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ON CLIQUISHNESS OF MAPS OF TWO VARIABLES

Abstract. We give some sufficient conditions under which a separately cliquish map of two variables is cliquish.

Throughout the paper $(X, \mathcal{T})$ and (Y, τ) are topological spaces, $(Z, \mathcal{V})$ a uniform space and $P_{\mathcal{V}}$ a saturated family of pseudometrics on Z inducing the given uniformity $\mathcal{V}$.

A function $g : X \to Z$ is called: *cliquish at* $x_0 \in X$ if for each $\rho \in P_{\mathcal{V}}$, $\varepsilon > 0$ and each neighbourhood U of x_0 there is an open nonempty set $U_1 \subset U$ such that $\rho(g(x'), g(x'')) < \varepsilon$ for $x', x'' \in U_1$; *cliquish*, if it is cliquish at each point [3]; (in the case of a metric space Z these definitions coincide with those given in [10]).

For a function $f : X \times Y \to Z$ and any $x \in X$, $y \in Y$ by f_x, f^y we will denote the x-section and y-section of f, i.e. the functions $f_x : Y \to Z$ and $f^y : X \to Z$ given by $f_x(y) = f(x, y) = f^y(x)$. It is known that a separately cliquish function (i.e. possessing all sections cliquish) need not be cliquish [5]. On the other hand the cliquishness of f does not imply this property for its sections. For instance, let us consider the subset $A = \{p = (x, y) \in R^2 : x \geq 0 \text{ and } y \geq 0\} \cup \{p = (q, 0) \in R^2 : q \in Q\}$ of the euclidean plane and let $f : R^2 \to R$ be the characteristic function of A. Then f is cliquish, but for $y = 0$ the section f^y is not cliquish at any point $p = (x, 0)$ with $x < 0$.

In this paper we give some sufficient conditions for cliquishness of functions and multivalued maps of two variables.

For a topological space $(X, \mathcal{T})$ let us put

$$\mathcal{T}_q = \{U \setminus H : U \in \mathcal{T}, H \text{ is of the first category in } (X, \mathcal{T})\}.$$

Then $\mathcal{T}_q$ is a topology on X, $\mathcal{T} \subset \mathcal{T}_q$; [6]. In what follows, topological notions referring to the topology $\mathcal{T}_q$ will be qualified by the prefix $\mathcal{T}_q$ or simply q

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to distinguish them from those pertaining to the initial topology $\mathcal{T}$. For example $Int_q A$ and $Cl_q A$ denote the $\mathcal{T}_q$-interior and $\mathcal{T}_q$-closure of a set A while $Int A$ and $Cl A$ are the interior and the closure with respect to the topology $\mathcal{T}$. Furthermore, for a function $g: X \to Z$ the symbols $C(g, \mathcal{T})$ and $C(g, \mathcal{T}_q)$ will be used to denote the sets of all points at which g is $\mathcal{T}$-continuous or $\mathcal{T}_q$-continuous, respectively.

It is easy to see that given topological spaces $(X, \mathcal{T})$ and (Y, τ) we have $\mathcal{T} \times \tau \subset \mathcal{T}_q \times \tau_q \subset (\mathcal{T} \times \tau)_q$ and — in general — these three topologies are different.

LEMMA 1 ([2, 6]). *Let $(X, \mathcal{T})$ be a Baire space, then:*

(a) *for each $\mathcal{T}_q$-closed set $A \subset X$ we have $Int_q A = Int A$;*

(b) *the spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}_q)$ have the same classes of the first category sets;*

(c) *$(X, \mathcal{T})$ and $(X, \mathcal{T}_q)$ have the same classes of sets with the Baire property;*

(d) *$(X, \mathcal{T}_q)$ is a Baire space.*

A function $g: X \to Z$ is said to be: *quasicontinuous at $x_0 \in X$* if for each neighbourhoods U of x_0 and V of $g(x_0)$ there is an open nonempty set $U_1 \subset U$ with $g(U_1) \subset V$; *quasicontinuous*, if it has this property at each point; [7].

It immediately follows from the definitions that if $g: X \to Z$ is continuous (quasicontinuous, cliquish) at x_0, then it is $\mathcal{T}_q$-continuous ($\mathcal{T}_q$-quasicontinuous, $\mathcal{T}_q$-cliquish) at x_0, but the converse is not true.

THEOREM 1. *Let $(X, \mathcal{T})$, (Y, τ) be locally second countable Baire spaces, $(Z, \mathcal{V})$ a uniform one and let $f: X \times Y \to Z$ be a function which all sections f_x, f^y, $x \in X$, $y \in Y$, are cliquish. If for each $(x, y) \in X \times Y$, f_x is τ_q-quasicontinuous at y or f^y is $\mathcal{T}_q$-quasicontinuous at x, then f is $\mathcal{T} \times \tau$-cliquish.*

Proof. Let us take $(x_0, y_0) \in X \times Y$, $\varepsilon > 0$, $\rho \in P_{\mathcal{V}}$ and let $U \times V$ be a neighbourhood of (x_0, y_0). Without loss of generality we can assume that U, V have countable bases $\{U_n : n = 1, 2, \ldots\}$ and $\{V_n : n = 1, 2, \ldots\}$, respectively. We put for each $n = 1, 2, \ldots$

$$A_n = \left\{ y \in V : \rho\left(f(x', y), f(x'', y)\right) < \frac{1}{8}\varepsilon \text{ for } x', x'' \in U_n \right\}.$$

Since for each point $(x, y) \in U \times V$, f^y is cliquish at x, there exists $n \geq 1$ such that $\rho(f(x', y), f(x'', y)) < \frac{1}{8}\varepsilon$ for $x', x'' \in U_n$; hence $V = \bigcup_{n=1}^{\infty} A_n$. The set V is of the second category in (Y, τ_q), so for some $n \geq 1$ we have

$\emptyset \neq Int_q Cl_q A_n = Int Cl_q A_n$. Now we put for each $j = 1, 2, \dots$

$$B_j = \Big\{ x \in U_n : V_j \subset Int Cl_q A_n \\ \text{and } \rho(f(x, y'), f(x, y'')) < \frac{1}{8}\varepsilon \text{ for } y', y'' \in V_j \Big\}.$$

For any points $x \in U_n$ and $y \in Int Cl_q A_n \cap V$ the function f_x is cliquish at y, thus there exists $V_j \subset Int Cl_q A_n \cap V$ such that $\rho(f(x, y'), f(x, y'')) < \frac{1}{8}\varepsilon$ for $y', y'' \in V_j$. This implies $\bigcup_{n=1}^{\infty} B_j = U_n$. The set U_n is of the second category in $(X, \mathcal{T}_q)$, so $\emptyset \neq Int_q Cl_q B_k = Int Cl_q B_k$ for some $k \geq 1$. In the consequence we obtain $A_n \cap V_k \neq \emptyset$, $B_k \cap Int Cl_q B_k \cap U_n \neq \emptyset$ and

(1) if $x \in B_k$, then $\rho(f(x, y'), f(x, y'')) < \frac{1}{8}\varepsilon$ for $y', y'' \in V_k$.

