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THE VITALI–HAHN–SAKS THEOREM FOR THE PRODUCT OF QUANTUM LOGICS

Abstract. We show as a main result that if each quantum logic of a given collection of quantum logics satisfies the Vitali–Hahn–Saks theorem, then so does their product. As a consequence we formulate a dual result for the sum of a collection of Dynkin systems.

Introduction

Recently quite a few mathematicians and physicists "have gone noncommutative", i.e. have exercised their effort in extending classic (commutative) theorems to obtain more general (noncommutative) results. This has occurred as much in algebra and geometry as in measure theory, often in an attempt to shed light on questions of quantum mechanics. We want to contribute to the measure-theoretic line by going on with the study of the Vitali–Hahn–Saks theorem (VHS) in quantum logics as previously investigated in [3], [4], [5], [6], [10], etc. Upon calling a quantum logic a VHS logic if the VHS theorem holds for it, we ask if the class of VHS logics is closed under the formation of countable products. This question announced itself naturally after the findings of [10] and [5] which established an abundance of the VHS logics and some of those which are not VHS. We answer this question in the positive, providing as a by-product new types of VHS logics. We then apply the result to "concrete logics" obtaining a result on the VHS theorem for Dynkin systems of sets.

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1. Basic notions on quantum logics

Our terminology is generally taken from [12]. Let us only review basic notions as we shall use them in the sequel.

DEFINITION 1.1. By a *quantum logic* (abbr., a logic) we mean a σ-complete orthomodular poset, i.e. a triple $(L, \leq, ')$ subject to the following requirements $(a, b \in L)$:

(i) L is a nonvoid set, $\leq$ is a partial ordering and $'$ is a unary operation,
(ii) L possesses a least and a greatest element, $0, 1$,
(iii) $a'' = a$,
(iv) if $a \leq b$, then $b' \leq a'$,
(v) $a \vee a' = 1$, $a \wedge a' = 0$,
(vi) if $a_i \in L$ $(i \in N)$ and $a_i \leq a_j'$ for each $i \neq j$, then $\bigvee_{i=1}^{\infty} a_i$ exists in L,
(vii) if $a \leq b$, then $b = a \vee (a' \wedge b)$ (the orthomodular law).

The prototype examples of logics are Boolean σ-algebras and lattices of projectors in a Hilbert space. Generally, logics do not have to be lattices and do not have to be distributive.

We shall deal with *states* on logics.

DEFINITION 1.2. By a *state* on L we mean a normalized measure on L. Formally, $s\colon L \to [0, 1]$ is said to be a state on L if

(i) $s(1) = 1$,
(ii) if $a_i \in L$ $(i \in N)$ and $a_i \leq a_j'$ $(i \neq j)$, then

$$s\Big(\bigvee_{i=1}^{\infty} a_i\Big) = \sum_{i=1}^{\infty} s(a_i).$$

Let us denote by $\mathscr{S}(L)$ the set of all states on L.

Our next definition introduces the key concept of this paper. Therein, the notion of absolutely continuous states and the notion of uniformly absolutely continuous states is given the standard meaning. Expressed formally, the state s is said to be *absolutely continuous* with respect to the state t if

$$\forall \varepsilon > 0\ \exists \delta > 0\colon t(a) < \delta \Rightarrow s(a) < \varepsilon \quad \text{for each} \quad a \in L.$$

Further, a sequence $(s_i)_{i \in N}$ of states on L is *uniformly absolutely continuous* with respect to the state t on L if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall i \in N\colon t(a) < \delta \Rightarrow s_i(a) < \varepsilon \quad \text{for each} \quad a \in L.$$

DEFINITION 1.3. A quantum logic, L, is said to be Vitali–Hahn–Saks (abbr. a VHS logic) if the Vitali–Hahn–Saks theorem is valid for L: Given states $s_i \in \mathscr{S}(L)$ $(i \in N)$ and a state $t \in \mathscr{S}(L)$ such that s_i are absolutely continuous with respect to t and, for each $a \in L$, the limit $\lim s_i(a)$ exists, then

(i) the function $s: L \to [0,1]$, $s(a) = \lim_i s_i(a)$ is a state on L,
(ii) s is absolutely continuous with respect to t,
(iii) the states s_i $(i \in N)$ are uniformly absolutely continuous with respect to t.

It can be shown [6] that for each logic the condition (i) is always fulfilled. This is easy to check since the σ-additivity of states verifies on orthogonal families only. The conditions (ii) and (iii) do *not* have to be fulfilled (see [10] and [5]). It is easy to see that (iii) implies (ii) and that the uniform convergence of s_i is equivalent to (iii). It should be noted that many important logics satisfy (iii). So do, for instance, all logics containing only finitely many maximal σ-Boolean subalgebras, the projector logic $L(H)$ for a Hilbert space H, many set-representable logics, etc. (see [10] and [5]). Thus, the class of VHS logics is reasonably large.

