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Ali A. Mehemmed

AN APPLICATION OF KWAPIEŃ'S THEOREM TO THE INVESTIGATION OF THE STRUCTURE OF L_0-SPACE

Abstract. Our aim in this paper is to use Kwapień's theorem to show that the space L_0 is prime. This was a longstanding open question posed by Pełczyński which was resolved by Kalton [3]. The proof given here uses some new lemmas which can be of interest.

Consider these notations and terminologies.

A metrizable topological vector space (or metric linear space) is called an *F-space* if it is complete for an invariant metric (for any invariant metric).

A sequence $\langle f_n \rangle$ in $L_0(\mu)$ is said to converge to f in *measure* if, given $\varepsilon > 0$, there is an integer N such that for all n we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon.$$

By Lemma (2.1) in [2] any subset $\{g_\alpha : \alpha \in I\}$ of the space $L_0(\mu)$ has a least upper bound, therefore, the space $L_0(\mu)$ is an F-space with F-norm $\|\cdot\|_0$ defined by $\|f\|_0 = \int_0^1 \frac{|f|}{1+|f|}\, d\mu$, for any $f \in L_0(\mu)$, and its invariant metric defined by $d(f,g) = d(0, f-g) = \int_0^1 \frac{|f-g|}{1+|f-g|}\, d\mu$.

Let X and Y be two F-spaces, the symbol $X \oplus Y$ denote the product of X and Y, i.e. $X \oplus Y = \{(x,y) : x \in X, y \in Y\}$ with F-norm $\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}$.

Let X_1, X_2, ... be a sequence of F-spaces with F-norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}, \ldots$, respectively. By $(X_1 \oplus X_2 \oplus \ldots)_E$ we denote the space of all sequences $\langle x_i \rangle$, where $x_i \in X_i$ such that $\langle \|x_i\|_{X_i} \rangle \in E$ with the norm $\|\langle x_i \rangle\| = \big\| \langle \|x_i\|_{X_i} \rangle \big\|_E$. Moreover, if $E, X_1, X_2, \ldots$ are Banach spaces, then $(X_1 \oplus X_2 \oplus \ldots)_E$ is also a Banach space.

A subspace Y of an F-space X is said to be *complemented* in X if there is a subspace Y_1 (a complement to Y) such that for each $x \in X$ there exist

y in Y and y_1 in Y_1 such that $x = y + y_1$ and if $0 = y + y_1$ then $y = 0$ and $y_1 = 0$.

See M. M. Day [8] in [4] for the proofs of the following:

1° The subspace Y is complemented in X iff there is a projection $P : X \xrightarrow{\text{onto}} Y$.

2° If Y has a complement Y_1 in X, then $X \approx Y \oplus Y_1$.

An F-space X is *prime* if it is isomorphic to each of its infinite-dimensional complemented subspaces (i.e. $X \approx X_1 \oplus X_2$ with X_1 infinite-dimensional, then $X \approx X_1$). And X is *primary* if $X \approx Y \oplus Z$ implies that either Y or $Z \approx X$. The space L_p $(1 \le p < \infty)$ is primary (see Kalton [3]).

We begin first by using Pełczyński's decomposition technique in proving the following.

LEMMA 1 [1]. *If $L_0 \approx X \oplus Y$ and $X \approx L_0 \oplus X_1$, then $X \approx L_0$.*

Proof. Notice that $\prod_{n=1}^{\infty} X_n = X_1 \oplus X_2 \oplus \ldots$ with the norm $\|x\| = \sum_{n=1}^{\infty} 2^{-n} \|X_n\|$, which induced the product topology. If in $[0,1]$ we construct a sequence $\langle E_n \rangle$ of pairwise disjoint subsets of positive measures such that $\coprod_{n=1}^{\infty} E_n = [0,1]$, then $L_0 \approx \bigcup_{n=1}^{\infty} L_0(E_n)$, and each $L_0(E_n) \approx L_0$.

