite family of unilaterally continuous jump functions.

Let $\mathcal{R}$ be the set of all reals. A function $f : \mathcal{R} \to \mathcal{R}$ is said to be a jump function if for each point $x \in \mathcal{R}$ there are the both finite unilateral limits

$$f(x+) = \lim_{t \to x^+} f(t) \quad \text{and} \quad f(x-) = \lim_{t \to x-} f(t).$$

It is known ([1]) that each jump function f may be discontinuous on countable set only, i.e. the set $D(f)$ of all discontinuity points of f is countable.

Observe that

REMARK 1. Let $f : \mathcal{R} \to \mathcal{R}$ be a function. If for each point $x \in \mathcal{R}$ there is at least one finite unilateral limit $f(x+)$ or $f(x-)$ then the set $D(f)$ of all discontinuity points of f is countable.

Proof. If x is a discontinuity point of f then there is a pair $(I(x), J(x))$ of closed intervals $I(x)$, $J(x)$ with rational endpoints such that

$$f(x) \in int(I(x)), \quad I(x) \subset int(J(x)) \quad \text{and} \quad x \in cl(f^{-1}(\mathcal{R} \setminus J(x))),$$

where $int(X)$ denotes the interior of the set X and $cl(X)$ is the closure of X. Assume to a contrary that the set $D(f)$ is not countable. Since the family of all pairs of closed intervals with rational endpoints is countable, there is a pair (I, J) of closed intervals with rational endpoints such that the set

$$A = \{x \in D(f) : I(x) = I \quad \text{and} \quad J(x) = J\}$$

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is not countable. Observe that if $u \in A$ is a bilateral condensation point of A then there are not unilateral limits $f(u+)$ and $f(u-)$, which contradicts to the hypothesis. This completes the proof of Remark 1.

Observe also that each countable set $A \subset \mathcal{R}$ is the set of all discontinuity points of some jump function $f : \mathcal{R} \to \mathcal{R}$. Of course, if $A = \{x_1, x_2, \ldots\}$, where $x_i \neq x_j$ for $i \neq j$, then the function

$$f(x_i) = \frac{1}{i} \quad \text{for} \quad i \geq 1 \quad \text{and} \quad f(x) = 0 \quad \text{otherwise on } \mathcal{R}$$

is a jump function with $D(f) = A$. Similarly, if

$$g(x) = \begin{cases} 0 & \text{for } x \leq \inf A \\ \sum_{x_i < x} \frac{1}{2^i} & \text{for } x > inf\ A \end{cases}$$

then g is a non-decreasing left-continuous function (so g is a jump function) and $D(g) = A$. Moreover the function

$$h(x) = \begin{cases} g(x) & \text{for } x \in \mathcal{R} \setminus A \\ \frac{1}{2}(g(x-) + g(x+)) & \text{for } x \in A \end{cases}$$

is a non-decreasing jump function which is not unilaterally continuous at points $x \in A$ and such that $D(h) = A$.

Obviously, the sum of two right-continuous (left-continuouns) functions is also right-continuous (left-continuous). However the sum of two unilaterally continuous jump functions may be discontinuous on the right and on the left hand at some points. For example the function

$$f(x) = 0 \text{ for } x \neq 0 \quad \text{and} \quad f(0) = 1$$

is the sum of functions

$$g(x) = 1 \text{ for } x \leq 0 \quad \text{and} \quad g(x) = 0 \text{ for } x > 0$$

and

$$h(x) = -1 \text{ for } x < 0 \quad \text{and} \quad h(x) = 0 \text{ for } x \geq 0,$$

which are jump and unilaterally continuous at each point $x \in \mathcal{R}$.

The sum of two jump functions is also a jump function, so the set $D(f+g)$ of all discontinuity points of the sum of two unilaterally continuous jump functions f and g is countable.