For any $(x, y), (u, v) \in (B_k \cap Int Cl_q B_k \cap U_n) \times (A_n \cap V_k)$ the inequality

$$\rho(f(x, y), f(u, v)) \leq \rho(f(x, y), f(u, y)) + \rho(f(u, y), f(u, v)) < \frac{1}{4}\varepsilon$$

holds, i.e. we have obtained

(2) $\rho(f(x, y), f(u, v))$
$< \frac{1}{4}\varepsilon$ for $(x, y), (u, v) \in (B_k \cap Int Cl_q B_k \cap U_n) \times (A_n \cap V_k)$.

Let us take $(x_1, y_1) \in (Int Cl_q B_k \cap U_n) \times V_k$. At first we suppose that f_{x_1} is τ_q-quasicontinuous at y_1. In this case there exists a nonempty open set $V' \subset V_k$ and an of the first category set $M' \subset Y$ such that

$$\rho(f(x_1, y), f(x_1, y_1)) < \frac{1}{8}\varepsilon \text{ for } y \in V' \setminus M'.$$

This follows from conditions $V' \setminus M' \in \tau_q$ and $V' \setminus M' \subset Int Cl_q A_n$ that $(V' \setminus M') \cap A_n \neq \emptyset$. Furthermore, we have $V_k \subset Int Cl_q A_n$. Hence for each $u \in B_k \cap Int Cl_q B_k \cap U_n$ and $v \in (V' \setminus M') \cap A_n$ it holds: $\rho(f(x_1, v), f(u, v)) < \frac{1}{8}\varepsilon$ and $\rho(f(x_1, v), f(x_1, y_1)) < \frac{1}{8}\varepsilon$; consequently $\rho(f(x_1, y_1), f(u, v)) < \frac{1}{4}\varepsilon$. So we have shown:

(3) if $(x_1, y_1) \in (Int Cl_q B_k \cap U_n) \times V_k$ and f_{x_1} is τ_q- quasicontinuous at y_1, then there exists a point $(u, v) \in (B_k \cap Int Cl_q B_k \cap U_n) \times (A_n \cap V_k)$ such that $\rho(f(x_1, y_1), f(u, v)) < \frac{1}{4}\varepsilon$.

Now, let $(x_1, y_1) \in (Int Cl_q B_k \cap U_n) \times V_k$ be a point such that f^{y_1} is $\mathcal{T}_q$-quasicontinuous at x_1. Then there exists a nonempty open set $U' \subset Int Cl_q B_k \cap U_n$ and an of the first category set $H' \subset X$ such that

$$\rho(f(x_1, y_1), f(x, y_1)) < \frac{1}{8}\varepsilon \quad \text{for } x \in U' \setminus H'.$$

Since B_k is $\mathcal{T}_q$-dense in $Int Cl_q B_k \cap U_n$ we obtain $(U' \setminus H') \cap B_k \cap Int Cl_q B_k \cap U_n \neq \emptyset$. Hence for each $u \in (U' \setminus H') \cap B_k \cap Int Cl_q B_k \cap U_n$, $v \in A_n \cap V_k$

we get $\rho(f(x_1,y_1),f(u,y_1)) < \frac{1}{8}\varepsilon$ and $\rho(f(u,y_1),f(u,v)) < \frac{1}{8}\varepsilon$; then $\rho(f(x_1,y_1),f(u,v)) < \frac{1}{4}\varepsilon$. Thus we have shown:

(4) if $(x_1,y_1) \in (IntCl_qB_k \cap U_n) \times V_k$ and f^{y_1} is $\mathcal{T}_q$-quasicontinuous at x_1, then there exists a point $(u,v) \in (B_k \cap IntCl_qB_k \cap U_n) \times (A_n \cap V_k)$ such that $\rho(f(x_1,y_1),f(u,v)) < \frac{1}{4}\varepsilon$.

Finally, from (3), (4) and (2) we have $\rho(f(x',y'),f(x'',y'')) < \varepsilon$ for (x',y'), $(x'',y'') \in (IntCl_qB_k \cap U_n) \times V_k$, so f is $\mathcal{T} \times \tau$-cliquish at (x_0,y_0) and the proof is completed. ∎

THEOREM 2. *Let $(X,\mathcal{T})$, (Y,τ) be locally second countable spaces such that $(X \times Y, \mathcal{T} \times \tau)$ is a Baire space and let $(Z,\mathcal{V})$ be a uniform one. Suppose that $f: X \times Y \to Z$ is a function with f_x τ_q-cliquish and f^y $\mathcal{T}_q$-cliquish for each $x \in X$, $y \in Y$. If for each $(x,y) \in X \times Y$ f_x is quasicontinuous at y or f^y is quasicontinuous at x, then f is $\mathcal{T}_q \times \tau_q$-cliquish.*

Proof. Let us take $(x_0,y_0) \in X \times Y$, $\varepsilon > 0$, $\rho \in P_{\mathcal{V}}$ and let $(U \setminus H_1) \times (V \setminus H_2)$ be a $\mathcal{T}_q \times \tau_q$ neighbourhood of (x_0,y_0). Without loss of generality we can suppose that U, V have countable bases $\{U_n : n = 1,2,\ldots\}$ and $\{V_n : n = 1,2,\ldots\}$, respectively. Let us put

$$E = \{(x,y) \in X \times Y : f_x \text{ is quasicontinuous at } y\},$$
$$E_1 = \{(x,y) \in X \times Y : f^y \text{ is quasicontinuous at } x\}.$$

Since $E \cup E_1 = X \times Y$ and $(U \setminus H_1) \times (V \setminus H_2)$ is of the second category in $(X \times Y, \mathcal{T} \times \tau)$ then at least one of the sets $E \cap ((U \setminus H_1) \times (V \setminus H_2))$, $E_1 \cap ((U \setminus H_1) \times (V \setminus H_2))$ is of the second category. Suppose cf. that the first one. Denote for each $n = 1,2,\ldots$

$$A_n = \left\{x \in U : \rho(f(x,y'),f(x,y'')) < \frac{1}{8}\varepsilon \text{ for } y',y'' \in V_n\right\}.$$

