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MULTIDIMENSIONAL RIEMANN–HILBERT TYPE PROBLEMS IN LEIBNIZ ALGEBRAS WITH LOGARITHMS

To the memory of Professor Eberhard Meister

Abstract. Riemann–Hilbert type problems in Leibniz algebras with logarithms have been studied in PR[8] (cf. Chapter 14). These problems correspond to such classical problems when the Cauchy transformation is an involution. It was shown that this involution is not multiplicative. On the other hand, in the same book equations with multiplicative involutions were considered. These results can be applied to equations with an involutive transformation of argument, in particular, to equations with transformed argument by means of a function of Carleman type. Riemann–Hilbert type problems with an additional multiplicative involution in commutative Leibniz algebras with logarithms are examined in PR[13]. Results obtained there can be applied not only to problems with a transformation of argument but also to problems with the conjugation (in the complex sense).

In the present paper there are considered similar problems in several variables with Riemann- Hilbert condition posed on each variable separately. For instance, these problems correspond in the classical case to problems for polyanalytic functions on polydiscs (cf. HD[1], Ms[1]).

Riemann–Hilbert type problems in Leibniz algebras with logarithms have been studied in PR[8] (cf. Chapter 14). These problems correspond to such classical problems when the Cauchy transformation is an involution. It was shown that this involution is **not** multiplicative. On the other hand, in the same book (cf. Chapter 16; also PR[9], PR[12]) equations with multiplicative involutions were considered. These results can be applied to equations with an involutive transformation of argument, in particular, to equations with transformed argument by means of a function of Carleman type.

Key words and phrases: Riemann–Hilbert problem, analytic function, polyanalytic function, Cauchy singular integral, algebra with unit, Leibniz condition, Carleman function, transformed argument, conjugation, involution, multiplicative involution, logarithmic mapping, antilogarithmic mapping.

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Riemann–Hilbert type problems with an additional multiplicative involution in commutative Leibniz algebras with logarithms are examined in PR[13]. Results obtained there can be applied not only to problems with a transformation of argument but also to problems with the conjugation (in the complex sense).

In the present paper there are considered similar problems in several variables with Riemann–Hilbert condition posed on each variable separately. For instance, these problems correspond in the classical case to problems for polyanalytic functions on polydiscs (cf. HD[1], Ms[1]).

In order to find solutions to a generalized Riemann–Hilbert problem, we consider algebras with logarithms induced by a linear operator D and with an involution S. In commutative algebras solutions will be obtained in a similar manner as in the classical case (cf. Example 2.1). In noncommutative algebras we shall need some additional assumptions and some modifications of considerations used in the commutative case. We should point out that logarithmic and antilogarithmic mappings are **non**-linear.

The next step is to consider such problems with an additional involution which, by assumption, is multiplicative. This involution may correspond in the classical case to an involutive transformation of argument and/or to the conjugation in the complex sense: $x \to \overline{x}$ (cf. PR[13], Examples 2.2, 2.3).

In order to consider multidimensional problems, we shall generalize problems mentioned above to a Cartesian product of a (finite) number of algebras with logarithms and involutions.

1. Preliminaries

Let X be a linear space over a field $\mathbb{F}$ of scalars of the characteristic zero. Recall that $L(X)$ is the set of all linear operators with domains and ranges in X and $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$.

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \text{dom } D$ implies $xy, yx \in \text{dom } D$, then we shall write $D \in \mathbf{A}(X)$. The set of all *commutative* algebras belonging to $\mathbf{A}(X)$ will be denoted by $\mathsf{A}(X)$. If $D \in \mathbf{A}(X)$ then

$$f_D(x,y) = D(xy) - c_D[xDy + (Dx)y] \qquad \text{for} \quad x, y \in \text{dom } D,$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component.* Non-Leibniz components have been introduced for right invertible operators $D \in \mathsf{A}(X)$ (cf. PR[6], Section 6.1). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x,y) \qquad \text{for} \quad x, y \in \text{dom } D.$$

If $D \in \mathsf{A}(X)$ is right invertible then the algebra X is said to be a *D-algebra.*

We shall consider in $\mathbf{A}(X)$ the following sets:

• the set of all *multiplicative* mappings (not necessarily linear) with domains and ranges in X: $M(X) = \{A : \text{dom } A \subset X,\ A(xy) = A(x)A(y)$ for $x, y \in \text{dom } A\}$;

• the set $I(X)$ of all invertible elements belonging to X;

• the set $R(X)$ of all right invertible operators belonging to $L(X)$;

• the set $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$ of all right inverses to a $D \in R(X)$;

• the set $\mathcal{F}_D = \{F \in L_0(X) : F^2 = F, \quad FX = \ker D$ and $\exists_{R \in \mathcal{R}_D} FR = 0\}$ of all *initial* operators for a $D \in R(X)$;

• the set $\mathcal{I}(X)$ of all invertible operators belonging to $L(X)$.

Clearly, if $\ker D \neq \{0\}$ then the operator D is right invertible, but not invertible. Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$. Elements of the kernel of a $D \in R(X)$ are said to be *constants.* If $D \in \mathcal{I}(X)$ then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$. We also have dom $D = RX \oplus \ker D$ independently of the choice of an $\mathcal{R}_D$ (cf. PR[6]).

It is well-known that F is an initial operator for a $D \in R(X)$ if and only if there is an $R \in \mathcal{R}_D$ such that $F = I - RD$ on dom D. Moreover, if F' is any projection onto $\ker D$ then F' is an initial operator for D corresponding to the right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$ (cf. PR[6]).

Suppose that $D \in \mathbf{A}(X)$. Let $\Omega_r, \Omega_l : \text{dom } D \longrightarrow 2^{\text{dom } D}$ be multifunctions defined as follows:

$$\Omega_r u = \{x \in \text{dom } D : Du = uDx\}, \qquad \Omega_l u = \{x \in \text{dom } D : Du = (Dx)u\} \tag{1.1}$$

for $u \in \text{dom } D$. The equations

$$Du = uDx \quad \text{for } (u,x) \in \text{graph } \Omega_r\ , \qquad Du = (Dx)u \quad \text{for } (u,x) \in \text{graph } \Omega_l \tag{1.2}$$

are said to be the *right* and *left basic equations*, respectively. Clearly,

$$\Omega_r^{-1} x = \{u \in \text{dom } D : Du = uDx\}, \quad \Omega_l^{-1} x = \{u \in \text{dom } D : Du = (Dx)u\}$$

for $x \in$ dom D. The multifunctions Ω_r, Ω_l are well-defined and dom $\Omega_r \cap$ dom $\Omega_l \supset \ker D$.

