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COMPUTING EQUIVALENT TABULAR FUNCTIONS

Abstract. Tabular functions were invented to form a formal framework for normal and inverted function tables used in documenting complex software systems. They are defined as maps on finite partitions of a given nonempty set X with values in a set of function symbols. It is shown that every tabular function is equivalent to a tabular function of degree one. The problem of equivalence of two tabular functions is reduced to the problem of equivalence of two sets of functional expressions derived from the set of initial function symbols.

1. Introduction

Function and relation tables [10, 11, 12] have been in use for the formal documenting of complex software systems for some time. They are matrix-like functional expressions with a set of predicates representing conditional expressions used for the indices and a corresponding set of functional expressions used for the entries. Every normal function table represents a function whose domain consists of those elements that are used as arguments for the table predicates. The table predicates represent conditional expressions which split the whole domain into disjoint sets. Each entry in the table represents the values assumed by the function represented by the table when its arguments satisfy the corresponding condition predicates. Given two function tables with the same domain and codomain, the natural problem arises: How to decide whether they represent the same function? The solution to this problem consists of an algorithm which when followed provides a "yes" or "no" answer in a finite number of steps. The main difficulty in arriving to the answer, in general, lies in the fact that we cannot make pointwise comparisons of the functions under consideration since the domain may be infinite. Nevertheless, if we go to the descriptions of the functions considered, then we can compare those objects in a finite number of steps without taking into account their domain, provided the descriptions themselves are not too complex. That is, they are "finite" in a certain sense. Tabular functions introduced in this paper make this distinction clear. The approach taken seems particularly suitable for functions that look like piece-wise analytic. The other approaches towards function and relation tables are taken in [2, 6, 13]. They are useful for their classification and for the efficient computation of values or conditions assumed or represented by the tables. In particular, the way of using efficiently function and relation tables in documenting well structured programs is presented in [11, 12]. We define tabular functions of order one as maps from the set of finite partitions of a given set X, and with values in a set of function symbols. The set of function symbols under consideration will be identified with the set of functional expressions, or functional terms, derived from a given basic set of function symbols by following certain derivation rules. Recursively, tabular functions of higher orders are defined as maps from the set of finite partitions of X, and with values in the set of function symbols obtained in previous steps. Finally, the union of all of those sets is defined as the set of tabular functions on X. Then, we introduce the natural equivalence relation in the set obtained and prove that every tabular function is equivalent to a tabular function of order one, obtained during the first step in the above recursive process. Thus, the problem of equivalence of two tabular functions is reduced to the problem of equivalence of two sets of functional expressions derived from the original set of function symbols. Given function $f : X \to Y$, we will commonly identify the functional expression $f(x)$ where x is a variable that assumes values from a given subset A of X, with the function $x \mapsto f(x)$, where $x \in A \subset X$.

2. Preliminaries

Given a nonempty set X, denote by $PRF(X)$ the set of all finite partitions of the set X. Given two partitions of the set X, α and β, we say that β is finer than α if every member of α is a union of some members of β. Equivalently, β is finer than α if every member of β is included in a certain member of α.If the partition β is finer than α, we will write $\alpha \preceq \beta$ and say that β is a refinement of α. The set $PRF(X)$ is a partially ordered set with respect to the relation of being a finer partition. The trivial partition, containing only one set, X itself, is the smallest element with respect to the relation $\preceq$. If the set X is finite, then the partition made of singletons of the set X is the greatest element in $PRF(X)$. Given $\alpha \epsilon PRF(X)$, denote by $|\alpha|$ its cardinality, that is, if $\alpha = \{A_1, \ldots, A_m\}$, then $|\alpha| = m$, where m is a positive integer. Finally, given two partitions α and β of the set X, we will denote by $\alpha\beta$ the partition that consists of all nonempty intersections $A \cap B$, for all A in α and B in β, respectively. The following proposition says that the set $PRF(X)$ is a directed set with respect to the refinement relation.