Then $(U \times V) \cap E \subset (\bigcup_{n=1}^{\infty} A_n) \times V \subset U \times V$, hence $(U \times V) \cap E = (\bigcup_{n=1}^{\infty} A_n \times V) \cap E$. Since $(U \times V) \cap E$ is of the second category, for some $n \geq 1$, then the set $(A_n \times V) \cap E$ and also $A_n \times V$ is of the second category. Under assumptions, $(X,\mathcal{T})$ and (Y,τ) are Baire spaces. According to Lemma 1, A_n is of the second category in $(X,\mathcal{T}_q)$. Thus $\emptyset \neq Int_qCl_qA_n = IntCl_qA_n$. For each $y \in V_n$ the function f^y is $\mathcal{T}_q$-cliquish at any point belonging to $IntCl_qA_n$, therefore for each $y \in V_n$ there is $U_{m(y)} \subset IntCl_qA_n$ and an of the first category set $H_y \subset X$ such that

(5) $\rho(f(x',y),f(x'',y)) < \frac{1}{8}\varepsilon$ for $x',x'' \in U_{m(y)} \setminus H_y$.

Let us put $B_m = \{y \in V_n : m(y) = m\}$ for $m = 1,2,\ldots$ Then $V_n = \bigcup_{m=1}^{\infty} B_m$. The set V_n is of the second category, so $IntClB_k \neq \emptyset$ for some $k \geq 1$. Since $B_k \cap IntClB_k \cap V_n$ is a dense second countable subspace of $IntClB_k \cap V_n$ so we can choose a countable set $D \subset B_k \cap IntClB_k \cap V_n$

which is dense in $IntClB_k \cap V_n$. Then $M = \bigcup_{y\in D} H_y$ is of the first category and from (5) we have

$$(6) \qquad \rho(f(x',y),f(x'',y)) < \frac{1}{8}\varepsilon \text{ for } y \in D,\ \ x',x'' \in U_k \setminus M.$$

Furthermore

$$(7) \qquad \rho(f(x,y'),f(x,y'')) < \frac{1}{8}\varepsilon \text{ for } x \in A_n,\ \ y',y'' \in IntClB_k \cap V_n.$$

So, it follows from (6) and (7) that

$$(8) \qquad \rho(f(x,y),f(u,v)) < \frac{1}{4}\varepsilon \text{ for } (x,y),(u,v) \in (U_k \cap A_n \setminus M) \times D.$$

Now, let $(x,y) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap E$. By the quasicontinuity of f_x at y there is a nonempty open set $W \subset IntClB_k \cap V_n$ such that $\rho(f(x,y),f(x,w)) < \frac{1}{8}\varepsilon$ for $w \in W$. From this inequality and (6) for each $v \in D \cap W$ and $u \in U_k \cap A_n \setminus M$ we get $\rho(f(x,y),f(x,v)) < \frac{1}{8}\varepsilon$ and $\rho(f(x,v),f(u,v)) < \frac{1}{8}\varepsilon$. Thus we have shown:

(9) for each $(x,y) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap E$ there is a point $(u,v) \in (U_k \cap A_n \setminus M) \times D$ such that $\rho(f(x,y),f(u,v)) < \frac{1}{4}\varepsilon$.

Finally, let $(x,y) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap E_1$. The function f^y is quasicontinuous at x, so a nonempty open set $W_1 \subset U_k$ can be choosen in such a way that $\rho(f(x,y),f(x',y)) < \frac{1}{8}\varepsilon$ for $x' \in W_1$. The set A_n is $\mathcal{T}_q$-dense in U_k, hence $(W_1 \setminus M) \cap A_n \neq \emptyset$. For each point $a \in (W_1 \setminus M) \cap A_n$ and $b \in D$ the following inequalities $\rho(f(a,b),f(a,y)) < \frac{1}{8}\varepsilon$ and $\rho(f(x,y),f(a,y)) < \frac{1}{8}\varepsilon$ are true and from this follows that

(10) for each $(x,y) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap E_1$ there exists a point $(a,b) \in (U_k \cap A_n \setminus M) \times D$ such that $\rho(f(x,y),f(a,b)) < \frac{1}{4}\varepsilon$.

As a consequence of (8), (9), (10) we have $\rho(f(x',y'),f(x'',y'')) < \varepsilon$ for $(x',y'),(x'',y'') \in (U_k \setminus (M \cup H_1)) \times (IntClB_k \cap V_n \setminus H_2) \subset (U \setminus H_1) \times (V \setminus H_2)$ which ends the proof. ∎

THEOREM 3. *Let $(X,\mathcal{T})$, (Y,τ) be locally second countable spaces such that $(X \times Y, \mathcal{T} \times \tau)$ is a Baire space and let $(Z,\mathcal{V})$ be a uniform space. Suppose that a function $f: X \times Y \to Z$ has all sections f_x τ_q-cliquish and all f^y $\mathcal{T}_q$-cliquish, $x \in X$, $y \in Y$. If at least one of the sets*

$$E = \{(x,y) \in X \times Y : f_x \text{ is quasicontinuous at } y\},$$
$$E_1 = \{(x,y) \in X \times Y : f^y \text{ is quasicontinuous at } x\}$$

is residual, then f is $(\mathcal{T} \times \tau)_q$-cliquish.

Proof. Let $(x_0,y_0) \in X \times Y$, $\varepsilon > 0$, $\rho \in P_{\mathcal{V}}$ and let $(U \times V) \setminus H$ be a $(\mathcal{T} \times \tau)_q$ neighbourhood of (x_0,y_0), i.e. $U \in \mathcal{T}$, $V \in \tau$ and H is of the first category in $X \times Y$. Suppose that E is residual. Using notations and arguments as in the proof of Theorem 2 we repeat that proof up to (9). Hence we get

$\rho(f(x',y'), f(x'',y'')) < \varepsilon$

for $(x',y'),(x'',y'') \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap (E \setminus H) \subset (U \times V) \setminus H$

and

$$((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap (E \setminus H) = [U_k \times (IntClB_k \cap V_n)] \setminus [(M \times (IntClB_k \cap V_n)) \cup (X \times Y \setminus E) \cup H] \in (\mathcal{T} \times \tau)_q$$

which completes the proof. ∎

LEMMA 2. *Let $(X,\mathcal{T})$ be a Baire space, (Z,d) a metric one and let $g: X \to Z$ be given*

(a) *The function g is cliquish if and only if $X \setminus C(g,\mathcal{T})$is of the first category* [5, 9].