THEOREM 1. *Every jump function $f : \mathcal{R} \to \mathcal{R}$ is the sum of two unilaterally continuous jump functions.*

Proof. Since f is a jump function, for each positive integer n the set

$$A_n = \{x : osc\ f(x) \geq \frac{1}{2^n}\}$$

is closed and every point $x \in A_n$ is isolated in A_n. Without loss of generality we can suppose that

$$A_1 \neq \emptyset \quad \text{and} \quad B_{n+1} = A_{n+1} \setminus A_n \neq \emptyset \quad \text{for} \quad n \geq 1.$$

For each point $x \in A_1$ let

$$r_1(x) = \inf\{|t-x| : t \in A_1 \setminus \{x\}\},$$

where we assume that $\inf \emptyset = \infty$, and let $s_1(x)$ be a positive real less than $\frac{r_1(x)}{3}$ such that $x + s_1(x) \in C(f) = \mathcal{R} \setminus D(f)$. Define

$$g_1(t) = \begin{cases} f(x) & \text{for } t \in [x, x+s_1(x)], \quad x \in A_1 \\ f(t) & \text{otherwise on } \mathcal{R} \end{cases}$$

and

$$h_1(t) = \begin{cases} f(t) - f(x) & \text{for } t \in [x, x+s_1(x)], \quad x \in A_1 \\ 0 & \text{otherwise on } \mathcal{R}. \end{cases}$$

Then g_1 and h_1 are jump functions unilaterally continuous at each point $x \in A_1$ and $g_1 + h_1 = f$.

In the second step for each point $x \in B_2$ we define

$$r_2(x) = \inf\{|t-x| : t \in A_2 \setminus \{x\}\}$$

and we find a positive real $s_2(x) < \frac{r_2(x)}{3}$ such that

$$\{x - s_2(x), x + s_2(x)\} \cap \{t + s_1(t) : t \in A_1\} = \emptyset$$

and

$$x - s_2(x), x + s_2(x) \in C(f)$$

and

$$|f(u) - f(v)| < \frac{1}{2} \quad \text{for } u, v \in [x - s_2(x), x + s_2(x)]$$

and for each $t \in A_1$ one of the following two conditions holds:

$$[x - s_2(x), x + s_2(x)] \subset [t, t + s_1(t)] \quad \text{or} \quad [x - s_2(x), x + s_2(x)] \cap [t, t + s_1(t)] = \emptyset.$$

Let

$$g_2(t) = \begin{cases} g_1(x) & \text{for } t \in [x, x+s_2(x)], \quad x \in B_2 \\ g_1(t) - h_1(x) + h_1(t) & \text{for } t \in [x - s_2(x), x], \quad x \in B_2 \\ g_1(t) & \text{otherwise on } \mathcal{R} \end{cases}$$

and

$$h_2(t) = \begin{cases} h_1(x) & \text{for } t \in [x - s_2(x), x], \quad x \in B_2 \\ h_1(t) - g_1(x) + g_1(t) & \text{for } t \in [x, x+s_2(x)], \quad x \in B_2 \\ h_1(t) & \text{otherwise on } \mathcal{R}. \end{cases}$$

Then g_2 and h_2 are jump functions unilaterally continuous at each point $x \in A_2$ and

$$g_2 + h_2 = g_1 + h_1 = f.$$

Generally, in n-th step for each point $x \in B_n$ we define

$$r_n(x) = \{|t - x| : t \in A_n \setminus \{x\}\}$$

and find a positive real $s_n(x) < \frac{r_n(x)}{3}$ such that

$$\{x - s_n(x), x + s_n(x)\} \cap \{t + s_1(t) : t \in A_1\} = \emptyset,$$

$$\{x - s_n(x), x + s_n(x)\} \cap \{t - s_k(t), t + s_k(t) : t \in B_k,\ 1 < k < n\} = \emptyset$$

and

$$x - s_n(x), x + s_n(x) \in C(f)$$

$$|f(u) - f(v)| < \tfrac{1}{2^{n-1}} \quad \text{for} \quad u, v \in [x - s_n(x), x + s_n(x)]$$

and for each $t \in A_1$ one of the following two conditions holds

$$[x - s_n(x), x + s_n(x)] \subset [t, t + s_1(t)] \quad \text{or} \quad [x - s_n(x), x + s_n(x)] \cap [t, t + s_1(t)] = \emptyset$$

and for each $t \in B_k$, $k \in \{2, \dots, n-1\}$ one of the following two conditions holds:

$$[x - s_n(x), x + s_n(x)] \subset [t - s_k(t), t + s_k(t)]$$

or

$$[x - s_n(x), x + s_n(x)] \cap [t - s_k(t), t + s_k(t)] = \emptyset.$$

Let

$$g_n(t) = \begin{cases} g_{n-1}(x) & \text{for } t \in [x, x + s_n(x)],\ x \in B_n \\ g_{n-1}(t) - h_{n-1}(x) + h_{n-1}(t) & \text{for } t \in [x - s_n(x), x],\ x \in B_n \\ g_{n-1}(t) & \text{otherwise on } \mathcal{R} \end{cases}$$

and

$$h_n(t) = \begin{cases} h_{n-1}(x) & \text{for } t \in [x - s_n(x), x],\ x \in B_n \\ h_{n-1}(t) - g_{n-1}(x) + g_{n-1}(t) & \text{for } t \in [x, x + s_n(x)],\ x \in B_n \\ h_{n-1}(t) & \text{otherwise on } \mathcal{R}. \end{cases}$$

Then g_n and h_n are jump functions unilaterally continuous at each point $x \in A_n$ and

$$g_n + h_n = g_{n-1} + h_{n-1} = \cdots = g_1 + h_1 = f.$$

Observe that

$$|g_n - g_{n-1}| < \frac{1}{2^{n-1}} \quad \text{and} \quad |h_n - h_{n-1}| < \frac{1}{2^{n-1}} \quad \text{for} \quad n \geq 2.$$

So the sequences (g_n) and (h_n) uniformly converge to functions g and h, respectively. Evidently

$$g + h = \lim_{n\to\infty} g_n + \lim_{n\to\infty} h_n = \lim_{n\to\infty} (g_n + h_n) = f.$$

If $x \in D(f)$ then there is a positive integer $n(x)$ such that the functions g_n and h_n are unilaterally continuous at x for $n > n(x)$, and consequently g and h have the same property at x. If $x \in C(f)$ then all functions g_n and h_n, $n \geq 1$, are continuous or at least unilaterally continuous at x as well as g and h. So g and h are unilaterally continuous at each point, since the uniform limits of sequences of jump functions are jump functions, g and h are jump functions. This completes the proof.

The product of two jump functions is a jump function. But there are jump functions which are not the products of any finite family of unilaterally continuous jump functions.

For example, let $\{a_n : n = 1, 2, \ldots\}$ be a dense countable set and let

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x = a_n, \quad n \geq 1 \\ 0 & \text{otherwise on } \mathcal{R}. \end{cases}$$

If $f = f_1 f_2 \cdots f_n$ then

$$f^{-1}(0) = \bigcup_{k=1}^{n} f_k^{-1}(0),$$

and there is a positive integer $k \leq n$ such that the set $f_k^{-1}(0)$ is of the second category. So there is an open interval $I \subset cl(f_k^{-1}(0))$. There is a positive integer m such that $a_m \in I$. Since $f(a_m) \neq 0$, we have $f_k(a_m) \neq 0$ and consequently the function f_k is not unilaterally continuous at a_m.

REMARK 2. Let $f : \mathcal{R} \to \mathcal{R}$ be a jump function such that $f(x) \neq 0$ for $x \in \mathcal{R}$. Then there are unilaterally continuous jump functions $g, h : \mathcal{R} \to \mathcal{R}$ with $f = gh$.

Proof. If f is a jump function then $|f|$ has also the same property. Let $\phi = \ln(|f|)$. Then ϕ is a jump function and, by Theorem 1, the function

$$\phi = g_1 + h_1,$$

where g_1 and h_1 are unilaterally continuous jump functions. Consequently, the functions

$$g_2 = e^{g_1} \quad \text{and} \quad h = e^{h_1}$$

are unilaterally continuous jump functions and

$$|f| = \exp \phi = e^{g_1 + h_1} = e^{g_1} e^{h_1} = g_2 h.$$

The function $g = sgn(f) g_2$ is an unilaterally continuous jump function and

$$f = sgn(f)|f| = (sgn(f) g_2) h = gh,$$

thus the proof is completed.

References

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