PROPOSITION 2.1. *Given two partitions α and β in $PRF(X)$, there is a partition γ in $PRF(X)$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.*

Proof. Take γ as the product $\alpha\beta$. ∎

Let X, Y be two nonempty sets. Denote by FS the set of total function symbols such that for every function denoted by a member of FS, its domain is equal to X and its range is in Y. The set FS is usually obtained from an initial, finite set of function symbols FS_0 by applying certain functional operations so that the set FS can be regarded as the closure of the set FS_0 with respect to those operations. The elements of the set FS_0 denote different functions and are identified with the functions they denote. Two different elements of the set FS may denote the same function $f : X \to Y$, and in this case they are called equivalent. The elements of the set FS will be called functional expressions. The set of all classes of equivalent functional expressions will be denoted by $FS^\wedge$ and identified with the set of all functions from the set X and with values in the set Y that are denoted (or represented) by elements of FS. A functional expression, that is an element of the set FS, may contain an indeterminate, say x, such that when x is assigned a value from the set X, the value the expression assumes is an element of the set Y, and thus represents the unique function from the set X to the set Y. The following example should make this distiction clear.

EXAMPLE 2.1. Let $X = \mathbb{R}$ (the field of real numbers), $FS_0 = \{1, x, \cos(x), \sin(x)\}$, and $FS^\wedge$ denote the linear algebra over $\mathbb{R}$ generated by the set FS_0. The elements of the set $FS^\wedge$ are all functions obtained from the finite set FS_0 by application of three operations: the addition of functions, the multiplication of functions by real scalars, and the multiplication of functions by functions. Thus the set FS consists of all functional expressions of the form:

$$\sum_{i=1}^{n} A_i * x^{a_i} * (\cos(x))^{b_i} * (\sin(x))^{c_i} * \exp(d_i * x), \tag{2.1}$$

where n is an arbitrary, positive integer; and given n, for each $i = 1, 2, \ldots, n$, every A_i is a real number, and all a_i, b_i, c_i and d_i are nonnegative integers. Then, by the standard trigonometric identity, the functional expressions: $[\cos(x)]^2 + [\sin(x)]^2$ and 1, are equivalent, and represent the function: $\mathbb{R} \ni x \to 1 \in \mathbb{R}$. ∎

We are not concerned in this paper with the structural properties of the set FS other than those related to tabular operations as defined in the sequel. Actually, the main result of this paper consists in proving that tabular functions are closed with respect to those (tabular) operations.

3. Tabular functions of degree one

Given two nonempty sets X, Y, denote by FS the set of total function symbols such that for every function denoted by a member of FS, its domain is equal to X and its range is in Y. Given finite partition $\alpha = \{A_1, \ldots, A_m\} \in PRF(X)$, define

$$TB^{(1)}\,(\alpha, FS) = \{F \,|F : \alpha \to FS\}. \tag{3.1}$$

That is, every member F of (3.1) is represented by a set of ordered pairs:

$$F = \{(A_1, f_1), (A_2, f_2), \ldots, (A_m, f_m)\}, \tag{3.2}$$

where $f_j \epsilon FS$ for $j = 1, 2, \ldots m$.

Alternatively, it is represented by a pair

$$F = (\alpha, f), \tag{3.3}$$

where α is a given partition and $f = (f_1, f_2, \ldots, f_m)$ is an m-tuple of function symbols that occur in (3.2). When following definition (3.3) we will assume that members of the partition α are written in an order determined by (3.2).

Next, define

$$TB^{(1)}(X, FS) = \bigcup_{\alpha \epsilon PRF(X)} TB^{(1)}(\alpha, FS). \tag{3.4}$$

Elements of the set (3.4) will be called tabular functions of degree one, and denoted by capital letters $F, G, H, \ldots$ For the sake of completeness, the members of the set FS will be referred to as tabular functions of degree 0. It is evident from (3.4) that the tabular function F is given by all pairs in (3.2). Given $F \in TB^{(1)}(X, FS)$, denote by $\alpha(F)$ the corresponding partition and by $f(F)$ the corresponding $|\alpha|$-tuple of function symbols, that is both $\alpha(F)$ and $f(F)$ satisfying

$$F = (\alpha(F), f(F)) \in TB^{(1)}\,(X, FS), \tag{3.5}$$

$$\alpha(F) = \{A_1, \ldots, A_{|\alpha|}\} \in PRF(X), \tag{3.6}$$

$$f(F) = (f_1, f_2, \ldots, f_{|\alpha|}) \in \prod_{i=1}^{|\alpha|} FS \tag{3.7}$$

(Cartesian product of FS $|\alpha|$ times).