(b) *Moreover, let (Z,d) be separable. Then g has the Baire property if and only if g is $\mathcal{T}_q$-cliquish.*

Proof. (b) Let $\{W_n : n = 1,2,\ldots\}$ be an open base of Z. Then we have

$$X \setminus C(g,\mathcal{T}_q) = \bigcup_{n=1}^{\infty} \left[g^{-1}(W_n) \setminus Int_q g^{-1}(W_n)\right].$$

If g has the Baire property, then $X \setminus C(g,\mathcal{T}_q)$ is of the first category in $(X,\mathcal{T}_q)$, so g is $\mathcal{T}_q$-cliquish. Conversely, let g be $\mathcal{T}_q$-cliquish. Then according to (a) the set $X \setminus C(g,\mathcal{T}_q)$ is of the first category. For each open set $W \subset Z$ we have $g^{-1}(W) = V \cup H$, where $V \in \mathcal{T}_q$ and $H \subset X \setminus C(g,\mathcal{T}_q)$. Thus $g^{-1}(W)$ has the Baire property, which completes the proof. ∎

As a simple consequence of this lemma and Theorems 2 and 3 we obtain:

THEOREM 4. *Let $(X,\mathcal{T})$, (Y,τ) be locally second countable spaces such that $(X \times Y, \mathcal{T} \times \tau)$ is a Baire space and let (Z,d) be a separable metric space. Suppose that a function $f: X \times Y \to Z$ has all sections f_x, f^y with the Baire property. If one of the following conditions is satisfied:*

(a) *for each $(x,y) \in X \times Y$, f_x is quasicontinuous at y or f_y is quasicontinuous at x;*

(b) *at least one of the sets $E = \{(x,y) \in X \times Y : f_x$ is quasicontinuous at $y\}$, $E_1 = \{(x,y) \in X \times Y : f^y$ is quasicontinuous at $x\}$ is residual;*

then f has the Baire property. ∎

COROLLARY 1 (1, Th. 3 and Th. 4). *Suppose that a function $f: R^2 \to R$ has all sections f_x, f^y cliquish (with the Baire property). If for each $(x,y) \in R^2$,*

f_x is quasicontinuous at y or f^y is quasicontinuous at x, then f is cliquish (has the Baire property). ■

But for real functions it is possible to establish the results similar to those in Theorems 1 and 4 taking some weaker assumptions on sections f_x and f^y. To begin with, we remind some definitions.

A function $f: X \to R$ is said to be upper (lower) quasicontinuous at $x_0 \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of x_0 there exists an open nonempty set $U_1 \subset U$ such that $f(x) < f(x_0) + \varepsilon$ (resp. $f(x_0) - \varepsilon < f(x)$) for $x \in U_1$, [4].

A function f is called upper (lower) quasicontinuous if it has this property at each point.

Each quasicontinuous function is upper and lower quasicontinuous; the inverse is not true. Furthermore, each upper (lower) quasicontinuous function defined on a Baire space is cliquish [4].

THEOREM 5. *Let $(X, \mathcal{T})$, (Y, τ) be locally second countable Baire spaces and let $f: X \times Y \to R$ be a function which all sections f_x and f^y are cliquish. If for each $(x, y) \in X \times Y$, f_x is upper and lower τ_q-quasicontinuous at y or f^y is upper and lower $\mathcal{T}_q$-quasicontinuous at x, then f is $\mathcal{T} \times \tau$-cliquish.*

Proof. Let $(x_0, y_0) \in X \times Y$, $\varepsilon > 0$ and let $U \times V$ be a neighbourhood of (x_0, y_0). Suppose that U, V have countable bases $\{U_n : n = 1, 2, \ldots\}$ and $\{V_n : n = 1, 2, \ldots\}$, respectively. Let us denote

$$A_n = \left\{ y \in V : |f(x', y) - f(x'', y)| < \frac{1}{8}\varepsilon \text{ for } x', x'' \in U_n \right\}.$$

Now, using the same notations and arguments as in the proof of Theorem 1 we repeat that proof up to (2), (taking the euclidean metric in R instead of ρ).

Let $(x_1, y_1) \in (IntCl_q B_k \cap U_n) \times V_k$. At first we suppose that f_{x_1} is upper and lower τ_q-quasicontinuous at y_1. By the upper τ_q-quasicontinuity there is an open nonempty set $V' \subset V_k$ and an of the first category set $M' \subset Y$ such that

$$f(x_1, y) < f(x_1, y_1) + \varepsilon \text{ for } y \in V' \setminus M'.$$

Since $V' \setminus M' \in \tau_q$ and $V' \setminus M' \subset IntCl_q A_n$ then we obtain $(V' \setminus M') \cap A_n \neq \emptyset$. Furthermore for each $u \in B_k \cap IntCl_q B_k \cap U_n$ and $v \in (V' \setminus M') \cap A_n$ we have $|f(x_1, v) - f(u, v)| < \frac{1}{8}\varepsilon$ and $f(x_1, v) < f(x_1, y_1) + \frac{1}{8}\varepsilon$. This implies $f(u, v) < f(x_1, v) + \frac{1}{8}\varepsilon < f(x_1, y_1) + \frac{1}{4}\varepsilon$. So we have shown:

(11) if $(x_1, y_1) \in (IntCl_q B_k \cap U_n) \times V_k$ and f_{x_1} is upper τ_q-quasicontinuous at y_1, then there exists $(u, v) \in (B_k \cap IntCl_q B_k \cap U_n) \times (A_n \cap V_k)$ such that

$$f(u, v) < f(x_1, y_1) + \frac{1}{4}\varepsilon.$$

Similarly, applying the lower τ_q-quasicontinuity of f_{x_1} at y_1 we obtain

(12) if $(x_1, y_1) \in (IntCl_qB_k \cap U_n) \times V_k$ and f_{x_1} is lower τ_q-quasicontinuous at y_1, then there exists $(u_1, v_1) \in (B_k \cap IntCl_qB_k \cap U_n) \times (A_n \cap V_k)$ such that

$$f(x_1, y_1) - \frac{1}{4}\varepsilon < f(u_1, v_1).$$

Take $(x_2, y_2) \in (IntCl_qB_k \cap U_n) \times V_k$ with f_{x_2} upper and lower τ_q-quasicontinuous at y_2. According to (11) and (12) points (u', v'), $(u'_1, v'_1) \in (B_k \cap IntCl_qB_k \cap U_n) \times (A_n \cap V_k)$ can be choosen in such a way that the inequalities

$$f(u', v') < f(x_2, y_2) + \frac{1}{4}\varepsilon \quad \text{and} \quad f(x_2, y_2) - \frac{1}{4}\varepsilon < f(u'_1, v'_1)$$

hold. Hence, using also (2) and (11), we get

$$\begin{aligned} f(x_1, y_1) - f(x_2, y_2) &< f(u_1, v_1) + \frac{1}{4}\varepsilon - f(u', v') + \frac{1}{4}\varepsilon \\ &\le |f(u_1, v_1) - f(u', v')| + \frac{1}{2}\varepsilon < \frac{3}{4}\varepsilon. \end{aligned}$$