Since

$$\begin{aligned} L_0 &\approx L_0 \oplus L_0 \oplus \ldots \\ &\approx (X \oplus Y) \oplus (X \oplus Y) \oplus \ldots \\ &\approx X \oplus (Y \oplus X) \oplus (Y \oplus X) \oplus \ldots \\ &\approx X \oplus L_0, \end{aligned}$$

we have

$$\begin{aligned} X &\approx L_0 \oplus X_1 \\ &\approx X \oplus L_0 \oplus X_1 \\ &\approx X \oplus L_0 \oplus L_0 \oplus X_1 \\ &\approx L_0 \oplus X \\ &\approx L_0. \ \blacksquare \end{aligned}$$

If A is a measurable subset of the unit interval $[0,1]$ we may define a projection on L_0 by $P_A(f) = \chi_A(f)$ for each $f \in L_0$.

LEMMA 2 [1]. *If P is a projection on L_0 and there exist measurable sets E and A in $[0,1]$ of positive measures such that $P_A \circ P|_{L_0(E)}$ is an isomorphism from $L_0(E)$ onto $L_0(A)$, then*

$$P(L_0) \approx L_0.$$

Proof. Since $P_A\big(P(L_0(E))\big) = L_0(A)$, i.e.

$$\begin{array}{ccc} L_0(E) & \xrightarrow{P} & P(L_0(E)) \\ & \searrow^{P_A P} & \downarrow{\scriptstyle P_A} \\ & & L_0(A), \end{array}$$

suppose $\langle f_n \rangle$ is a sequence in $L_0(E)$ and $P(f_n) \to 0$. Then $P_A(P(f_n)) \to 0$, so that $f_n \to 0$. Thus $P|_{L_0(E)} : L_0(E) \overset{\text{onto}}{\longrightarrow} P(L_0(E))$ is an isomorphism. We claim that $P(L_0(E)) \oplus L_0(-A) = L_0$.

Let $g \in P(L_0(E)) \cap L_0(-A)$, then there exists $f \in L_0(E)$ such that $P(f) = g$, and $g \in L_0(-A)$ so that $P_A(P(f)) = P_A(g) = \chi_A(g) = 0$. But since $P_A \circ P$ is an isomorphism on $L_0(E)$, $f = 0$. Thus $P(L_0(E)) \cap L_0(-A) = \{0\}$.

Now suppose $g \in L_0$. Then there exists an $f \in L_0(E)$ such that $P_A(P(f)) = P_A(g)$. Thus $g = P(f) + (g - P(f))$. Since $P_A(g - P(f)) = P_A(g) - P_A(P(f)) = 0$, $g - P(f) \in L_0(-A)$, i.e. $P(L_0(E)) \oplus L_0(-A) = L_0$.

Consequently, $P(L_0(E)) \oplus \big(P(L_0) \cap L_0(-A)\big) = P(L_0)$. Since $P(L_0(E)) \approx L_0(A) \approx L_0$, so by Lemma 1 we have $P(L_0) \approx L_0$. ∎

In [2] we modificate Kwapień's theorem as follows.

THEOREM 1 [2]. *If $T \in \mathcal{L}(L_0)$, then $T(f) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} g_{in} f \circ \varphi_{in}$ for each $f \in L_0$ where*

1) *$\langle A_n \rangle$ is a pairwise disjoint collection of sets of positive measure on $[0,1]$.*
2) *$\{E_{1n}, \ldots, E_{K_n n}\}$ is a partition of $[0,1]$ into sets of positive measure.*
3) *$\varphi_{in} : \operatorname{supp} g_{in} \to E_{in}$.*

In particular, $T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n$ where

i — each $g_n \in L_0$,

ii — each $\varphi_n : \operatorname{supp} g_n \to [0,1]$ is a non-singular measurable mapping,

iii — for almost all x in $[0,1]$, $g_n \neq 0$ for only finitely many n.