Each F given by (3.5)–(3.7) determines in a natural and unique way a function $F^\wedge$ from the set X and with values in the set Y, given as follows:

$$F^\wedge(x) = f_j(x) \text{ for } x \in A_j \text{ and } j = 1, 2, \ldots, |\alpha|. \tag{3.8}$$

Since the sets $A_1, \ldots, A_{|\alpha|}$ are pairwise disjoint and their union is all of X, each $x \in X$ belongs to exactly one of them and the value (3.8) is defined by exactly one of the $|\alpha|$ function symbols that occur in (3.7).

DEFINITION 3.1. Two tabular functions $F, G \in TB^{(1)}(X, FS)$ are said to be *equivalent* if they determine the same function on X, following (3.5)–(3.8). If the tabular functions F and G are equivalent, we will write $F \backsimeq G$, that is

$$F \backsimeq G \quad \text{iff} \quad F^{\wedge} = G^{\wedge}. \tag{3.9}$$

PROPOSITION 3.1. *The relation "$\backsimeq$" is an equivalence relation on $TB^{(1)}(X, FS)$.*

Proof. Already, the relation " $\backsimeq$ " has been defined in terms of equivalence classes. Given $F \in TB^{(1)}(X)$, the equivalence class $[F]$ with respect to the relation will be identified with the function $F^{\wedge}$ as given by (3.8). ∎

EXAMPLE 3.1. Let $X = \mathbb{R}$, $FS_0 = \{1, \exp(x)\}$,and let $FS^{\wedge}$ denote the linear algebra over $\mathbb{R}$ generated by the set FS_0, with the corresponding set of functional expressions FS. Next, consider the partitions of $\mathbb{R}$: $\alpha = \{(-\infty, 0), [0, 1), [1, \infty)\})$, and $\beta = \{(-\infty, 1), [1, \infty)\}$, and let $f = (1, 1, \exp(x))$, $g = (1, \exp(x))$, $F = (\alpha, f)$, and $G = (\beta, g)$. Clearly, the tabular functions F and G are different as members of $TB^{(1)}(\mathbb{R})$, and equivalent, as both determine the fuction $\Psi : \mathbb{R} \to \mathbb{R}$, given by

$$\Psi(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ \exp(x) & \text{for } x > 1. \end{cases} \tag{3.10}$$

EXAMPLE 3.2. Let $X = \mathbb{R}$ and FS denote the set of functional expressions like in Example 2.1. Next, consider the partitions $\alpha = \{(-\infty, 0), [0, 1), [1, \infty)\})$, and $\beta = \{(-\infty, 1), [1, \infty)\}$, of the set $\mathbb{R}$ and define $f = (\sin^2(x) + \cos^2(x), 1, \exp(x))$, $g = (1, \exp(x))$, and $F = (\alpha, f)$, $G = (\beta, g)$. Clearly, following a similar argument like in Example 2.1, we can see that $F^{\wedge} = G^{\wedge} = 1$. Nevertheless, the tabular functions $F = (\alpha, f)$ and $G = (\beta, g)$ regarded as maps $F, G : PRF(\mathbb{R}) \to FS$, are different by (3.5)–(3.7). ∎

The equivalence in Example 3.1 follows from the fact that for every argument the functions $F^{\wedge}$ and $G^{\wedge}$ are represented by the same functional expressions from FS over all members of an appriopriate partition. The equivalence in Example 3.2 involves equivalence among the members of the set FS itself. In this case the functions are represented by functional expressions that are different on the interval $(-\infty, 0)$, though they are equivalent in the set FS. The Example 3.2 is universal in that sense that the problem of equivalence of two tabular functions can be reduced to the problem of equivalence of members of the set FS over an appriopriate partition. In the sequel we will assume that the set FS has been fixed and write $TB^{(1)}(X)$ for $TB^{(1)}(X, FS)$.