2. Results

On the ground of the investigations carried on in [5] and [10], a natural question occurs of whether the class of VHS logics is closed under the formation of countable products. There is also a need to clarify this question because of the important position within quantum mechanics of the logic $\Pi_{i=1}^{\infty} L_i$, where L_i is either a Boolean σ-algebra or the projector logic $L(H)$.

THEOREM 2.1. *Let L_i $(i \in N)$ be quantum logics. Let $L = \Pi_{i=1}^{\infty} L_i$ be the direct product of L_i's. Then L is VHS if, and only if, each L_i $(i \in N)$ is a VHS logic.*

Before we launch on the proof of Th. 2.1, let us collect the properties of the product in question and relate them to the VHS theorem.

PROPOSITION 2.2. *Let L_i $(i \in N)$ be quantum logics and let $L = \Pi_{i=1}^{\infty} L_i$.*

(i) *The set $E(L)$ of all elements of L whose coordinates consist of 0's and 1's in the respective L_i's $(i \in N)$ forms a Boolean subalgebra of L which is Boolean isomorphic to the complete Boolean algebra* $\exp N$ *of all subsets of the set of natural numbers.*
(ii) *Denote by e_j $(j \in N)$ the element of $E(L)$ $(E(L) \subset L)$ which consists of all 0's on all but the j-th coordinates, the j-th coordinate being 1. Let $u \in \mathscr{S}(L)$ and $M_u = \{j \in N \mid u(e_j) > 0\}$. Then u can be written as*

$$u = \sum_{j \in M_u} u(e_j) \cdot u^j,$$

where u^j is the state on L defined by setting

$$u^j(a) = \frac{u(a \wedge e_j)}{u(e_j)}.$$

Notice that the state u is a σ-convex combination of u^j, $j \in M_u$, because

$$1 = u(1) = u\Big(\bigvee_{j \in M_u} e_j\Big) = \sum_{j \in M_u} u(e_j).$$

(iii) *If $f: P \to Q$ is a quantum logic morphism onto Q and if P is a VHS logic, then so is Q. Consequence 1: Each Boolean σ-algebra is a VHS logic. Consequence 2: If $\Pi_{i=1}^{\infty} L_i$ is VHS, then each L_i is VHS.*

Proof. The property (i) follows easily — $E(L)$ is obviously a subset of the centre of $\Pi_{i=1}^{\infty} L_i$. (The centre of L is the set of all "absolutely compatible" elements of L, see e.g. [12].) To check (ii), observe that $\sum_{j \in M_u} u(e_j) = 1$ because u is a state on the Boolean σ-algebra $E(L) \cong \exp N$ when restricted to $E(L)$. The rest is easy (for details, see [8]). Property (iii) is verified in a straightforward manner. For Consequence 1, one uses the Loomis–Sikorski theorem and the classic VHS theorem for σ-algebras of sets, for Consequence 2 one uses the projection of $\Pi_{i=1}^{\infty} L_i$ onto L_i. ∎

Let us return to the proof of Th. 2.1. The condition is obviously necessary: If $\Pi_{i=1}^{\infty} L_i$ is VHS, then each L_i $(i \in N)$ has to be VHS (Prop. 2.2(iii), Consequence 2). Let us show the sufficiency. Let each L_i $(i \in N)$ be a VHS logic. For the product $\Pi_{i=1}^{\infty} L_i$, write $L = \Pi_{i=1}^{\infty} L_i$. Let us suppose that s_i $(i \in N)$ and s are states on L such that, for each $a \in L$, $\lim_i s_i(a) = s(a)$. Our goal is to show that this convergence is uniform. This would complete the proof that L is a VHS logic (as mentioned before, the uniform convergence is easily seen to imply the uniform absolute continuity).

Let us fix an $\varepsilon, \varepsilon > 0$. As in Prop. 2.2, denote by e_j $(j \in N)$ the element of $E(L)$ $(E(L) \subset L)$ which consists of all 0's on all but the j-th coordinates, the j-th coordinate being 1. Then we obtain

$$1 = s(1) = s\Big(\bigvee_{j \in N} e_j\Big) = \sum_{j \in N} s(e_j).$$

As a consequence, there exists a finite subset, M, of the set $M_s = \{j \in N \mid s(e_j) > 0\}$ such that

