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FIXED POINT THEOREMS USING DIAMETERS OF LEVEL SETS AND THE NOTION OF R.G.I. MAPPING

Abstract. In this paper, we generalize and improve a recent result established by W. A. Kirk and L. M. Saliga (see [3], Theorem 4.3, p. 149) by using a result of W. Walter (see [4]). Indeed, We prove that the conclusions of Kirk-Saliga theorem are not only satisfied but, moreover, they are equivalent for a wide class of contractive gauge functions. Our main result contains also a new equivalent conclusion (see Property (4) in Theorem 2.2 below). As a consequence, we recapture (and improve the results of) a theorem proved by M. Angrisani and M. Clavelli in [2]. Theorem 2.2 completes and improves the main result of [1].

1. Introduction

In a recent paper (see [3], Theorem 4.3, p. 149), W. A. Kirk and L. M. Saliga have proved the following theorem

THEOREM 1.1 ([3]). *Let (M,d) be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies: there exists $\alpha < 1$ such that*

$$(K,S) \qquad d(Tx,Ty) \leq \alpha\, diam(O(x,y)) \quad \text{for all } x,y \in M.$$

Then we have:
(1) *T has a unique fixed point $z \in M$, and $\lim_{k\to+\infty} T^k(x) = z$ for each $x \in M$.*
(2) *$\lim_{c\to 0^+} diam(L_c) = 0$, and the mapping $F : x \mapsto d(x,Tx)$ is an r.g.i..*
(3) *For each sequence $\{x_n\} \subset M$; $\lim_n d(x_n,Tx_n) = 0$ if and only if $\{x_n\}$ converges to z.*

Here, $O(x,y) = O(x) \cup O(y)$, where $O(x) := \{x, Tx, T^2x, ...\}$ for all $x,y \in M$; and $L_c := \{x \in M : F(x) \leq c\}$ for all $c \geq 0$. We recall (see [2] and [3]) that a function $G : M \to \mathbb{R}$ is said to be a regular-global-inf

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(r.g.i.) at $x \in M$ if $G(x) > \inf_M(G)$ implies there exist $\epsilon > 0$ such that $\epsilon < G(x) - \inf_M(G)$ and a neighborhood N_x of x such that $G(y) > G(x) - \epsilon$ for each $y \in N_x$. If this condition holds for each $x \in M$, then G is said to be an r.g.i. on M. To prove Theorem 1.1, the authors have used the following result of W. Walter (see [4]).

THEOREM 1.2 ([4]). *Let (M, d) be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies the following condition:*

$$(W) \qquad d(Tx, Ty) \leq \phi(diam(O(x, y))) \quad \text{for all } x, y \in M,$$

where ϕ is a contractive gauge function on M. This means that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing and satisfies $\phi(s) < s$ for all $s > 0$. Then T has a unique fixed point $z \in M$, and $\lim_{k\to+\infty} T^k(x) = z$ for each $x \in M$.

A natural question was addressed by W.A. Kirk and L.M Saliga in [3]. Does the conclusion of Theorem 1.1 remain valid under the weaker assumption of Theorem 1.2 ? Another question may also be addressed: are the conclusions of Theorem 1.1 equivalent?

The main goal of this paper is to discuss these questions. The main contribution is stated in Theorem 2.2, where we establish that the conclusions of Theorem 1.1 are not only satisfied but, in fact, they are equivalent for a wide class of contractive gauge functions. Moreover, we can add a new equivalent property (see Property (4) in Theorem 2.2 below). Thus, our result improves and generalizes Theorem 1.1. As a consequence, we recapture (and improve the results of) a theorem proved by M. Angrisani and M. Clavelli in [2]. This paper improves and completes also the main result of the paper [1].

This paper is organized as follows. Section 2 contains the main result. In Section 3, as application of the main result, we provide a result improving and generalizing a theorem proved by M. Angrisani and M. Clavelli in [2].