DEFINITION 3.2. Let F be a tabular function of degree one with $\alpha(F) = \{A_1, \ldots, A_m\}$, and $f(F) = (f_1, \ldots, f_m)$, and let $\beta = \{B_1, B_2, \ldots, B_n\}$ be a finer partition than $\alpha(F)$, that is $\alpha(F) \preceq \beta$. Consider the following n-tuple $g = (g_1, g_2, \ldots, g_n)$ of function symbols given by

$$g_j = f_i, \quad \text{if } B_j \subset A_i, \quad \text{for } j = 1, 2, \ldots n;\ i = 1, 2, \ldots, m. \tag{3.11}$$

The tabular function $(\beta, g) \in TB^{(1)}(X, FS)$ will be denoted by $F(\alpha \uparrow \beta)$, and called a *lift of F from α to β.*

PROPOSITION 3.2. *For every $F \in TB^{(1)}(X)$, and every $\beta \in PRF(X)$ such that $\alpha \preceq \beta$,*

$$F \simeq F(\alpha \uparrow \beta). \tag{3.12}$$

Proof. F and $F(\alpha \uparrow \beta)$ determine the same function on X. ∎

It is clear that the equivalent tabular functions F and $F(\alpha \uparrow \beta)$ that occur in (3.12) are different as members of $TB^{(1)}(X)$ provided the partitions α and β are different. As we mentioned already, given two tabular functions $F, G \in TB^{(1)}(X)$, the difficulty in deciding whether they are equivalent lies in the fact that we cannot, in general, compare the values assumed by the functions $F^\wedge$ and $G^\wedge$, since their domain, the set X, may be infinite. Thus, we are led to deal with their descriptions. Then we have the following characterization.

PROPOSITION 3.3. *Any $F, G \in TB^{(1)}(X)$ are equivalent if and only if there is $\gamma \in PRF(X)$ such that $\alpha(F) \preceq \gamma$ and $\alpha(G) \preceq \gamma$, and $F(\alpha(F) \uparrow \gamma) \simeq G(\alpha(G) \uparrow \gamma)$.*

Proof. ($\Rightarrow$) Let $F, G \in TB^{(1)}(X)$ be equivalent. By Proposition 2.1 there is a $\gamma \in PRF(X)$ such that $\alpha(F) \preceq \gamma$ and $\alpha(G) \preceq \gamma$. Following Proposition 3.2,we obtain $F \simeq F(\alpha(F) \uparrow \gamma)$ and $G \simeq G(\alpha(G) \uparrow \gamma)$. Then, by Proposition 3.1, $F(\alpha(F) \uparrow \gamma) \simeq G(\alpha(G) \uparrow \gamma)$.

($\Leftarrow$) Let $F, G \in TB^{(1)}(X)$ and $\gamma \in PRF(X)$ such that $\alpha(F) \preceq \gamma$ and $\alpha(G) \preceq \gamma$, and $F(\alpha(F) \uparrow \gamma) \simeq G(\alpha(G) \uparrow \beta)$. As before, $F \simeq F(\alpha(F) \mapsto \gamma)$ and $G \simeq G(\alpha(G) \mapsto \gamma)$ so, by transitivity of "$\simeq$", $F \simeq G$. ∎

EXAMPLE 3.3. Let $X = \mathbb{R}$, $FS_0 = \{x, abs(x)\}$. Let FS_1 denote the linear algebra over $\mathbb{R}$ generated by the set FS_0, with the corresponding set $FS_1^\wedge$ of functional expressions. Next, consider the partitions $\alpha = \{(-\infty, 0), [0, \infty)\})$ and $\beta = \{(-\infty, 0), [0, 2), [2, \infty)\}$. Then, define $f = (-x, x)$, $g = (abs(x), x, x))$, $F = (\alpha, f)$, and $G = (\beta, g)$. Clearly, the tabular functions