Similarly we obtain $f(x_2, y_2) - f(x_1, y_1) < \frac{3}{4}\varepsilon$, so

(13) $|f(x_1, y_1) - f(x_2, y_2)| < \frac{3}{4}\varepsilon$ for all $(x_i, y_i) \in (IntCl_qB_k \cap U_n) \times V_k$ such that f_{x_i} is upper and lower τ_q-quasicontinuous at y_i, $i = 1, 2$.

Now, suppose that $y_1 \in V_k$ and f^{y_1} is upper and lower $\mathcal{T}_q$-quasicontinuous at $x_1 \in IntCl_qB_k \cap U_n$. Then there exists a nonempty open set $U' \subset IntCl_qB_k \cap U_n$ and an of the first category set H' such that $f(x, y_1) < f(x_1, y_1) + \frac{1}{8}\varepsilon$ for $x \in U' \setminus H'$. Since B_k is $\mathcal{T}_q$-dense in $IntCl_qB_k \cap U_n$ then we have $(U' \setminus H') \cap B_k \cap IntCl_qB_k \cap U_n \neq \emptyset$. Therefore for all $u \in (U' \setminus H') \cap B_k \cap IntCl_qB_k \cap U_n$ and $v \in A_n \cap B_k$ the inequalities $|f(u, y_1) - f(u, v)| < \frac{1}{8}\varepsilon$ and $f(u, y_1) < f(x_1, y_1) + \frac{1}{8}\varepsilon$ hold, hence $f(u, v) < f(u, y_1) + \frac{1}{8}\varepsilon < f(x_1, y_1) + \frac{1}{4}\varepsilon$. Thus

(14) if $(x_1, y_1) \in (IntCl_qB_k \cap U_n) \times V_k$ and f^{y_1} is upper $\mathcal{T}_q$-quasicontinuous at x_1, then there exists $(u, v) \in (B_k \cap IntCl_qB_k \cap U_n) \times (A_n \cap V_k)$ such that

$$f(u, v) < f(x_1, y_1) + \frac{1}{4}\varepsilon.$$

Analogously, by the symmetry, using the lower $\mathcal{T}_q$-quasicontinuity of f^{y_1} at x_1 we have

(15) if $(x_1, y_1) \in (IntCl_qB_k \cap U_n) \times V_k$ and f^{y_1} is lower $\mathcal{T}_q$-quasicontinuous at x_1, then there is $(u_1, v_1) \in (B_k \cap IntCl_qB_k \cap U_n) \times (A_n \cap V_k)$ such that

$$f(x_1, y_1) - \frac{1}{4}\varepsilon < f(u_1, v_1).$$

The properties (14) and (15) imply

(16) $|f(x_1,y_1)-f(x_2,y_2)|<\frac{3}{4}\varepsilon$ for all $(x_i,y_i)\in(IntCl_qB_k\cap U_n)\times V_k$ such that f^{y_i} is upper and lower $\mathcal{T}_q$-quasicontinuous at x_i, $i=1,2$.

Finally, applying (11), (12), (14) and (15), similarly as in in the proof of the property (13), we get $|f(x,y)-f(x',y')|<\varepsilon$ for all points $(x,y),(x',y')\in(IntCl_qB_k\cap U_n)\times V_k\subset U\times V$ which completes the proof. ∎

THEOREM 6. *Let $(X,\mathcal{T})$, (Y,τ) be locally second countable Baire spaces and let $f\colon X\times Y\to R$ be a function with f_x upper quasicontinuous and f^y lower quasicontinuous (or conversely) for all $x\in X$, $y\in Y$. Then f is cliquish.*

Proof. Following [4, Coroll. 9] all sections f_x and f^y for $x\in X$, $y\in Y$ are cliquish. Let $(x_0,y_0)\in X\times Y$, $\varepsilon>0$ and let $U\times V$ be a neighbourhood of (x_0,y_0). Using notations and arguments as in the proof of Theorem 1 we repeat that proof up to (2). Thus we get open sets $U_n\subset U$, $V_k\subset V$ and of the second category sets A_n, B_k such that $B_k\cap IntCl_qB_k\cap U_n\neq\emptyset$ and $A_n\cap V_k\neq\emptyset$. Then, similarly as in the proof of Theorem 5, we show that for the the function f the conditions (11) and (15) are satisfied. This implies $|f(x,y)-f(u,v)|<\frac{1}{4}\varepsilon$ for all $(x,y),(u,v)\in(B_k\cap IntClB_k\cap U_n)\times(A_n\cap V_k)$. Furthermore for each $(x,y)\in(IntClB_k\cap U_n)\times V_k$ there exist points $(u,v),(u_1,v_1)\in(B_k\cap IntClB_k\cap U_n)\times(A_n\cap V_k)$ such that $f(u,v)-\frac{1}{4}\varepsilon<f(x,y)<f(u_1,v_1)+\frac{1}{4}\varepsilon$. Hence $|f(x,y)-f(x',y')|<\varepsilon$ for each $(x,y),(x',y')\in(IntClB_k\cap U_n)\times V_k\subset U\times V$, which completes the proof. ∎

THEOREM 7. *Let $(X,\mathcal{T})$, (Y,τ) be locally second countable spaces such that $(X\times Y,\mathcal{T}\times\tau)$ is a Baire space and let $f\colon X\times Y\to R$ be a function such that f_x is τ_q-cliquish and f^y is $\mathcal{T}_q$-cliquish for each $x\in X$, $y\in Y$. If for each $(x,y)\in X\times Y$, f_x is upper and lower quasicontinuous at y or f^y is upper and lower quasicontinuous at x, then f is $\mathcal{T}_q\times\tau_q$-cliquish.*