Suppose that $(u_r, x_r) \in$ graph Ω_l, $(u_l, x_l) \in$ graph Ω_r, L_r, L_l are selectors of Ω_r, Ω_l, respectively, and E_r, E_l are selectors of Ω_r^{-1}, Ω_l^{-1}, respectively. By definitions, $L_r u_r \in$ dom Ω_r^{-1}, $E_r x_r \in$ dom Ω_r, $L_l u_l \in$

dom Ω_l^{-1}, $E_l x_l \in$ dom Ω_l and the following equations are satisfied:

$$Du_r = u_r D L_r u_r, \qquad DE_r x_r = (E_r x_r) D x_r;$$
$$Du_l = (DL_l u_l) u_l, \quad DE_l x_l = (Dx_l)(E_l x_l).$$

Any invertible selector L_r of Ω_r is said to be a *right logarithmic mapping* and its inverse $E_r = L_r^{-1}$ is said to be a *right antilogarithmic mapping.* If $(u_r, x_r) \in$ graph Ω_r and L_r is an invertible selector of Ω_r then the element $L_r u_r$ is said to be a *right logarithm* of u_r and $E_r x_r$ is said to be a *right antilogarithm* of x_r. By $G[\Omega_r]$ we denote the set of all pairs (L_r, E_r), where L_r is an invertible selector of Ω_r and $E_r = L_r^{-1}$. Respectively, any invertible selector L_l of Ω_l is said to be a *left logarithmic mapping* and its inverse $E_l = L_l^{-1}$ is said to be a *left antilogarithmic mapping.* If $(u_l, x_l) \in$ graph Ω_l and L_l is an invertible selector of Ω_l then the element $L_l u$ is said to be a *left logarithm* of u_l and $E_l x_l$ is said to be a *left antilogarithm* of x_l. By $G[\Omega_l]$ we denote the set of all pairs (L_l, E_l), where L_l is an invertible selector of Ω_l and $E_l = L_l^{-1}$.

If $D \in \mathsf{A}(X)$ then $\Omega_r = \Omega_l$ and we write $\Omega_r = \Omega$ and $L_r = L_l = L$, $E_r = E_l = E$, $(L, E) \in G[\Omega]$. Selectors L, E of Ω and Ω^{-1} are said to be *logarithmic* and *antilogarithmic* mappings, respectively. For any $(u, x) \in G[\Omega]$ elements Lu, Ex are said to be *logarithm* of u and *antilogarithm* of x, respectively (cf. PR[8]).

Clearly, by definition, for all $(L_r, E_r) \in G[\Omega_r]$, $(u_r, x_r) \subset$ graph Ω_r, $(L_l, E_l) \in G[\Omega_l]$, $(u_l, x_l) \in$ graph Ω_l we have

$$(1.3) \qquad E_r L_r u_r = u_r, \; L_r E_r x_r = x_r; \quad E_l L_l u_l = u_l, \; L_l E_l x_l = x_l;$$

$$(1.4) \qquad DE_r x_r = (E_r x_r) D x_r, \quad Du_r = u_r D L_r u_r;$$
$$DE_l x_l = (Dx_l)(E_l x_l)\,, \quad Du_l = (DL_l u_l) u_l.$$

A right (left) logarithm of zero is not defined. If $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$ then $L_r(\ker D \setminus \{0\}) \subset \ker D$, $E_r(\ker D) \subset \ker D$, $L_l(\ker D \setminus \{0\}) \subset \ker D$, $E_l(\ker D) \subset \ker D$. In particular, $E_r(0), E_l(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant.

If $D \in A(X)$ and if D satisfies the *Leibniz condition:* $D(xy) = xDy + (Dx)y$ for $x, y \in$ dom D then X is said to be a *Leibniz algebra.*

Let $D \in \mathsf{A}(X)$ and let $(L, E) \in G[\Omega]$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in$ dom Ω. If L is of the exponential type then $E(x+y) = (Ex)(Ey)$ for $x, y \in$ dom Ω. We have proved that a logarithmic mapping L is of the exponential type if and only if X is a *Leibniz commutative algebra* (cf. PR[8])). In Leibniz commutative algebras with $D \in R(X)$ a necessary and sufficient conditions for $u \in$ dom Ω is that $u \in I(X)$ (cf. PR[8]).

By $\mathbf{Lg_r}(D)$, $\mathbf{Lg_l}(D)$, $\mathbf{Lg}(D)$ we denote the classes of these algebras with $D \in R(X)$ and with unit $e \in \text{dom}\,\Omega$ for which there exist invertible selectors of Ω_r, Ω_l, Ω, respectively, i.e. there exist $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G(\Omega_l)$, respectively.

In the sequel we shall consider multidimensional Leibniz algebras, i.e. a Cartesian product of finite number of Leibniz algebras with logarithms.

Suppose then that we are given n commutative algebras X_j (over the field $\mathbb{F}$) with $D_j \in R(X_j)$ and with units $e_j \in X_j$ and multifunctions Ω_j $(j = 1, \dots, n)$. We assume that $X_j \in L_j(D_j)$, $(L_j, E_j) \in G[\Omega_j]$ and $e_j \in \text{dom}\,\Omega_j$ $(j = 1, \dots, n)$. Consider the Cartesian product

$$X = X_1 \times \ldots \times X_n$$

with the coordinatewise addition and multiplications of elements, multiplication of elements by scalars and coordinatewise operations of all mappings acting on X_j, i.e. for all $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $t = (t_1, \dots, t_n) \in \mathbb{F}^n$, $T_j \in L(X_j)$ $(j = 1, \dots, n)$ we have

$$x + y = (x_1 + y_1, \dots, x_n + y_n), \quad xy = (x_1 y_1, \dots, x_n y_n), \tag{1.5}$$

$$tx = (t_1 x_1, \dots, t_n x_n), \tag{1.6}$$

$$Tx = (T_1 x_1, \dots, T_n x_n) \quad \text{whenever } x_j \in \text{dom}\,T_j.$$

Clearly, X has the unit e. Namely, $e = (e_1, \dots, e_n)$.