F and G are different as members of $TB^{(1)}(\mathbb{R})$, and equivalent, as both determine the function $\Phi : \mathbb{R} \to \mathbb{R}$, given by

$$\Phi(x) = abs(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{3.13}$$

This time, in opposite to Example 3.2, there is no corresponding equivalence among the members of the set FS_1 (as total functions over $\mathbb{R}$). It is because the function $abs(x)$ is already represented by a tabular function from $TB^{(1)}(\mathbb{R}, FS)$ with $FS^\wedge$ being a linear algebra over $\mathbb{R}$ generated by the identity function $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$, alone.Consider, for instance,

$$\alpha = \{(-\infty, 0), [0, \infty)\}, \quad \text{and} \quad f = (-x, x). \tag{3.14}$$

Then, clearly, the tabular function

$$ABS = (\alpha, f) \tag{3.15}$$

represents the function $abs(x)$.

The last example shows that in order to be able to decide that two tabular functions are equivalent, the elements of the set FS that are not equivalent should remain so when reduced to members of the partition under consideration. (Except for, maybe, finite subsets). Such a property characterizes for instance analytic functions. From this point of view the functions represented by tabular functions of degree one can be looked upon as "piecewise analytic".

4. Tabular functions of higher degrees

We will need the following characterization of tabular functions of degree one.

PROPOSITION 4.1. *The set $TB^{(1)}(X)$ is equal to the following set*

$$Z = \{(\alpha, f) | \alpha \in PRF(X),\ f = (f_1, \dots, f_{|\alpha|}),\ f_j \in FS,\ j = 1, 2, \dots, |\alpha|\}. \tag{4.1}$$

Proof. Let $\alpha \in PRF(X)$. It is clear that $TB^{(1)}(\alpha, FS) \subset Z$. And vice versa, each $(\alpha, f) \in Z$ belongs to $TB^{(1)}(\alpha, FS)$.Thus, the assertion holds. ∎

Now we are in a position to extend the notion of a tabular function of degree one, defined in Section 3, to include higher degrees. To do so, we need to consider all pairs (α, f), where α is, as before, an element of $PRF(X)$, and $f = (f_1, f_2, \dots, f_{|\alpha|})$ is an $|\alpha|$-tuple of another tabular functions. Let us define recursively the following sets

$$PTB^{(0)}(X) = TB^{(0)}(X) = FS. \tag{4.2}$$

Inductively, for $k = 1, 2, 3, \dots$, we set

$$PTB^{(k)}(X) = \{(\alpha, f) | \alpha \in PRF(X), \quad f = (f_1^{(k-1)}, \dots, f_{|\alpha|}^{(k-1)})\}, \tag{4.3}$$

where

$$f_j^{(k-1)} \in PTB^{(k-1)}(X), \quad \text{for} \quad j = 1, 2, \ldots, |\alpha|. \tag{4.4}$$

Next,

$$TB^{(k)}(X) = \{(\alpha, f) | \alpha \in PRF(X), f = (f_1, f_2, \ldots, f_{|\alpha|})\}, \tag{4.5}$$

where

$$f_j \in \bigcup_{j=0}^{k-1} PTB^{(j)}(X). \tag{4.6}$$

Finally,

$$PTB(X) = \bigcup_{k=0}^{\infty} PTB^{(k)}(X), \tag{4.7}$$

$$TB(X) = \bigcup_{k=0}^{\infty} TB^{(k)}(X). \tag{4.8}$$

Given $F \in TB(X)$ we have that either $F \in FS$, or that there is a partition $\alpha \in PTF(X)$, say of m members, $\alpha = \{A_1, A_2, \ldots, A_m\}$, and a corresponding an m-tuple $f = (f_1, f_2, \ldots, f_m)$, with each $f_j \in TB(X)$, such that $F = (\alpha, f)$. Then we define its degree, denoted by $\deg(F)$, as follows:

$$\deg(F) = \begin{cases} 0, & \text{if } F \in TB^{(0)}(X) = FS, \text{ and} \\ 1 + \max\limits_{1 \le j \le m} (\deg(f_j), & \text{otherwise.} \end{cases} \tag{4.9}$$