2. Main result

2.1. Before stating the main result, we need to introduce some notations and make some remarks. We note $\mathcal{G}$ the set of contractive gauge functions. We denote $\mathcal{G}_1$ the set of contractive gauge functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ for which there exist a positive number $\beta \in]0, \infty]$ and an associated non-negative function ψ defined on $[0, \beta[$ such that the two following properties are satisfied:

(P 1) $\lim_{t\to 0} \psi(t) = 0$, and

(P 2) $\forall t \in [0, \beta[, \ \forall s \geq 0, \ s - \phi(s) \leq t \Longrightarrow s \leq \psi(t)$.

REMARK 2.1.1. Let ϕ be a contractive gauge function, then by a classical result, it satisfies $\lim_n \phi^n(s) = 0$ for all $s \geq 0$.

REMARK 2.1.2. Let $\mathcal{G}_2$ be the class of functions $\phi \in \mathcal{G}$ such that the mapping $\theta : x \mapsto x - \phi(x)$ from $[0, +\infty[$ onto $[0, +\infty[$ is strictly increasing. Let $\phi \in \mathcal{G}_2$ and let ψ be the inverse of θ on $[0, +\infty[$. Then, it is easy to see that the properties (P 1) and (P 2) are satisfied by the mappings ϕ and ψ with associated number $\beta = \infty$. Thus the class $\mathcal{G}_2$ is contained in the class $\mathcal{G}_1$. Moreover, it is easy to see that this inclusion is strict.

We point out that the main result of the paper [1] consists in proving that if the contractive gauge function ϕ belongs to the class $\mathcal{G}_2$ then the conclusions of Theorem 1.1 are true. The aim of this paper is to generalize and complete the main result of [1] by proving the following theorem in which we establish that the conclusions of Theorem 1.1 are satisfied and equivalent and that they are equivalent to the fact that $D : x \mapsto diam(O(x))$ is an r.g.i. mapping on M. More precisely, we have

THEOREM 2.2. *Let (M, d) be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies the following condition:*

$$(W) \qquad d(Tx, Ty) \leq \phi(diam(O(x, y))) \quad \text{for all } x, y \in M,$$

where $\phi \in \mathcal{G}_1$. Then the following assertions are satisfied and equivalent:

(1) *T has a unique fixed point $z \in M$, and $\lim_{k \to +\infty} T^k(x) = z$ for each $x \in M$.*
(2) *$\forall c > 0$, the set L_c is nonvoid, $\lim_{c \to 0^+} diam(L_c) = 0$, and the mapping $F : x \mapsto d(x, Tx)$ is an r.g.i. on M.*
(3) *There exists a unique point $z \in M$, such that, for each sequence $\{x_n\} \subset M$; $\lim_n d(x_n, Tx_n) = 0$ if and only if $\{x_n\}$ converges to z.*
(4) *The mapping $D : x \mapsto diam(O(x))$ is an r.g.i. on M.*

In the proof of this theorem, we shall use the following lemma:

LEMMA 2.3. *Let $\phi \in \mathcal{G}_1$ and let $\beta > 0$ and ψ denote the associated function defined on $[0, \beta[$ such that (P 1) and (P 2) are satisfied. Let (M, d) be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies the condition:*

$$(W) \qquad d(Tx, Ty) \leq \phi(diam(O(x, y))) \quad \text{for all } x, y \in M,$$

Suppose that T has a fixed point, say, z in M. Then z is unique and the following property is satisfied,

$$(Q) \quad \forall \varepsilon \in]0, \beta[\ \forall u \in M, \quad \min\{d(u, T(u)), d(u, z)\} \leq \varepsilon \\ \Longrightarrow \phi\left(diam(O(u) \cup \{z\})\right) \leq \psi\left(\varepsilon\right).$$

Proof. Let $\phi \in \mathcal{G}_1$ and let ψ denote the associated function defined on $[0, \beta[$ such that (P 1) and (P 2) are satisfied. The general line of arguments follows [1] and [3]. Let z be a fixed point of T. Since ϕ is a gauge contractive function, it is clear that z must be the unique fixed point of T. Let $\varepsilon \in]0, \beta[$ and let $u \in M$ such that $\min\{d(u, T(u)), d(u, z)\} \leq \varepsilon$. Suppose, for example, that $d(u, z) \leq \varepsilon$. Then since $T(z) = z$,

$$d(u, Tu) \leq d(u, z) + d(T(u), T(z)) \leq \varepsilon + \phi\left(diam(O(u) \cup \{z\})\right).$$

To simplify the notations, we set $\tau := diam(O(u) \cup \{z\})$. We distinguish two cases.