Consequently, we shall write

$$I(X) = \big(I(X_1), \dots, I(X_n)\big), \quad \Omega = (\Omega_1, \dots, \Omega_n), \tag{1.7}$$

$$(L, E) = \big((L_1, E_1, \dots, (L_n, E_n)\big), \quad G[\Omega] = \big(G[\Omega_1], \dots, G[\Omega_n]\big),$$

$$\mathbf{L}(D) = \big(\mathbf{L}_1(D_1), \dots, \mathbf{L}_n(D_n)\big), \quad \mathsf{A}(X) = \big(\mathsf{A}(X_1), \dots, \mathsf{A}(X_n)\big).$$

Similar denotations are admitted in non-commutative cases. Namely,

$$(L_r, E_r) = \big((L_{r1}, E_{r1}, \dots, (L_{rn}, E_{rn})\big), \; G[\Omega_r] = \big(G[\Omega_{r1}], \dots, G[\Omega_{rn}]\big), \tag{1.8}$$

$$\mathbf{L_r}(D) = \big(\mathbf{L_{r1}}(D_1), \dots, \mathbf{L_{rn}}(D_n)\big), \quad \mathbf{A}(X) = \big(\mathbf{A}(X_1), \dots, \mathbf{A}(X_n)\big).$$

$$(L_l, E_l) = \big((L_{l1}, E_{l1}, \dots, (L_{ln}, E_{ln})\big), \quad G[\Omega_l] = \big(G[\Omega_{l1}], \dots, G[\Omega_{ln}]\big),$$

$$\mathbf{L_l}(D) = \big(\mathbf{L_{l1}}(D_1), \dots, \mathbf{L_{ln}}(D_n)\big).$$

If $D_j \in \mathbf{A}(X_j)$ $(j = 1, \dots, n)$ satisfy the Leibniz condition then $D \in \mathbf{A}(X)$ also satisfy that condition and X is said to be a *Leibniz algebra* (of dimension n).

If in order to prove some statement it is enough to prove it for an arbitrarily fixed j $(j = 1, \dots, n)$, we shall omit that proof and we refer either to PR[8] (Chapter 14) or to PR[13], respectively, i.e. to one-dimensional case: $n = 1$.

We shall denote the identity operators in all spaces $X_1, \ldots, X_n$ by the same letter I, since it does not lead here to any misunderstanding.

2. Commutative case

Here and in the sequel we shall consider an n-dimensional commutative Leibniz algebra $X \in \mathbf{L}(D)$ (cf. (1.5), (1.6), (1.7)).

THEOREM 2.1 (cf. PR[3] for $n = 1$). *Let $n \in N$ and let $j = 1, \ldots, n$. Let X_j be linear spaces over an algebraically closed field $\mathbb{F}$ with involutions $S_j \in L_0(X_j)$ (i.e. $S_j^2 = I$, $S_j \neq I$). Then for an arbitrarily fixed $1 \leq j \leq n$ $P_j^+ = \frac{1}{2}(I + S_j)$ and $P_j^- = \frac{1}{2}(I - S_j)$ are disjoint projectors giving partition of unity: $(P^\pm)_j^2 = P_j^\pm$; $P_j^+ P_j^- = P_j^- P_j^+ = 0$ and $P_j^+ + P_j^- = I$. Moreover, $P_j^+ - P_j^- = S_j$ and $S_j P_j^\pm = P_j^\pm S_j = \pm P_j^\pm$. So that, if we write $X_j^\pm = P_j^\pm X_j$, $x_j^\pm = P_j^\pm x_j$ for $x_j \in X_j$, then we have*

$$X_j = X_j^+ \oplus X_j^-; \quad x_j^\pm \in X_j^\pm; \quad S_j x_j^+ = x_j^+;\ S_j x_j^- = -x_j^- \quad \textit{for } x_j \in X_j.$$

COROLLARY 2.1. *Let all assumptions of Theorem 2.1 be satisfied. Let $1 \leq j \leq n$ be arbitrarily fixed. If $u_j \in X_j$, $u_j^\pm = P_j^\pm u_j$ and*

$$u_j^+ - u_j^- = v_j, \qquad \textit{where } v_j \in X_j, \tag{2.1}$$

then $u_j = S v_j$. In particular, if $v_j = 0$ then $u_j = 0$.

LEMMA 2.1 (cf. PR[8] for $n = 1$). *Let $n \in \mathbb{N}$. Let $D \in \mathsf{A}(X)$, where $X \in \mathbf{Lg}(D)$ is an n-dimensional Leibniz algebra with unit e and with an involution $S = (S_1, \ldots, S_n) \in L_0(X)$. Let $(L, E) \in G[\Omega]$. Then the following conditions are equivalent:*

(i) $Lu^\pm \in X^\pm$ *for all* $u \in \operatorname{dom} \Omega$;
(ii) $Ev^\pm \in X^\pm$ *for all* $v \in \operatorname{dom} \Omega^{-1}$;
(iii) $P^\pm L u^\pm = \pm L u^\pm$ *for all* $u \in \operatorname{dom} \Omega$;
(iv) $P^\pm E v^\pm = \pm E v^\pm$ *for all* $v \in \operatorname{dom} \Omega^{-1}$.

Let $n \in \mathbb{N}$ be arbitrarily fixed. We admit the following condition:

$(\mathsf{A})_n$ Let $\mathbb{F}$ be algebraically closed. Let $D \in \mathsf{A}(X)$, where $X \in \mathbf{Lg}(D)$ is an n-dimensional Leibniz algebra with unit e and with an involution $S \in L_0(X)$, $(L, E) \in G[\Omega]$ and

$$L(X^+ \cap \operatorname{dom} \Omega) \subset X^+, \quad L(X^- \cap \operatorname{dom} \Omega) \subset X^-. \tag{2.2}$$

In particular, (2.2) implies that $X^\pm \cap \operatorname{dom} \Omega \subset \operatorname{dom} \Omega^{-1}$.