COROLLARY 4.1.

$$\deg(F) = 0 \text{ if and only if } F \in FS = PTB^{(0)}(X), \tag{4.10}$$

$$\deg(F) = 1 \text{ if and only if } F \in PTB^{(1)}(X), \tag{4.11}$$

$$\deg(F) = k \text{ for any } F \in PTB^{(k)}(X), \text{ for } k = 0, 1, 2, \ldots \quad \blacksquare \tag{4.12}$$

Elements of the sets (4.3) will be called pure tabular functions of degree k on X, elements of the set (4.7) will be called pure tabular functions on X, and elements of the set (4.8) will be called tabular functions on X.

The following proposition summarizes the relationships between the sets we have introduced.

PROPOSITION 4.2.

$$PTB^{(0)}(X) = TB^{(0)}(X) = FS, \tag{4.13}$$

$$PTB^{(1)}(X) = TB^{(1)}(X), \tag{4.14}$$

$$PTB^{(k)}(X) \subset TB^{(k)}(X), \text{ for } k = 2, 3, \ldots \text{ (strict inclusion).} \tag{4.15}$$

P r o o f. The identities (4.13) and (4.14) follow directly from (4.2)–(4.6) and Proposition 4.1. The inclusion (4.15) follows from the fact that if we restrict ourselves in (4.5)-(4.6) to the elements of $PTB^{(k-1)}(X)$ only, then we obtain exactly the set $PTB^{(k)}(X)$. Moreover, if we consider other elements, for instance from $PTB^{(0)}(X)$, then we obtain elements from the outside of $PTB^{(k)}(X)$, for $k \geq 2$. ■

COROLLARY 4.2.

$$PTB(X) \subset TB(X) \text{ (proper inclusion).} \blacksquare \tag{4.16}$$

From now on unless explicitly stated we will assume that the tabular functions are of degree ≥ 1. By induction, following Definition 3.1, we extend the notion of equivalency of two tabular functions, from $TB^{(1)}(X)$ to $TB(X)$. Similarly, given tabular function $F \in TB(X)$, with its natural partition $\alpha = \alpha(F)$, and given partition $\beta \in PRF(X)$, such that $\alpha \preceq \beta$, we extend the notion of a lift of F from α to β, again denoted by $F(\alpha(F) \uparrow \gamma)$. Given $F \in TB(X)$, and $A \in \alpha(F)$, we will write $f(F)^{(A)}$ to indicate the component of the tuple $f(F)$ referring to the set A, if the tuple itself is not explicitly displayed. Let us also introduce the notion of a *joint* of two tabular functions defined on two different sets.

DEFINITION 4.1. Given two tabular functions $F_1 \in TB(X_1)$, and $F_2 \in TB(X_2)$, where $X_1 \cap X_2 = \emptyset$, their joint, denoted $F_1 \oplus F_2$, is defined as an element of $TB(X_1 \cup X_2)$ as follows:

$$\alpha(F_1 \oplus F_2) = \alpha(F_1) \cup \alpha(F_2), \quad \text{and} \tag{4.17}$$

$$f(F_1 \oplus F_2)^{(A)} = \begin{cases} f(F_1)^{(A)}, & \text{if } A \in \alpha(F_1), \\ f(F_2)^{(A)}, & \text{if } A \in \alpha(F_2). \end{cases} \tag{4.18}$$

It follows that the operation *joint* preserves equivalence and degree. Namely, we have the following proposition.