(a) $diam(O(u) \cup \{z\}) = \sup_p d(T^p(u), z)$. In this case let $\rho > 0$ be arbitrary and choose p so that $\sup_p d(T^p(u), z) \leq d(T^p(u), z) + \rho$. Then if $p = 0$, we have

$$\begin{aligned} diam(O(u) \cup \{z\}) &\leq d(u, z) + \rho \\ &\leq \varepsilon + \phi\left(diam(O(u) \cup \{z\})\right) + \rho, \end{aligned}$$

from which we get $\tau - \phi(\tau) \leq \varepsilon + \rho$. On the other hand, if $p \geq 1$,

$$\begin{aligned} diam(O(u) \cup \{z\}) &\leq d(T^p(u), T(z)) + \rho \\ &\leq \phi\left(diam(O(T^{p-1}(u) \cup \{z\})\right) + \rho \\ &\leq \phi\left(diam(O(u) \cup \{z\})\right) + \rho. \end{aligned}$$

Hence, we get $\tau - \phi(\tau) \leq \rho$. Therefore, in the two cases, we obtain $\tau - \phi(\tau) \leq \rho + \varepsilon$, from which (since $\rho > 0$ is arbitrary) $\tau - \phi(\tau) \leq \varepsilon$. By assumption, we must have $\tau \leq \psi(\varepsilon)$. It follows that $\phi(\tau) \leq \phi \circ \psi(\varepsilon) \leq \psi(\varepsilon)$.

(b) $diam(O(u) \cup \{z\}) = \sup_p d(T^p(u), u)$. Since $\lim_p d(T^p(u), u) = d(z, u)$, if one has $\sup_p d(T^p(u), u) = \lim_p d(T^p(u), u)$ then

$$diam(O(u) \cup \{z\}) = d(z, u) \leq \varepsilon + \phi\left(diam(O(u) \cup \{z\})\right).$$

Thus we get $\tau - \phi(\tau) \leq \epsilon$, which gives as before $\phi(\tau) \leq \psi(\varepsilon)$. Hence we may assume there exists $q \geq 1$ such that $diam(O(u) \cup \{z\}) = diam\,(O(u)) = d(T^q(u), u)$. In this case we have

$$\begin{aligned} diam(O(u) \cup \{z\}) &= diam\,(O(u)) = d(T^q(u), u) \\ &\leq d(u, z) + d(T(z), T^q(u)) \\ &\leq \varepsilon + \phi\left(diam(O(T^{q-1}(u)) \cup \{z\})\right) \\ &\leq \varepsilon + \phi\left(diam(O(u) \cup \{z\})\right). \end{aligned}$$

Thus the number $\tau = diam\,(O(u) \cup \{z\})$ satisfies $\tau - \phi(\tau) \leq \varepsilon$. It follows as before that $\phi(\tau) \leq \psi(\varepsilon)$.

If one supposes that $d(u,Tu) \leq \varepsilon$, then by a similar argument, we can prove that $\phi(\tau) \leq \psi(\varepsilon)$. Thus, Lemma 2.3 is completely proved. ∎

Proof of Theorem 2.2. Let $\phi \in \mathcal{G}_1$ and let ψ denote the associated function defined on $[0,\beta[$ such that (P 1) and (P 2) are satisfied.