LEMMA 2.2 (cf. PR[8] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied. Then*

(i) $uv \in X^\pm$ *whenever* $u, v \in X^\pm \cap \operatorname{dom} \Omega$;
(ii) $(u^\pm)^{-1} \in I(X^\pm)$ *whenever* $u^\pm \in I(X^\pm)$.

On the other hand, we have

PROPOSITION 2.1 (cf. PR[8] for $n = 1$). *Suppose that X is a commutative algebra over a field $\mathbb{F}$ with a multiplicative involution $S \in L_0(X$, i.e. $S(xy) = (Sx)(Sy)$ for $x, y \in X$. Then $x^\pm y^\pm \in X^+$, $x^+y^-, x^-y^+ \in X^-$ for all $x, y \in X$.*

PROPOSITION 2.2 (cf. PR[8] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied. Then the involution S is not multiplicative.*

Proposition 2.2 is important, since there are several examples of functional-differential equations with a multiplicative involution which can be also solved by means of algebraic methods (cf. PR[4], PR[5], PR[6]). Namely, any transformation of argument is a multiplicative operation (cf. also PR[8], PR[9], PR[12], PR[13]). However, by Proposition 2.2, any involution under consideration cannot be multiplicative if Condition $(\mathsf{A})_n$ is assumed.

Homogeneous n-dimensional Riemann–Hilbert problem. Suppose that Condition $(\mathsf{A})_n$ is satisfied. Find an $x_0 = (x_{01}, \ldots, x_{0n}) \in \operatorname{dom} \Omega$ such that

$$x_0^+ = a x_0^-, \quad \text{where } a \in \operatorname{dom} \Omega \text{ is given.} \tag{2.3}$$

THEOREM 2.2 (cf. PR[8] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied. If $a \neq \pm e$ then the homogeneous Riemann–Hilbert problem (2.3) has a solution*

$$x_0 = E(P^+La) + E(-P^-La) \quad \text{and} \quad x_0^\pm = E(\pm P^\pm La) \in I(X). \tag{2.4}$$

If $a = \pm e$ then the only solution of (2.3) is $x_0 = 0$.

Clearly, the solution (2.4) is dependent on the choice of selectors L.

COROLLARY 2.2 (cf. PR[8] for $n = 1$). *If all assumptions of Theorem 2.2 are satisfied then the solution to the problem (2.3) can be written in the form: $x_0 = (a + e)E(-P^-La)$.*

Consider now the set of all elements from $Y \subset X$ having *k-th roots*:

$$I_k(Y) = \{x \in Y : \exists_{y \in I(Y)} \quad y^k = x\} \qquad (k \in \mathbb{N}).$$

Here $n \in \mathbb{N}$ is fixed (for $n = 1$ cf. PR[8]). If $x \in I_k(Y)$ and $y^k = x$ then we write $y = x^{1/k}$ $(k \in \mathbb{N})$. By definition, $x \in I(Y)$.

COROLLARY 2.3 (cf. PR[8] for $n = 1$). *Suppose that all assumptions of Theorem 2.2 are satisfied and $a \in I_2(\operatorname{dom} \Omega)$. Then the solution to the problem (2.3) can be written in the form: $x_0 = (a^{\frac{1}{2}} + a^{-\frac{1}{2}})[E(SLa)]^{\frac{1}{2}}$.*

Nonhomogeneous n-dimensional Riemann–Hilbert problem. Suppose that Condition $(\mathsf{A})_n$ is satisfied. Find an $x \in \operatorname{dom} \Omega$ such that

$$x^+ = ax^- + b, \quad \text{where } a \in \operatorname{dom} \Omega,\ b \in X \text{ are given.} \tag{2.5}$$

THEOREM 2.3 (cf. PR[8] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied. If $a \neq \pm e$ and x_0 is a solution of the homogeneous Riemann–Hilbert problem (2.3) then the nonhomogeneous Riemann–Hilbert problem (2.5) has a solution*

$$x = x_0 + \frac{1}{2}(x_0 S y_0 + y_0 S x_0), \tag{2.6}$$
$$\text{where } y_0 = a^{-1} b E(P^- La) = a^{-1} b (P^- x_0)^{-1}.$$

If $a = e$ then a solution to (2.5) is $x = Sb$. If $a = -e$ then a solution to (2.5) is $x = b$.

COROLLARY 2.4 (cf. PR[8] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied. If $a \neq \pm e$ and x_1, x_2 are two solutions to the nonhomogeneous Riemann–Hilbert problem (2.5) then their difference $\tilde{x} = x_1 - x_2$ is a solution to the homogeneous problem (2.3). If $a = e$ then the problem (2.5) has a unique solution $x = Sb$. If $a = -e$ then (2.5) has a unique solution $x = b$.*

COROLLARY 2.5 (cf. PR[13] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied and $a \neq \pm e$, x_0 is a solution of the homogeneous Riemann–Hilbert problem (2.3) then the nonhomogeneous Riemann–Hilbert problem (2.5) has solutions of the form*

$$x = x_0 + S(x_0 y_0), \quad \text{where } y_0 = a^{-1} b E(P^- La) = a^{-1} b (P^- x_0)^{-1}. \tag{2.7}$$

If $a = e$ then the problem (2.5) has a unique solution $x = Sb$. If $a = -e$ then (2.5) has a unique solution $x = b$.

EXAMPLE 2.1. Let $n \in N$. Let $j = 1, \ldots, n$. Let $\mathbf{\Omega_j} \subset \mathbb{C}$ be domains with the boundaries $\partial \mathbf{\Omega}_j = \Gamma_j$ which are pairwise disjoint (for $n > 1$) closed regular arcs, i.e $\Gamma_j = \{z = z_j(t) : \alpha_j \leq t \leq \beta_j,\ z_j(\alpha_j) = z_j(\beta_j)\}$, where the functions $z_j \in C^1(\alpha_j, \beta_j)$, are one-to-one, $z_j'(t) \neq 0$ for $t \in (\alpha_j, \beta_j)$ and $\lim_{t \to \alpha_j + 0} z_j'(t) = \lim_{t \to \beta_j - 0} z_j'(t) \neq 0$.