Proof. Let $(x_0,y_0)\in X\times Y$, $\varepsilon>0$ and $(U\setminus H_1)\times(V\setminus H_2)$ be a $\mathcal{T}_q\times\tau_q$-neighbourhood of (x_0,y_0), i.e. $U\in\mathcal{T}$, $V\in\tau$, H_1, H_2 are of the first category in X and Y respectively. Without loss of generality we can assume that U, V have countable bases $\{U_n:n=1,2,\ldots\}$ and $\{V_n:n=1,2,\ldots\}$. We put

$$S=\{(x,y)\in X\times Y: f_x \text{ is upper and lower quasicontinuous at } y\}$$
$$S_1=\{(x,y)\in X\times Y: f^y \text{ is upper and lower quasicontinuous at } x\}.$$

In the next we use the same notations and arguments as in the proof of Theorem 2 and we repeat that proof up to (8), taking S, S_1 instead of E and E_1, and replacing ρ by the usual metric in R. Now, let $(x_1,y_1)\in((U_k\setminus M)\times(IntClB_k\cap V_n))\cap S$. Since f_{x_1} is upper quasicontinuous at y_1, there exists an open nonempty set $W\subset IntClB_k\cap V_n$ such that $f(x_1,w)<f(x_1,y_1)+\frac{1}{8}\varepsilon$ for $w\in W$. Then for any $u\in U_k\cap A_n\setminus M$ and $v\in D\cap W$

we have $f(x_1, v) < f(x_1, y_1) + \frac{1}{8}\varepsilon$, and from (6), $|f(x_1, v) - f(u, v)| < \frac{1}{8}\varepsilon$. Hence we have shown $f(u, v) < f(x_1, y_1) + \frac{1}{4}\varepsilon$. Similarly, using the lower quasicontinuity of f_{x_1} at y_1 we get the inequality: $f(x_1, y_1) - \frac{1}{4}\varepsilon < f(p, q)$ for some $(p, q) \in (U_k \cap A_n \setminus M) \times D$. Thus we have

(17) for each point $(x_1, y_1) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap S$ there exist points $(u_1, v_1), (p_1, q_1) \in (U_k \cap A_n \setminus M) \times D$ such that $f(u_1, v_1) < f(x_1, y_1) + \frac{1}{4}\varepsilon$ and $f(x_1, y_1) - \frac{1}{4}\varepsilon < f(p_1, q_1)$.

If $(x_2, y_2) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap S_1$, then the upper quasicontinuity of f^{y_2} at x_2 implies the existence of an open nonempty set $U' \subset U_k$ such that $f(x', y_2) < f(x_2, y_2) + \frac{1}{8}\varepsilon$ for $x' \in U'$. Since A_n is $\mathcal{T}_q$-dense in U_k we get $U' \cap A \setminus M \neq \emptyset$. Thus from the last inequality and (7) we obtain $f(u, y_2) < f(x_2, y_2) + \frac{1}{8}\varepsilon$ and $|f(u, y_2) - f(u, v)| < \frac{1}{8}\varepsilon$ for all $u \in U' \cap A_n \setminus M$ and $v \in D$; in the consequence $f(u, v) < f(x_2, y_2) + \frac{1}{4}\varepsilon$. In the similar way, by the lower quasicontinuity of f^{y_2} at x_2 we obtain the "symmetric" result; so we have proved:

(18) for each point $(x_2, y_2) \in ((U_k \setminus M) \times (IntClB_k \cap V_n)) \cap S_1$ there exist points $(u_2, v_2), (p_2, q_2) \in (U_k \cap A_n \setminus M) \times D$ such that $f(u_2, v_2) < f(x_2, y_2) + \frac{1}{4}\varepsilon$ and $f(x_2, y_2) - \frac{1}{4}\varepsilon < f(p_2, q_2)$.

Finally, let $(x_1, y_1), (x_2, y_2) \in (U_k \setminus (M \cup H_1)) \times (IntClB_k \cap V_n \setminus H_2)$. According to (17) and (18) the points $(u_i, v_i), (p_i, q_i) \in (U_k \cap A_n \setminus M) \times D$ can be choosen in such a way that $f(u_i, v_i) < f(x_i, y_i) + \frac{1}{4}\varepsilon$ and $f(x_i, y_i) - \frac{1}{4}\varepsilon < f(p_i, q_i)$, $i = 1, 2$. Then applying (8) we get $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$ which completes the proof. ∎

THEOREM 8. *Let $(X, \mathcal{T})$ be a Baire space, (Y, τ) a locally second countable one and let $f: X \times Y \to R$ be a function with f_x cliquish for each $x \in X$ and f^y upper and lower quasicontinuous for $y \in Y$. Then f is cliquish.*

Proof. Let $(x_0, y_0) \in X \times Y$, $\varepsilon > 0$ and let $U \times V$ be a neighbourhood of (x_0, y_0). We can assume that V has a countable base $\{V_n : n = 1, 2, \ldots\}$.

Let $A_n = \{x \in U : |f(x, y') - f(x, y'')| < \frac{1}{8}\varepsilon$ for all $y', y'' \in V_n\}$. For $x \in U$, f_x is cliquish, so there is $n_x \geq 1$ such that $|f(x, y') - f(x, y'')| < \frac{1}{8}\varepsilon$ for $y', y'' \in V_{n_x}$. Hence we obtain $U = \bigcup_{n=1}^{\infty} A_n$. Since U is of the second category, $IntClA_n \neq \emptyset$ for some $n \geq 1$. Let us take $(a, b) \in (U \cap IntClA_n) \times V_n$. According to [2, Coroll. 9] f^b is cliquish. Thus a nonempty open set $U_1 \subset U \cap IntClA_n$ can be choosen in such a way that $|f(x', b) - f(x'', b)| < \frac{1}{8}\varepsilon$ for $x', x'' \in U_1$. Then for any $(x, y) \in U_1 \times V_n$ and $(s, t) \in (A_n \cap U_1) \times V_n$ we have $|f(s, b) - f(s, t)| < \frac{1}{8}\varepsilon$. Furthermore, f^y is upper quasicontinuous at x, therefore there is a nonempty open set $U' \subset U_1$ with $f(x', y) < f(x, y) + \frac{1}{8}\varepsilon$

for $x' \in U'$. Hence for any $x_1 \in U' \cap A_n$ we have $f(x_1, y) < f(x, y) + \frac{1}{8}\varepsilon$, $|f(x_1, y) - f(x_1, b)| < \frac{1}{8}\varepsilon$ and $|f(x_1, b) - f(s, b)| < \frac{1}{8}\varepsilon$, which leads to the inequality $f(s,t) < f(x,y) + \frac{1}{2}\varepsilon$. Similarly, by using the lower quasicontinuity of f^y at x we get $f(x,y) < f(s,t) + \frac{1}{2}\varepsilon$. Thus we have shown that $|f(x,y) - f(s,t)| < \frac{1}{2}\varepsilon$ for all $(x,y) \in U_1 \times V_n$ and $(s,t) \in (A_n \cap U_1) \times V_n$. In the consequence we obtain $|f(x',y') - f(x'',y'')| < \varepsilon$ for every $(x',y'), (x'',y'') \in U_1 \times V_n$, which ends the proof. ■