PROPOSITION 4.3. *Let $F_i, G_i \in TB^{(1)}(X_i)$, and $F_i \simeq G_i$ for $i = 1, 2$. Then,*

$$F_1 \oplus F_2 \simeq G_1 \oplus G_2, \quad \text{and} \tag{4.19}$$

$$\deg(F_1 \oplus F_2) = \max(\deg(F_1), \deg(F_2)). \tag{4.20}$$

P r o o f. Directly follows from Definitions 3.1 and 4.1, and from formula (4.9). ■

Now we are in a position to state and prove the anticipated result on equivalency of arbitrary tabular functions.

THEOREM 4.1. *For every $F \in TB(X)$ there is $G \in TB^{(1)}(X)$ such that $F \simeq G$.*

Proof. By induction with respect to $\deg(F)$. Let $F \in TB(X)$. If $\deg(F)=1$, the theorem clearly holds. Let us now assume that whenever $\deg(F) \leq k$ for an integer $k \geq 1$, theorem holds, and assume that $\deg(F) = k+1$. Let

$$\alpha = \alpha(F) = \{A_1, A_2, \dots, A_m\} \quad \text{and} \tag{4.21}$$

$$f = f(F) = (f_1, f_2, \dots, f_m) \tag{4.22}$$

denote the natural partition for F and the corresponding m-tuple of function symbols, with each f_j in $TB(X)$ and each one of degree $\leq k$.

Denote by m_0 the number of elements in the tuple (4.22) that are of degree 0, where $0 \leq m_0 < m$.

First, consider the case $m_0 = 0$, that is $1 \leq \deg(f_j) \leq k$ for each $j = 1, 2, \dots, m$. By hypothesis, there are m tabular functions of degree 1, say $F_1^{(1)}, F_2^{(1)}, \dots, F_m^{(1)}$, with the corresponding partitions $\beta_1, \beta_2, \dots, \beta_m$ such that $f_j \simeq F_j^{(1)}$, for $j = 1, 2, \dots, m$. Let

$$\beta_i = \{B_{i,1}, B_{i,2}, \dots, B_{i,|\beta_i|}\} \quad \text{and} \tag{4.23}$$

$$f(F_i^{(1)}) = (f_{i,1}^{(0)}, f_{i,2}^{(0)}, \dots, f_{i,|\beta_i|}^{(0)}\}, \quad \text{for } i = 1, 2, \dots, m. \tag{4.24}$$

Next, define for every $i = 1, 2, \dots, m$, the following collections

$$\delta(A_i) = \{A_i \cap B | B \in \beta_i \quad \text{and} \quad A_i \cap B \neq \emptyset\}. \tag{4.25}$$

Finally, let

$$\delta(\alpha) = \bigcup_{i=1}^{m} \delta(A_i). \tag{4.26}$$

It is clear that every collection (4.25) forms a finite and nonempty partition for A_i, and thus (4.26) forms a finite partition for X, that is $\delta(\alpha) \in PRF(X)$. Moreover,

$$\alpha \preceq \delta(\alpha) \quad \text{and} \quad \beta_i \preceq \delta(\alpha), \quad \text{for } i = 1, 2, \dots m. \tag{4.27}$$

Let us enumerate the partition members in (4.25):

$$\delta(A_i) = \{A_{i,1}, A_{i,2}, \dots, A_{i,p_i}\}, \tag{4.28}$$

where $1 \leq p_i \leq |\beta_i|$ for $i = 1, 2, \dots, m$.

Now, for every $1 \leq i \leq m$, define the following p_i function symbols of degree 0:

$$g_{i,s}^{(0)} = f_{i,r}^{(0)}, \text{ if } A_{i,s} = A_i \cap B_{i,r}, \text{ for } s=1,2,\dots p_i;\ r=1,2,\dots,|\beta_i|. \tag{4.29}$$

By the above definition, for every $1 \leq i \leq m$, the p_i-tuples $(g_{i,1}^{(0)}, g_{i,2}^{(0)}, \ldots$ $\ldots, g_{i,p_i}^{(0)})$ given by (4.27) and $|\beta_i|$-tuples (4.24) represent the same function on A_i. Therefore, the pair

$$G = (\delta(\alpha), g), \text{ with the member } g \text{ given below,} \tag{4.30}$$

$$g = (g_{1,1}^{(0)}, g_{1,2}^{(0)}, \ldots, g_{1,p_1}^{(0)}, \ldots, g_{2,p_2}^{(0)}, \ldots, g_{m,1}^{(0)}, g_{m,2}^{(0)}, \ldots, g_{m,p_m}^{(0)}), \tag{4.31}$$

defines an element of $TB^{(1)}(X)$, such that $G \simeq F(\alpha(F) \uparrow \delta(\alpha))$. Thus $F \simeq G$, which proves the theorem for the case $m_0 = 0$.