We assume that the system $\Gamma = (\Gamma_1, \ldots, \Gamma_2)$ is *oriented* (cf. for instance, PR[5]). It means that the plane is divided into components $\mathbf{\Omega_0}$, $\mathbf{\Omega_1}, \ldots, \mathbf{\Omega_n}$ and we associate the sign "-" with the component $\mathbf{\Omega_0}$ containing the point ∞, and the sign "+" with the components having a common boundary with $\mathbf{\Omega_0}$. Next, we associate the sign "-" with the components having a common boundary with components having the sign "+", but not with $\mathbf{\Omega_0}$, and so on. Hence on the left of each of these arcs lies a domain with the sign "+", and on the right, a domain with the sign "-". If $n = 1$ then $\mathbf{\Omega_0} = \mathbb{C} \setminus \mathbf{\Omega_1}$ (cf. PR[8], Chapter 14; PR[13]).

We have to find a vector function $\Phi = (\Phi_1, \ldots, \Phi_n)$ with Φ_j piecewise analytic in the domains $\boldsymbol{\Omega}_j^+ = \boldsymbol{\Omega}_j$ and $\boldsymbol{\Omega}_j^- = \mathbb{C}\backslash\overline{\boldsymbol{\Omega}_j}$, bounded at infinity and such that their boundary values Φ^+ and Φ^- satisfy the following condition on the oriented system Γ:

$$\Phi_j^+(t) = G_j(t)\Phi_j^-(t) + g_j(t) \quad \text{for } t \in \Gamma_j, \quad (j = 1, \ldots, n) \tag{2.8}$$

where functions g_j, G_j are given. The solution of the problem is well-known (for $n = 1$ cf. for instance, Michlin Mi[1], Pogorzelski P[1], Meister Me[1], Wegert Wg[1],[2], Wendland Wn[1]; for $n > 1$, cf. for instance, BD[1], Ms[1]). In order to solve (2.8), we have to use properties of logarithmic and exponential functions (cf. Anosov and Bolibruch AB[1]) and of singular integral operators S_j $(j = 1, \ldots, n)$ defined by the *Cauchy principal value* of an integral, namely,

$$(S_j\varphi)(t_j) = \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int\limits_{\Gamma_j \backslash \{z \in \mathbb{C}\,:\, |z - t_j| < \varepsilon\}} \frac{\varphi(\tau)}{\tau - t_j} d\tau \overset{\text{def}}{=} \tag{2.9}$$

$$\overset{\text{def}}{=} \frac{1}{\pi i} \int\limits_{\Gamma_j} \frac{\varphi(\tau)}{\tau - t_j} d\tau \quad (t_j \in \Gamma_j) \quad (j = 1, \ldots, n).$$

In the case, when g_j, G_j belong to the space $H^\mu(\Gamma_j)$ of functions satisfying the Hölder condition on Γ_j with an exponent μ, $0 < \mu < 1$, $(j = 1, \ldots, n)$, the singular integral operators S_j defined by (2.9) are involutions in the space $X_j = H^\mu(\Gamma_j)$: $S_j^2 = I$ on X_j. Thus there are disjoint projectors P_j^+ and P_j^- giving the partition of unit and such that $\Phi_j^+ = P_j^+ x_j$, $\Phi_j^- = P_j^- x_j$ for an $x_j \in X_j$ $(j = 1, \ldots, n)$. Clearly, X_j are also commutative algebras over $\mathbb{C}$ with the pointwise multiplication. It not difficult to verify that the operators S_j defined by (2.9), the operators $D_j = \frac{\mathrm{d}}{\mathrm{d}t_j}$ and the usual logarithmic functions satisfy Condition $(\mathsf{A})_n$ with $S = (S_1, \ldots, S_n)$ and $D = (D_1, \ldots, D_n)$. Thus $x = \Phi$ is to be found if we apply logarithms to the homogeneous problem (2.8) (i.e. with $g_j = 0$ for $j = 1, \ldots, n$).

Then the non-homogeneous problem is solved by a use of the already found solution to the homogeneous problem (cf. Formula (2.7)).

This problem can be also formulated and solved in the same manner if some of Γ_j are oriented systems of finite sets of closed regular arcs. ∎

NOTE 2.1. If X is a commutative Leibniz algebra for a $D \in L(X)$ then X is a Leibniz algebra for a $D' = dD$, where $d \in X \setminus \{0\}$. Indeed, for all $x, y \in \operatorname{dom} D' = \operatorname{dom} D$ we have

$$D'(xy) = dD(xy) = d(xDy + yDx) = xdDy + ydDx = xD'y + yD'x.$$

In particular, if $D \in R(X)$, $R \in \mathcal{R}_D$ and $g = Re \in I(X)$, then X is a Leibniz algebra for $D_n = g^n D$ $(n \in \mathbb{N})$. If $n = 1$, then antilogarithms induced by

$D' = gD$ are $E'(\lambda g) = g^\lambda = E(\lambda Lg)$. Observe that in the classical case of the operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ and $R = \int_0^t$ in $C[0,T]$ we have $D_n = t^n \frac{\mathrm{d}}{\mathrm{d}t}$. The corresponding logarithms are (up to an additive constant) $Lx = \ln x$, as in the case of the operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ and antilogarithms are ct^λ ($c \in \mathbb{R}$). ∎

Multidimensional Riemann–Hilbert problem with multiplicative involution. Suppose that Condition $(\mathsf{A})_n$ is satisfied, $T_j \in L_0(X)$ are multiplicative involutions with projectors $Q_j^\pm$ giving the partition of unity ($j = 1, \dots, n$), $T = (T_1, \dots, T_n)$ and $a(t) = a_0 + a_1 t$, where $a_0, a_1, b \in \operatorname{dom} \Omega$ are given. By definition, T is also a multiplicative involution. Find an $x \in \operatorname{dom} \Omega$ such that

$$x^+ = a(T)x + b. \tag{2.10}$$

We shall use the following theorem, different than results obtained before for equations with involutions (cf. PR[3], PR[5], PR[8], PR[12]).