The results obtained for real functions we will apply to multivalued maps. To begin with, we stand some notions and notations. Given a uniform space $(Z, \mathcal{V})$ for any $p \in Z$, $M, M_1 \subset Z$, $\rho \in P_{\mathcal{V}}$ and $r > 0$ we denote

$$B(p,\rho,r) = \{z \in Z : \rho(p,z) < r\}, \qquad B(M,\rho,r) = \bigcup\{B(p,\rho,r) : p \in M\},$$
$$\rho(p,M) = \inf_{z \in M} \rho(p,z) \quad \text{and} \quad \rho(M_1, M) = \sup_{z \in M_1} \rho(z, M).$$

One can easily see the following property:

(19) if M_1 is compact, then $\rho(M_1, M) < r$ iff $M_1 \subset B(M, \rho, r)$.

In a uniform space $(Z, \mathcal{V})$ by $\mathcal{Z}$ we denote the family of all nonempty compact subsets of Z. Then the sets

$$\{(M_1, M_2) \in \mathcal{Z} \times \mathcal{Z} : M_1 \subset B(M_2, \rho, r) \text{ and } M_2 \subset B(M_1, \rho, r)\}, \quad \rho \in P_{\mathcal{V}}, r > 0,$$

form a base of a uniformity $\overline{\mathcal{V}}$ on $\mathcal{Z}$. Furthermore, for each $\rho \in P_{\mathcal{V}}$, the Hausdorff pseudometric $\tilde{\rho}$ is given by $\tilde{\rho}(M_1, M_2) = \max\{\rho(M_1, M_2), \rho(M_2, M_1)\}$, and then $P_{\tilde{\mathcal{V}}} = \{\tilde{\rho} : \rho \in P_{\mathcal{V}}\}$.

A multivalued map $F: X \to Z$ with compact values is said to be cliquish at a point $x \in X$ if the function $F: X \to (\mathcal{Z}, \tilde{\mathcal{V}})$ is cliquish at x [3]. F is called cliquish if it has this property at each point.

A multivalued map $F: X \to Z$ is said to be upper (lower) quasicontinuous at $x_0 \in X$ if for each open set $W \subset Z$ with $F(x_0) \subset W$ (resp. $F(x_0) \cap W \neq \emptyset$) and for each neighbourhood U of x_0 there is an open nonempty set $U_1 \subset U$ such that $F(x) \subset W$ (resp. $F(x) \cap W \neq \emptyset$) for $x \in U_1$, [3, 8]. F is called upper (lower) quasicontinuous if it has this property at each point.

Let us take $z \in Z$, a finite set $L \subset Z$ and $\rho \in P_{\mathcal{V}}$. For a multivalued map $F: X \to Z$ we have adjoint functions $\varphi_{L,\rho}, \psi_{z,\rho}: X \to R$ defined by $\psi_{z,\rho}(x) = \rho(z, F(x))$ and $\varphi_{L,\rho}(x) = \rho(F(x), L)$.

In the sequel by $\mathcal{L}(D)$ we denote the family of all finite subsets of a set $D \subset Z$ and we will write $\mathcal{L}$ instead of $\mathcal{L}(Z)$. Then the following holds:

LEMMA 3. *Let X be a topological space, $(Z,\mathcal{V})$ a uniform one and $F\colon X \to Z$ be a multivalued map with compact values. Then:*

(a) *if F is lower quasicontinuous at x_0, then for each $L \in \mathcal{L}$, $\rho \in P_\mathcal{V}$ the function $\varphi_{L,\rho}$ is lower quasicontinuous at x_o;*

(b) *if F is upper quasicontinuous at x_0, then for each $z \in Z$, $\rho \in P_\mathcal{V}$ the function $\psi_{z,\rho}$ is lower quasicontinuous at x_o.*

Proof. (a) We fix $\rho \in P_\mathcal{V}$, $L \in \mathcal{L}$, a neighbourhood U of x_0 and $a \in R$ with $a < \varphi_{L,\rho}(x_0)$. Then we choose $\varepsilon > 0$ satisfying $a < r - 2\varepsilon$, where $r = \varphi_{L,\rho}(x_0)$. Since the set $F(x_0)$ is compact, then some $z \in F(x_0)$ can be choosen in such a way that $\rho(z,w) > r - \varepsilon$ for each $w \in L$. Thus we have $F(x_0) \cap B(z,\rho,\varepsilon) \neq \emptyset$, so there exists an open nonempty set $U_1 \subset U$ with $F(x) \cap B(z,\rho,\varepsilon) \neq \emptyset$ for $x \in U_1$. Hence for all $y \in B(z,\rho,\varepsilon) \cap F(x)$, $x \in U_1$ and $w \in L$ we have $r - \varepsilon < \rho(z,w) \leq \rho(z,y) + \rho(y,w) < \varepsilon + \rho(y,w)$, i.e. $\rho(y,w) > r - 2\varepsilon$. In the consequence $\rho(y,L) > r - 2\varepsilon$ for each $y \in B(z,\rho,\varepsilon) \cap F(x)$, $x \in U_1$ which gives $\rho(F(x),L) > r - 2\varepsilon$. Thus we have obtained $\varphi_{L,\rho}(x) > a$ for $x \in U_1$.

(b) Let $\rho \in P_\mathcal{V}$, $z \in Z$, $\varepsilon > 0$ and a neighbourhood U of x_0 be given; we put $r = \psi_{z,\rho}(x_0)$. There exists an open nonempty set $U_1 \subset U$ such that $F(x) \subset B(F(x_0),\rho,\frac{1}{3}\varepsilon)$ for $x \in U_1$. Given $x \in U_1$, $y \in F(x)$ there exists $z_1 \in F(x_0)$ with $\rho(y,z_1) < \frac{1}{3}\varepsilon$. Hence $r - \frac{1}{3}\varepsilon < \rho(z,F(x_0)) \leq \rho(z,z_1) \leq \rho(z,y) + \rho(y,z_1) < \rho(z,y) + \frac{1}{3}\varepsilon$, so $r - \frac{2}{3}\varepsilon < \rho(y,z)$ for each $y \in F(x)$. The latter means that $r - \varepsilon < \psi_{z,\rho}(x)$ for $x \in U_1$ and the proof is completed. ∎

We will use some results presented in [3] which here are stated as the following lemmas.