(i) We prove that (1) $\Longrightarrow$ (2). Let z be the unique fixed point of T. It is clear that $z \in L_c$ for every $c > 0$. Let $\varepsilon \in]0,\beta[$ and let $u \in L_\varepsilon$. Then by Lemma 2.3, we have

$$\begin{aligned} d(u,z) &\leq d(u,T(u)) + d(T(u),T(z)) \\ &\leq \varepsilon + \phi\left(diam(O(u) \cup \{z\})\right) \leq \varepsilon + \psi(\varepsilon). \end{aligned}$$

We deduce

$$u,v \in L_\varepsilon \Rightarrow d(u,v) \leq d(u,z) + d(v,z) \leq 2\left(\varepsilon + \psi\left(\varepsilon\right)\right),$$

and since $\lim_{\varepsilon \to 0^+} \psi\left(\varepsilon\right) = 0$ this proves the first part of (2). To prove that F is an r.g.i., we use Proposition 1.2, of [3]. Let $\{x_n\}$ be a sequence such that $\lim_n F(x_n) = \inf_M(F) = 0$ and $\lim_n x_n = x$. By virtue of Lemma 2.3, we must have $x = z$. This implies that F is an r.g.i. function on M.

(ii) Let us prove (2) $\Longrightarrow$ (3). Consider $\{c_n\}$ a strictly decreasing sequence of positive numbers converging to zero, and set $A := \cap_n \overline{L_{c_n}}$, (where $\overline{L_{c_n}}$ means the closure of L_{c_n}). Then an application of Cantor's intersection theorem implies the existence of a unique element $z \in A$. For every nonzero integer n, since $z \in \overline{L_{c_n}}$, we can find $y_n \in L_{c_n}$ such that $d(y_n,z) \leq \frac{1}{n}$. Therefore $\{y_n\}$ converges to z. For each integer n, we have $0 \leq F(y_n) \leq c_n$. Hence $\lim_n F(y_n) = 0$. Since F is supposed to be regular, then $F(z) = \inf_M F = 0$. Thus z is a fixed point of T, it is unique since (W) is verified. Let $\{x_n\}$ be a sequence in M such that $\lim_n F(x_n) = 0$. Let $\varepsilon \in]0,\beta[$, and let $c_1 > 0$ such that $diam(L_{c_1}) < \varepsilon$. There exists an integer N_1 such that $F(x_n) < c_1$ for every integer $n \geq N_1$. Then $z, x_n \in L_{c_1}$. Therefore, $d(x_n,z) < \varepsilon$, for each integer $n \geq N_1$. Thus $\{x_n\}$ converges to the fixed point z. Conversely, Let $\{x_n\}$ be a sequence in M converging to the fixed point z. We have

$$F(x_n) \leq d(x_n,z) + d(T(x_n),T(z)) \leq d(x_n,z) + \phi\left(diam(O(x_n) \cup \{z\})\right).$$

Let $\varepsilon \in]0,\beta[$. By property (P 1), we can find at least a positive number $\delta < \varepsilon$ such that $\psi(\delta) < \varepsilon$. Let N_δ be an integer such that $d(x_n,z) \leq \delta$ for all integer $n \geq N_\delta$. Then according to Lemma 2.3, for all integer $n \geq N_\delta$, we shall have

$$\phi\left(diam(O(x_n) \cup \{z\})\right) \leq \psi(\delta) < \varepsilon.$$

Thus we have shown that the sequence $\{F(x_n)\}$ converges to zero.

(iii) Let us prove (3) $\Longrightarrow$ (4). We start by remarking that the point z involved in the assumption (3) must be fixed under T. Since ϕ is a contractive gauge function, z is the unique fixed point of T. Then, we deduce that $\inf_M D = 0$. To prove that D is an r.g.i., we use Proposition 1.2, of [3]. Let $\{x_n\}$ be a sequence such that $\lim_n D(x_n) = 0$ and $\lim_n x_n = x$. Since $F(x) \le D(x)$ for all $x \in M$, we get $\lim_n F(x_n) = 0$. By assumption, we must have $\lim_n x_n = z$. Then, $z = x$. Thus D is an r.g.i. function on M.