THEOREM 2.4 (cf. PR[13] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied and X is a commutative algebra over $\mathbb{F}$ with unit e, $T \in L_0(X)$ is a multiplicative involution with projectors $Q^\pm$ giving the partition of unity, $a(t) = a_0 + a_1 t$, where $a_0, a_1 \in X$ are given and either $Ta_k = a_k$ or $Ta_k = a_k T$, $k = (0,1)$.*

(i) *If $a(\pm 1) = a_0 \pm a_1 \in I(X)$ then the equation*

$$a(T)x = y, \qquad y \in X \tag{2.11}$$

has a unique solution

$$x = (a_0^2 - a_1^2)^{-1} a(-T) y. \tag{2.12}$$

(ii) *If $a_0 = \pm a_1$ and $a(\pm 1) = a_0 \pm a_1 \in I(X)$, then $a(T) = a(\pm 1) Q^\pm$ and all solutions of Equation* (2.12) *are of the form*

$$x = [a(\pm 1)]^{-1} y + x^\mp, \quad \textit{where } x^\mp \in X^\mp \textit{ are arbitrary.} \tag{2.13}$$

COROLLARY 2.6 (cf. PR[13] for $n = 1$). *Theorem* 2.4 *holds without the assumption that T is multiplicative if, in particular, a_0, a_1 are scalar multiples of unit e.*

NOTE 2.2. Suppose that the involution T appearing in Theorem 2.4 satisfies the condition: $Ta_k - a_k T \in J$ ($k = 0, 1$), where J is a proper two-sided ideal in the algebra X. Then we may apply Theorem 2.4 to the quotient algebra X/J and we obtain similar results up to an additive component belonging to J (cf. PR[5], PR[13] for $n = 1$). ∎

THEOREM 2.5 (cf. PR[13] for $n = 1$). *Suppose that Condition $(\mathsf{A})_n$ is satisfied and $a \neq \pm e$, $T_j \in L_0(X)$ are multiplicative involutions with projectors $Q_j^\pm$ giving the partition of unity ($j = 1, \dots, n$), $T = (T_1, \dots, T_n)$, $a(t) = a_0 + a_1 t$,* *where* $a_0, a_1 \in \operatorname{dom} \Omega$ *are given and either* $Ta_k = a_k$ *or* $Ta_k = a_k T$ $(k = 0, 1)$. *Let*

$$x_0 = E(P^+ L a_0) + E(-P^- L a_0), \tag{2.14}$$

$$a_0' = e,\ a_1' = a_1(e + a_0)^{-1}),\ a'(t) = a(t) - eI - a_1' t,$$

$$b' = (e + a_0^{-1})b. \tag{2.15}$$

If $a'(t)$ *satisfies all assumptions of Theorem* 2.4, *then the problem* (2.10) *has solutions of the form:*

$$x = a'(T)x^- + b' + [(a_0 - e)^2 - (a_1 - a_1')^2]^{-2} a'(-T)(b' - b). \tag{2.16}$$

EXAMPLE 2.2 (cf. PR[13] for $n = 1$). Let $\Gamma_j = \{z : |z| = r_j\}$, $(r_j > 0)$ $(j = 1, \dots, n)$. Suppose that X and S are defined as in Example 2.1. Consider the following problem: Find a vector function $\Phi = (\Phi_1, \dots, \Phi_n)$ with Φ_j piecewise analytic in the domains $\boldsymbol{\Omega}_j^+ = \boldsymbol{\Omega}_j$ and $\boldsymbol{\Omega}_j^- = \mathbb{C} \setminus \overline{\boldsymbol{\Omega}_j}$, bounded at infinity and such that their boundary values Φ_j^+ and Φ_j^- satisfy the following condition:

$$\Phi_j^+(t) = G_{j_0}(t)\Phi_j^-(t) + G_{j_1}(t)\Phi_j^+(h_j(t)) + G_{j_2}(t)\Phi_j^-(h_j(t)) + g_j(t) \quad \text{for } t \in \Gamma_j,\ (j = 1, \dots, n), \tag{2.17}$$

where functions g_j, h_j, G_{kj} $(k = 0, 1, 2;\ j = 1, \dots, n)$ are given, $h_j(\Gamma_j) \subset \Gamma_j$, $h_j(h_j(t)) \equiv t$, $h_j'(t) \neq 0$ for $t \in \Gamma_j$, $G_k(h_j(t)) = G_k(t)$ $(k = 0, 1, 2; j = 1, \dots, n)$. Write: $(Tx)(t) = (x_1(h_1(t), \dots, x_n(h_n(t)))$ for $x \in X$, $t \in \Gamma_j$ $(j = 1, \dots, n$. Clearly, T is a multiplicative involution and $TG_k = G_k$ $(k = 0, 1, 2)$. Thus we can apply Theorem 2.5 in order to solve the problem (2.17).

In particular, if for a j we have $\Gamma_j = \mathbb{R}$ (i.e. $r_j = \infty$), $h_j(t) = -t$ for $t \in \Gamma_j$, then the functions G_{j_k} are even. Indeed, for $k = 0, 1, 2$ we have $G_{j_k}(t) = (TG_{j_k})(t) = G_{j_k}(-t)$. ∎

EXAMPLE 2.3 (cf. PR[13] for $n = 1$). Let $j = 1, \dots, n$. Let $\overline{\Gamma_j} = \Gamma_j$, i.e. $\overline{z} \in \Gamma_j$ whenever $z \in \Gamma_j$. Suppose that X and S are defined as in Example 2.1. Consider the following problem: Find a vector function $\Phi = (\Phi_i, \dots, \Phi_n)$ with $\Phi_1, \dots, \Phi_n$ piecewise analytic in the domains $\boldsymbol{\Omega}_{\mathrm{j}}{}^+ = \boldsymbol{\Omega}_j$ and $\boldsymbol{\Omega}_j^- = \mathbb{C} \setminus \overline{\boldsymbol{\Omega}_j}$, bounded at infinity and such that their boundary values Φ_j^+ and Φ_j^- satisfy the following condition:

$$\Phi_j^+(t) = G_{j_0}(t)\Phi_j^-(t) + G_{j_1}(t)\overline{\Phi_j^+(t)} + G_{j_2}(t)\overline{\Phi_j^-(t)} + g_j(t) \quad \text{for } t \in \Gamma_j, \quad (j = 1, \dots, n), \tag{2.18}$$

where functions g_j, G_{j_k} $(k = 0, 1, 2)$ are given, $\overline{G_{j_k}} = G_{j_k}$ $(k = 0, 1, 2)$. Clearly, the functions G_{j_k} are real. Write: $(Tx) = \overline{x} = (\overline{x_1}, \dots, \overline{x_n})$ for $x \in X$. Clearly, T is a multiplicative involution and $TG_{j_k} = \overline{G_{j_k}} = G_{j_k}$