$$\sum_{j \in M} s(e_j) > 1 - \frac{\varepsilon}{4}.$$

We therefore have the inequality

$$\sum_{j\in N\setminus M} s(e_j) < \frac{\varepsilon}{4}.$$

Since

$$\sum_{j\in M} s_i(e_j) \to \sum_{j\in M} s(e_j)$$

when $i \in N$ approaches $+\infty$, there is a number $k_1 \in N$ such that the following implication holds true: If $i \geq k_1$, then $\sum_{j\in M} s_i(e_j) > 1 - \frac{\varepsilon}{4}$. We therefore have, for $i \geq k_1$,

$$\sum_{j\in N\setminus M} s_i(e_j) < \frac{\varepsilon}{4}.$$

Consider now the states s_i^j and s^j induced by s_i $(i \in N)$ and s (see Prop. 2.2 (ii)). Obviously, $\lim_i s_i^j = s^j$. This is correct since if $s(e_j) > 0$ then at most finitely many s_i^j are undefined. By our assumption, each L_j $(j \in N)$ satisfies the VHS theorem. Since L_j is isomorphic to $[0, e_j]$, we infer that the convergence $s_i^j \xrightarrow{i} s^j$ must be uniform. Taking advantage of this, we see that for each $j \in M$ we can find an index $n_j \in N$ such that, for each $i \geq n_j$, we obtain the following inequality (it is supposed that $a \in L$ and that m denotes the number of elements of M):

$$\left|s_i^j(a) - s^j(a)\right| < \frac{\varepsilon}{4m}.$$

Let $k_2 = \max\{n_j \mid j \in M\}$. Let us view the Boolean σ-algebra $E(L)$ as a sublogic of L. The VHS theorem holds true for Boolean σ-algebras and therefore when we restrict all s_i and s to $E(L)$, then the sequence of the restricted s_i converges uniformly to the restricted s. Thus, there is a number $k_3 \in N$ such that, for each $i \geq k_3$ and each $e_j \in E(L)$,

$$|s_i(e_j) - s(e_j)| < \frac{\varepsilon}{4m}.$$

Let us set $k = \max\{k_1, k_2, k_3\}$. In order to verify that $s_i \to s$ uniformly, let us assume that $i \geq k, a \in L$, and let us consider $|s_i(a) - s(a)|$. In the attempt to estimate it above, we first have (Prop. 2.2 (ii))

$$\begin{aligned}|s_i(a) - s(a)| &= \Big|\sum_{j\in N} s_i(e_j)s_i^j(a) - \sum_{j\in N} s(e_j)s^j(a)\Big| \\ &\leq \sum_{j\in M} |s_i(e_j)s_i^j(a) - s(e_j)s^j(a)| \\ &\quad + \sum_{j\in N\setminus M} s_i(e_j)s_i^j(a) + \sum_{j\in N\setminus M} s(e_j)s^j(a).\end{aligned}$$

We have obtained three terms to estimate. First,

$$\begin{aligned}
&\sum_{j\in M} |s_i(e_j)s_i^j(a) - s(e_j)s^j(a)| \\
&= \sum_{j\in M} |s_i(e_j)s_i^j(a) - s_i(e_j)s^j(a) + s_i(e_j)s^j(a) - s(e_j)s^j(a)| \\
&\le \sum_{j\in M} s_i(e_j)\cdot|s_i^j(a) - s^j(a)| + \sum_{j\in M} s^j(a)\cdot|s_i(e_j) - s(e_j)| \\
&\le \sum_{j\in M} |s_i^j(a) - s^j(a)| + \sum_{j\in M} |s_i(e_j) - s(e_j)| \\
&\le m\left(\frac{\varepsilon}{4m} + \frac{\varepsilon}{4m}\right) = \frac{\varepsilon}{2}.
\end{aligned}$$

Second,

$$\sum_{j\in N\setminus M} s_i(e_j)s_i^j(a) \le \sum_{j\in N\setminus M} s_i(e_j) < \frac{\varepsilon}{4}.$$

Third,

$$\sum_{j\in N\setminus M} s(e_j)s^j(a) \le \sum_{j\in N\setminus M} s(e_j) < \frac{\varepsilon}{4}.$$

We see that, for each $i \ge k$ and $a \in L$, we have obtained $|s_i(a) - s(a)| < \varepsilon$. This shows that L is a VHS logic and the proof is complete. ∎

As a consequence of the previous theorem, let us explicitly formulate the result the validity of which actually stood as the main motivation for this article. It involves the product logic of factors which are of considerable importance in quantum theories. Before stating the result, recall that the projection logic $L(H)$, for a separable Hilbert space H, is a VHS logic (see [10]) and each Boolean σ-algebra is a VHS logic (including those which may not be set-representable—see Prop. 2.2 (iii); such Boolean σ-algebras play a distinguished role within quantum theories, see e.g. [3] and [12]).