(iv) Let us prove that (4) $\Longrightarrow$ (1). Let x_0 be any arbitrary point in M, and set $x_{n+1} := T(x_n)$ for all integer n. By using mathematical induction, it is easy to prove the following

$$(R) \qquad d(T^n(x), T^n(y)) \le \phi^n \left(diam\left(O(x,y)\right)\right) \quad \forall x, y \in M.$$

We deduce from (R) two conclusions. The first one is that the sequence $\{x_n\}$ is Cauchy. The second one (according to Remark 2.1.1) is that $\lim_n D(x_n)=0$. Since (M,d) is complete, then there exists an element $z \in M$ such that $\lim_n x_n = z$. It is clear from (R) that this limit z is independent from x_0. Since D is an r.g.i. function, we deduce that $D(z) = \inf_M(D) = 0$, wich implies that z is the unique fixed point under T and that all the Picard sequences converge to z.

(v) Thus the four properties are equivalent. To complete the proof we observe that (1) is satisfied by virtue of Walter's theorem. ∎

3. A related result

M. Angrisani and M. Clavelli have proved in [2] the following result.

THEOREM 3.1 [2]. *Let (M,d) be a complete metric space and suppose T is a self-mapping of M satisfying: there exists $\alpha < 1$ such that for each $x, y \in M$*

$$(3.1) \quad d(Tx,Ty) \le \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

Then $\inf_M(F) = 0$, $\lim_{c\to 0^+} diam(L_c) = 0$ *and F is an r.g.i. on M.*

Using our main result we provide the following generalization and improvement of this theorem. More precisely, we have the following

THEOREM 3.2. *Let (M,d) be a complete metric space and suppose T is a self-mapping of M satisfying: there exists $\phi \in \mathcal{G}_1$ with associated number $\beta = +\infty$ such that for each $x, y \in M$*

$$(A,C) \quad d(Tx,Ty) \le \phi\left(\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}\right).$$

Then the self-mapping T has bounded orbits, and the following properties are satisfied and equivalent:

(1) *T has a unique fixed point $z \in M$, and* $\lim_{k\to+\infty} T^k(x) = z$ *for each $x \in M$.*

(2) $\forall c > 0$, *the set L_c is nonvoid,* $\lim_{c\to 0^+} diam(L_c) = 0$, *and the mapping $F : x \mapsto d(x,Tx)$ is an r.g.i. on M.*

(3) *There exists a unique point* $z \in M$, *such that, for each sequence* $\{x_n\} \subset M$; $\lim_n d(x_n, Tx_n) = 0$ *if and only if* $\{x_n\}$ *converges to* z.
(4) *The mapping* $D : x \mapsto diam(O(x))$ *is an r.g.i. on* M.

Proof. (a) Let $x \in M$. For each integer n we set $O_n(x) := \{x, Tx, \ldots, T^n(x)\}$. It is easy to verify that

$$diam(O_n(Tx)) \leq \phi(diam(O_{n+1}(x))) \tag{3.2.1}$$

and that for each integer $n \geq 1$ there exists an integer $k_n \in \{1, 2, \ldots, n\}$ such that

$$diam(O_n(x)) = d(x, T^{k_n}(x)). \tag{3.2.2}$$

Then, with the help of (3.2.1) and (3.2.2), we obtain

$$\begin{aligned} diam(O_n(x)) &= d(x, T^{k_n}(x) \leq d(x, Tx) + d(Tx, T^{k_n}(x)) \\ &\leq d(x, Tx) + diam(O_{n-1}(Tx)) \\ &\leq d(x, Tx) + \phi(diam(O_n(x))). \end{aligned}$$

By property (P 2), we deduce that $diam(O_n(x)) \leq \psi\,(d(x, Tx))$, for every integer $n \geq 1$. Since $O(x) = \cup_n O_n(x)$, we conclude that

$$diam(O(x)) = \sup_n diam(O_n(x)) \leq \psi\,(d(x, Tx)) < \infty.$$

This proves that T has bounded orbits.

(b) Since T has bounded orbits and satisfies condition (A,C), it follows immediately that T must satisfy also the condition (W). Therefore, by using Theorem 2.2, the four properties are satisfied and equivalent. This completes the proof of Theorem 3.2. ∎

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