To prove the case $0 < m_0 < m$, assume, for the sake of simplicity, that the elements of degree 0 in the tuple $f(F)$ occupy the last m_0 positions in the tuple. Then, consider the following collections of sets:

$$X_1 = \bigcup_{i=1}^{m-m_0} A_i, \tag{4.32}$$

$$X_2 = \bigcup_{j=1}^{m_0} A_{m-m_0+j}, \tag{4.33}$$

$$\alpha_1 = \{A_1, A_2, \ldots, A_{m-m_0}\}, \tag{4.34}$$

$$\alpha_2 = \{A_{m-m_0+1}, A_{m-m_0+2}, \ldots, A_m\}. \tag{4.35}$$

Obviously, the sets given by (4.32) and (4.33) are disjoint, their union equals X, and the collections given by (4.34) and (4.35) form partitions for them.

Now, consider the following two tuples of tabular functions:

$$f_{(1)} = (f_1, f_2, \ldots, f_{m-m_0}) \quad \text{and} \tag{4.36}$$

$$f_{(2)} = (f_{m-m_0+1}, f_{m-m_0+2}, \ldots, f_m). \tag{4.37}$$

We use them to form two tabular functions defined on X_1 and X_2 respectively:

$$F_1 = (\alpha_1, f_{(1)}), \tag{4.38}$$

$$F_2 = (\alpha_2, f_{(2)}). \tag{4.39}$$

Then, we have

$$F_1 \in TB(X_1) \quad \text{and} \quad 1 \leq \deg(F_1) \leq k+1, \tag{4.40}$$

$$F_2 \in TB(X_2) \quad \text{and} \quad \deg(F_2) = 1, \tag{4.41}$$

$$F = F_1 \oplus F_2. \tag{4.42}$$

Proceeding exactly like in the case $m_0 = 0$, we get a tabular function of degree one, say $G_1 \in TB^{(1)}(X_1)$, such that $G_1 \simeq F_1$. Then, consider the joint $G = G_1 \oplus F_2$. It follows from (4.9) and (4.20) that

$$\text{(4.43)} \qquad \deg(G) = \deg(G_1 \oplus F_2) = \max(\deg(G_1), \deg(F_2)) = 1.$$

Moreover, since $F_1 \simeq G_1$, then $F = F_1 \oplus F_2 \simeq G_1 \oplus F_2 = G$ which completes the proof. ∎

5. Conclusions

Given two tabular functions $F, G \in TB(X)$, of degree ≥ 1, there are, by Theorem 4.1, tabular functions of degree one, say $F^{(1)}, G^{(1)} \in TB^{(1)}(X)$, such that $F^{(1)} \simeq F$, and $G^{(1)} \simeq G$. Thus, by Propositions 2.1 and 3.2, there are equivalent lifts, say $F'^{(1)}$ and $G'^{(1)}$,of degree one, of $F^{(1)}$ and $G^{(1)}$respectively, defined on a partitition $\gamma \in PRF(X)$. Let $\gamma = \{C_1, C_2, \ldots, C_m\}$, $f(F'^{(1)}) = (f_1, f_2, \ldots, f_m)$ and $f(G'^{(1)}) = (g_1, g_2, \ldots, g_m)$. Then the equivalency of F and G follows from the equivalency of the function symbols $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$ (in FS) respectively. Based on this result we can construct effective provers for piecewice analytic functions in complex or real variables. In particular we can extend the capabilities of the existing computer algebra systems when it comes to piecewise defined functions.

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