($k = 0, 1, 2$). Thus we can apply Theorem 2.5 in order to solve the problem (2.18). ∎

3. Noncommutative case

In this section instead of Condition $(\mathsf{A})_n$ we shall admit either the condition

$(\mathbf{A})^l_n$, where $n \in \mathbb{N}$ (cf. PR[8], PR[13] for $n = 1$). Let $\mathbb{F}$ be algebraically closed. Let $D \in \mathbf{A}(X)$. Suppose that $X \in \mathbf{Lg_l}(D)$ is a Leibniz algebra with unit e and with an involution $S \in L_0(X)$, $uv \in X^\pm$ whenever $u, v \in X^\pm$ and $DS = SD$ on dom D, $(L_l, E_l) \in G[\Omega_l]$ and

$$L_l(X^+ \in \text{dom } \Omega_l) \subset X^+, \quad L_l(X^- \cap \text{dom } \Omega_l) \subset X^-. \tag{3.1}$$

or the condition

$(\mathbf{A})^r_n$, where $n \in N$ (cf. PR[8], PR[13] for $n = 1$). Let $\mathbb{F}$ be algebraically closed. Let $D \in \mathbf{A}(X)$. Suppose that $X \in \mathbf{Lg_r}(D)$ is a Leibniz algebra with unit e and with an involution $S \in L_0(X)$, $uv \in X^\pm$ whenever $u, v \in X^\pm$ and $DS = SD$ on dom D, $(L_r, E_r) \in G[\Omega_r]$ and

$$L_r(X^+ \cap \text{dom } \Omega_r) \subset X^+, \quad L_r(X^- \cap \text{dom } \Omega_r) \subset X^-. \tag{3.2}$$

In particular, (3.1) implies that $X^\pm \cap \text{dom } \Omega_l \subset \text{dom } \Omega_l^{-1}$, (3.2) implies that $X^\pm \cap \text{dom } \Omega_r \subset \text{dom } \Omega_r^{-1}$.

NOTE 3.1 (cf. PR[13] for $n = 1$). Let either Condition $(\mathbf{A})^l_n$ or Condition $(\mathbf{A})^r_n$ be satisfied. The assumed condition that $uv \in X^\pm$ whenever $u, v \in X^\pm$ in the commutative case is proved by Lemma 2.2. It is so, because in that case logarithms are of the exponential type. Without this property, we cannot prove a lemma corresponding to Lemma 2.2.

We should point out also that here, in the noncommutative algebras, we shall need essentially the property that D and S commute each with another. Thus Condition $(\mathsf{A})_n$ is not a particular case of Conditions $(\mathbf{A})^l_n$ and $(\mathbf{A})^r_n$. ∎

LEMMA 3.1 (cf. PR[8], PR[13] for $n = 1$). *Let $D \in \mathbf{A}(X)$. Suppose that $X \in \mathbf{Lg_l}(D)$ is a Leibniz algebra with unit e and with an involution $S \in L_0(X)$ and $(L_l, E_l) \in G[\Omega_l]$ ($X \in \mathbf{Lg_r}(D)$ and $(L_r, E_r) \in G[\Omega_r]$, respectively). Then the following conditions are equivalent for $u_l \in \text{dom } \Omega_l$, $u_r \in \text{dom } \Omega_r$, $v_l \in \text{dom } \Omega_l^{-1}$, $v_r \in \text{dom } \Omega_r^{-1}$:*

(i) *$L_l u_l^\pm \subset X^\pm$ ($L_r u_r^\pm \subset X^\pm$, respectively);*
(ii) *$E_l v_l^\pm \subset X^\pm$ ($E_r v_r^\pm \subset X^\pm$, respectively.);*
(iii) *$P^\pm L_l u_l^\pm = \pm L_l u_l^\pm$ ($P^\pm L_r u_r^\pm = \pm L_r u_r^\pm$, respectively);*
(iv) *$P^\pm E_l v_l^\pm = \pm E_l v_l^\pm$ ($P^\pm E_r v_r^\pm = \pm E_r v_r^\pm$, respectively).*

Homogeneous n-dimensional Riemann–Hilbert problem. Let either Condition $(\mathbf{A})_n^l$ or Condition $(\mathbf{A})_n^r$ be satisfied. Find an $x_0 = (x_{01}, \ldots, x_{0n}) \in X$ such that

$$x_{0j}^+ = a x_{0j}^-, \quad (j = 1, \ldots, n), \tag{3.3}$$

where $a \in I(X) \cap \operatorname{dom} \Omega_l$. ($a \in I(X) \cap \operatorname{dom} \Omega_r$, *respectively*).