THEOREM 2.3. *Let L_i ($i \in N$) be either a Boolean σ-algebra or the lattice $L(H)$ of projectors in a separable Hilbert space. Let $L = \Pi_{i=1}^{\infty} L_i$. Then L satisfies the Vitali–Hahn–Saks theorem.*

If we inspect the proof of Th. 2.1, we find out that Th. 2.1 allows for certain "cardinal generalizations" as regards the number of the factors in the product. Some of the generalizations are, however, subject to the foundation of the set theory we work with. We nevertheless have one "absolute" result.

THEOREM 2.4. *Let I be a set of the first uncountable cardinality and let each L_i ($i \in I$) satisfy the Vitali–Hahn–Saks theorem. Then so does the product $\Pi_{i\in I} L_i$.*

P r o o f. Let $L = \Pi_{i \in I} L_i$. Obviously, the Boolean σ-algebra $E(L)$ is isomorphic to $\exp I$ and, as known (see e.g. [7]), each state on $\exp I$ must live on a countable subset of I. Thus, any countable collection of states must live on a countable subset of I, and we then apply Th. 2.1. ∎

In a similar vein we obtain the following result. (By CH we abbreviate the assumption of continuous hypothesis and by $\neg M$ the assumption of nonexistence of measurable cardinals. As known, both of these assumptions are consistent with the standard ZFC theory of sets.)

THEOREM 2.5. *Under $\neg M$, a product of logics each of which satisfies the Vitali–Hahn–Saks theorem is a logic which satisfies the Vitali–Hahn–Saks theorem. Under CH, a product of continuum many logics each of which satisfies the Vitali–Hahn–Saks theorem is a logic which satisfies the Vitali–Hahn–Saks theorem.*

P r o o f. Under $\neg M$, for each set I any state on $\exp I$ must live on a countable set (see [7]). We then easily reduce the problem to a countable product. Under CH, the continuum is the first uncountable cardinality and Th. 2.4 applies. ∎

In the second part we apply our results to the investigation of states on Dynkin systems of sets. Since Dynkin systems of sets have proved to be important in investigations of stochastic nature ([1], [2], [9]), our result may be relevant to classical or generalized probability. Let us first establish the link of Dynkin systems and quantum logics.

DEFINITION 2.6. A quantum logic which allows for a set representation is called a *Dynkin system (of sets)*. Thus, a collection Δ of subsets of a set S is said to be a Dynkin system if

(i) $\emptyset \in \Delta$,

(ii) whenever $A \in \Delta$, then $S \setminus A \in \Delta$,

(iii) whenever $A_i \in \Delta$ $(i \in N), A_i \cap A_j = \emptyset$, then $\bigcup_{i=1}^{\infty} A_i \in \Delta$.

By a state on the Dynkin system (S, Δ) is meant a probability measure on Δ. Obviously, a Dynkin system is a quantum logic with the inclusion relation on S for $\leq$ and the set complementation on S for $'$, and the notions of state coincide for quantum logics and Dynkin systems.

The product of quantum logics transforms into the sums of Dynkin systems.

DEFINITION 2.7. Let (S_i, Δ_i) $(i \in I)$ be a collection of Dynkin systems. Let $S = \bigcup_{i \in I} S_i$ be the disjoint union of S_i's. Let $\Delta = \{A \subset S \mid A \cap S_i \in$

Δ_i for each $i \in I\}$. Then we call (S, Δ) *the sum of the Dynkin systems* (S_i, Δ_i).

PROPOSITION 2.8. *Suppose that* (S, Δ) *is the sum of the Dynkin systems* (S_i, Δ_i) $(i \in I)$. *Then* (S, Δ) *is a Dynkin system. Moreover, if* (S_i, Δ_i) *is viewed as a quantum logic, say* L_i $(i \in I)$, *then the sum* (S, Δ) *is isomorphic (in the category of quantum logics) to the product* $\Pi_{i \in I} L_i$.

Proof. The proof reduces to a straightforward verification. ∎

We are in the position to translate our result into the Dynkin system setup. Observe that, in general, the VHS theorem does not have to hold for Dynkin systems though, on the other hand, the paper [5] establishes a large class of Dynkin systems for which the VHS theorem does hold.

THEOREM 2.9. *Let the cardinality of* I *do not exceed the first uncountable cardinal. Let* (S_i, Δ_i) $(i \in I)$ *be a collection of Dynkin systems and let each* (S_i, Δ_i) *satisfies the Vitali–Hahn–Saks theorem. Then the sum* (S, Δ) *of* (S_i, Δ_i) $(i \in I)$ *also satisfies the Vitali–Hahn–Saks theorem. Under* $\neg M$, *any sum of a collection of Dynkin systems each of which satisfies the Vitali–Hahn–Saks theorem also satisfies the Vitali–Hahn–Saks theorem.*

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