LEMMA 4 ([3, Th. 3]). *Let X be a topological space, $(Z,\mathcal{V})$ a uniform one and let $F\colon X \to Z$ be a multivalued map.*

(a) *If F is lower quasicontinuous at x_0, then for each $z \in Z$, $\rho \in P_\mathcal{V}$ the function $\psi_{z,\rho}$ is upper quasicontinuous at x_0.*

(b) *If there exists a dense set $D \subset Z$ such that for each $z \in D$, $\rho \in P_\mathcal{V}$ the function $\psi_{z,\rho}$ is upper quasicontinuous at x_0, then F is lower quasicontinuous at x_0.*

LEMMA 5 ([3, Th. 4]). *Let X be a topological space, $(Z,\mathcal{V})$ a uniform one and let $F\colon X \to Z$ be a multivalued map with compact values.*

(a) *If F is upper quasicontinuous at x_0, then for each $L \in \mathcal{L}$, $\rho \in P_\mathcal{V}$ the function $\varphi_{L,\rho}$ is upper quasicontinuous at x_0.*

(b) *If there exists a dense set $D \subset Z$ such that for each $L \in \mathcal{L}(D)$, $\rho \in P_\mathcal{V}$ the function $\varphi_{L,\rho}$ is upper quasicontinuous at x_0, then F is upper quasicontinuous at x_0.*

LEMMA 6 ([3, Th. 5]). *Assume that a multivalued map $F: X \to (Z, \mathcal{V})$ is cliquish at a point $x_0 \in X$. Then*

(a) *for each $z \in Z$, $\rho \in P_{\mathcal{V}}$ the function $\psi_{z,\rho}$ is cliquish at x_0;*

(b) *if F has compact values, then for each $L \in \mathcal{L}$, $\rho \in P_{\mathcal{V}}$ the function $\varphi_{L,\rho}$ is cliquish at x_0.*

LEMMA 7 ([3, Th. 6]). *Let X be a Baire space, $(Z, \mathcal{V})$ a separable uniform one and let $F: X \to Z$ be a multivalued map with compact values. If for each $\rho \in P_{\mathcal{V}}$, $z \in D$, $L \in \mathcal{L}(D)$ functions $\psi_{z,\rho}$, $\varphi_{L,\rho}$ are cliquish, where D is a countable dense subset of Z, then F is cliquish.*

THEOREM 9. *Let $(X, \mathcal{T})$, (Y, τ) be locally second countable spaces such that $(X \times Y, \mathcal{T} \times \tau)$ is a Baire space and let $(Z, \mathcal{V})$ be a separable uniform space. Suppose that a multivalued map $F: X \times Y \to Z$ with compact values has all sections F_x, F^y, $x \in X$, $y \in Y$ cliquish. If for each $(x, y) \in X \times Y$, F_x is upper and lower quasicontinuous at y or F^y is upper and lower quasicontinuous at x, then F is cliquish.*

Proof. Let D be a dense countable subset of Z. Following Lemma 6 for each $w \in Z$, $\rho \in P_{\mathcal{V}}$ and $L \in \mathcal{L}(D)$ all functions $(\psi_{w,\rho})_x$, $(\psi_{w,\rho})^y$, $(\varphi_{L,\rho})_x$, $(\varphi_{L,\rho})^y$ are cliquish. Applying Lemmas 3, 4, 5 we have that all sections $(\psi_{w,\rho})_x$, $(\varphi_{L,\rho})_x$ are upper and lower quasicontinuous at y or $(\psi_{w,\rho})^y$, $(\varphi_{L,\rho})^y$ are upper and lower quasicontinuous at x for $(x, y) \in X \times Y$. Under assumptions $(X, \mathcal{T})$ and (Y, τ) are Baire spaces, so according to Theorem 5 all functions $\psi_{w,\rho}$, $\varphi_{L,\rho}$ are cliquish. Finally using Lemma 7 we obtain the conclusion. ∎

THEOREM 10. *Let $(X, \mathcal{T})$, (Y, τ) be locally second countable spaces such that $(X \times Y, \mathcal{T} \times \tau)$ is a Baire space and let $(Z, \mathcal{V})$ be a separable uniform space. If $F: X \times Y \to Z$ is a multivalued map with compact values such that F_x is upper quasicontinuous and F^y is lower quasicontinuous (or F_x is lower quasicontinuous and F^y is upper quasicontinuous) for $x \in X$, $y \in Y$ then F is cliquish.*

Proof. According to Lemmas 3–5 all sections $(\varphi_{L,\rho})_x$, $(\psi_{z,\rho})^y$ are upper quasicontinuous and $(\varphi_{L,\rho})^y$, $(\psi_{z,\rho})_x$ are lower quasicontinuous for each $z \in Z$, $L \in \mathcal{L}$, $\rho \in P_{\mathcal{V}}$. Hence by Theorem 6 all functions $\varphi_{L,\rho}$, $\psi_{z,\rho}$ are cliquish. Now applying Lemma 7 we have that F is cliquish. ∎

THEOREM 11. *Let $(X, \mathcal{T})$ be a topological space, (Y, τ) locally second countable such that $(X \times Y, \mathcal{T} \times \tau)$ is a Baire space and $(Z, \mathcal{V})$ a separable uniform one. If $F: X \times Y \to Z$ is a multivalued map with compact values such that F_x is cliquish and F^y upper and lower quasicontinuous for all $x \in X$, $y \in Y$ then F is cliquish.*

Proof. It follows from Lemma 6 that all sections $(\varphi_{L,\rho})_x$, $(\psi_{w,\rho})_x$ are cliquish and by Lemmas 3-5 all $(\varphi_{L,\rho})^y$, $(\psi_{w,\rho})^y$ are upper and lower quasi-continuous for each $w \in Z$, $L \in \mathcal{L}$, $\rho \in P_{\mathcal{V}}$. Applying Theorem 8 we get that all functions $\varphi_{L,\rho}$, $\psi_{w,\rho}$ are cliquish, so Lemma 7 implies that F is cliquish. ■

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