THEOREM 3.1 (cf. PR[8], PR[13] for $n = 1$). *Suppose that Condition* $(\mathbf{A})_n^l$ *(Condition* $(\mathbf{A})_n^r$*, respectively) holds. Then the Riemann–Hilbert problem* (3.3) *has a solution if and only if there a* u *such that*

$$u^+ a - a u^- = Da \qquad (a u^+ - u^- a = Da, \quad \textit{respectively}). \tag{3.4}$$

If it is the case and $D \in R(X)$*, then the solution, we are looking for, is*

$$x_0 = E_l P^+(Ru + z) + E_l P^-(Ru + z), \quad \textit{where } z \in \ker D \tag{3.5}$$
$$(x_0 = E_r P^+(Ru + z) + E_r P^-(Ru + z), \quad \textit{respectively}).$$

Moreover, $x_0^\pm = E_l P^\pm(Ru + z) \in I(X)$, $x_0^\pm = E_l P^\pm(Ru + z) \in I(X)$, *respectively.*

THEOREM 3.2 (cf. PR[8], PR[13] for $n = 1$). *Suppose that either Condition* $(\mathbf{A})_n^l$ *or Condition* $(\mathbf{A})_l^r$ *holds. If*

$$a X^- a^{-1} \subset X^- \quad (a^{-1} X^+ a \subset X^+, \textit{ respectively}), \tag{3.6}$$

then the Riemann–Hilbert problem (3.3) *has a solution of the form* (3.5)*, where*

$$u = D L_l P^+ a - a^{-1}(D L_l P^- a) a; \tag{3.7}$$
$$(u = a(D L_r P^+ a) a^{-1} - D L_r P^{-1} a, \textit{ respectively}).$$

COROLLARY 3.1 (cf. PR[8], PR[13] for $n = 1$). *Suppose that all assumptions of Theorem* 3.2 *are satisfied. Then* $P^- a \in I(X) \cap \operatorname{dom} \Omega_r$ $(P^- a \in I(X) \cap \operatorname{dom} \Omega_l$*, respectively) and*

$$u = D L_l P^+ a + a^{-1}[D L_r (P^- a)^{-1}] a$$
$$(u = a(D L_l P^+ a) a^{-1} + D L_r (P^- a)^{-1}, \textit{ respectively}).$$

Nonhomogeneous n-dimensional Riemann–Hilbert problem. Let either Condition $(\mathbf{A})_n^l$ or Condition $(\mathbf{A})_n^r$ be satisfied. Let $a \in I(X) \cap \operatorname{dom} \Omega_l$, ($a \in I(X) \cap \operatorname{dom} \Omega_r$, respectively). Find an $x = (x_1, \ldots, x_n) \in X$ such that

$$x_j^+ = a_j x^- + b_j, \qquad \text{where } b_j \in X, \ (j = 1, \ldots, n). \tag{3.8}$$

THEOREM 3.3 (cf. PR[8], PR[13] for $n = 1$). *Suppose that all assumptions of Theorem* 3.2 *are satisfied,* $a \neq \pm e$ *and* x_0 *is a solution of the homogeneous problem* (3.3)*. Then the nonhomogeneous n-dimensional Riemann–Hilbert* *problem* (3.8) *has a solution of the form*

$$x = x_0 + \frac{1}{2}[x_0 S y_0 + (Sx_0)y_0], \quad y_0 = (x_0^+)^{-1}a^{-1}b. \tag{3.9}$$

If either $a = e$ *or* $a = -e$ *then* (3.8) *has a unique solution, namely* $x = Sb$ *or* $x = b$*, respectively.*

COROLLARY 3.2 (cf. PR[8], PR[13] for $n = 1$. *Suppose that either Condition* $(\mathbf{A})_n^l$ *or Condition* $(\mathbf{A})_n^r$ *is satisfied. If* $a \neq \pm e$ *and* x', x'' *are two solutions to the non-homogeneous n-dimensional Riemann–Hilbert problem* (3.8) *then their difference* $x = x' - x''$ *is a solution to the homogeneous problem* (3.3). *If* $a = e$ *then the problem* (3.8) *has a unique solution* $x = Sb$ *(cf. Corollary* 2.4*). If* $a = -e$ *then* (3.8) *has a unique solution* $x = b$.

COROLLARY 3.3 (cf. PR[8], PR[13] for $n = 1$. *Suppose that either Condition* $(\mathbf{A})_n^l$ *or Condition* $(\mathbf{A})_n^r$ *is satisfied. Then the involution* S *is not multiplicative* (cf. Proposition 2.2).

The already obtained solutions to the Riemann–Hilbert problem can be used in order to solve linear equations with involutions in Leibniz algebras with logarithms. Namely, we have

EXAMPLE 3.1 (cf. PR[13] for $n = 1$). Suppose that either Condition $(\mathsf{A})_n$ or Condition $(\mathbf{A})_n^r$ or Condition $(\mathbf{A})_n^l$ is satisfied. Consider the equation:

$$(aI + bS)x = y, \quad \text{where } a, b, y \in X. \tag{3.10}$$

Since $aI + bS = a(P^+ + P^-) + b(P^+ - P^-) = (a + b)P^+ + (a - b)P^-$, Equation (3.10) can be rewritten as follows

$$(a + b)x^+ + (a - b)x^- = y. \tag{3.11}$$

If $a = \pm b$ and $a \pm b \in I(X)$ then solutions to this equations exist if and only if $(a \pm b)^{-1}y \in X^\pm$. If it is the case, then $x^\pm = (a \pm b)^{-1}y + x^\mp$, where $x^\mp \in X^\mp$ is arbitrary.

Suppose now that $(a + b)^{-1}(a - b) \in I(X) \in \text{dom}\,\Omega$ (dom Ω_r, dom Ω_l, respectively). Then Equation (3.11) can be written as

$$x^+ = (a + b)^{-1}(a - b)x^{-1} + (a + b)^{-1}y,$$

i.e. we have a Riemann–Hilbert problem $x^+ = \tilde{a}x^- + \tilde{y}$ with $\tilde{a} = (a+b)^{-1}(a - b)$ and $\tilde{b} = (a + b)^{-1}y$.

Solutions to Equation (3.10) have been obtained earlier in another way under other assumptions, for instance, that a, b commute (anticommute) with the involution S (cf. PR[3]-PR[5], PR[12], PR[13] for $n = 1$).

Having already solved Equation (3.10), we can solve the equation

$$(aI + bS + cT)x = y, \quad \text{where } a, b, c, y \in X, \tag{3.12}$$

where $T \in L_0(X)$ is a multiplicative involution satisfying all assumptions of Theorem 2.5. ■

This method without any essential change can be applied to problems with the Hilbert transform, a singular integral with the Cauchy kernel on a curve closed at infinity (cf. PR[2]), also with the cotangent Hilbert transform in appropriate spaces of functions.

The dependence of solutions on the choice of selectors has been examined in PR[8] for $n = 1$.

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