Conversely, every mapping defined by $T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n$ is a continuous linear operator on L_0.

LEMMA 3 [1]. *Suppose $T \in \mathcal{L}(L_0)$ is defined by $T(f) = \sum_{k=1}^{\infty} g_k f \circ \varphi_k$, and $T(f) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} g_{in} f \circ \varphi_{in}$ is a canonical representation. Let $E \subset E_{in}$ and $A \subset A_n$ such that E and A have positive measures and $\varphi_{in} : A \to E$ and there exists an integer N such that $g_k(x) = 0$, for all $x \in A$ when $k \geq N+1$. Then there exists $A_0 \subset A$ such that A_0 has a positive measure and $\varphi_{in} = \varphi_j$ for some j, $1 \leq j \leq N$.*

Proof. By assumption $P_A \circ T : L_0(E) \to L_0(A)$, and for $f \in L_0(E)$

$$P_A(T(f)) = g_{in} f \circ \varphi_{in} = P_A\Big(\sum_{k=1}^{N} g_k f \circ \varphi_k\Big).$$

Suppose that $m(\{x \in A : \varphi_{in}(x) = \varphi_1(x)\}) = 0$. Then

$$A = \bigcup_{r \in Q \cap (0,1)} \{x \in A : \varphi_{in}(x) < r < \varphi_1(x)\}$$
$$\cup \bigcup_{r \in Q \cap (0,1)} \{x \in A : \varphi_1(x) < r < \varphi_{in}(x)\} \text{ a.e.}$$

Thus, without loss of generality there exists $r \in Q \cap (0,1)$ such that if $I_1 = [0,r)$, $J_1 = (r,1]$ and $A_1 = A \cap \varphi_{in}^{-1}(I_1) \cap \varphi_1^{-1}(J_1)$, then $m(A_1) > 0$. Continuing inductively we can choose intervals $I_1, I_2, \dots, I_N$ and $J_1, J_2, \dots, J_N$, each with positive length, and sets $A_1, A_2, \dots, A_N$ of positive measures such that $I_1 \supset I_2 \supset \dots \supset I_N$, $I_k \cap J_k = \emptyset$, $\varphi_{in}(A_k) \subset I_k$, and $\varphi_k(A_k) \subset J_k$. Thus $I_N \cap J_N = \emptyset$ for all $k = 1, \dots, N$ $\varphi_{in}(A_N) \subset I_N$, and $\varphi_k(A_N) \subset J_k$. Hence

$$P_{A_N} \circ T(\chi_{I_N}) = \chi_{A_N} g_{in} \chi_{\varphi_{in}^{-1}(I_N)} \neq 0.$$

But $P_{A_N} \circ T(\chi_{I_N}) = \chi_{A_N} \sum_{k=1}^{N} g_k \chi_{\varphi_k^{-1}(I_N)} = 0$, because $\varphi_k^{-1}(I_N) \cap A_N = \emptyset$, i.e. $\varphi_k(A_N) \subset J_k$ and $I_N \cap J_N = \emptyset$.

This is a contradiction. ∎

Now we give our proof of the Kalton theorem [3] which shows that the space L_0 of all measurable functions on the unit interval is prime. In our proof we will use the canonical representation form of the projection $P : L_0 \xrightarrow{\text{onto}} X$, $P(f) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} g_{in} f \circ \varphi_{in}$ beside the form $P(f) = \sum_{j=1}^{\infty} g_j f \circ \varphi_j$.

THEOREM 2 [3]. *The space L_0 is prime.*

Proof. Suppose $L_0 \approx X_1 \oplus X_2$, and $P : L_0 \xrightarrow{\text{onto}} X_1$ is a projection. Let $P(f) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} g_{in} f \circ \varphi_{in}$ be a canonical form for P. We may write $P(f) = \sum_{j=1}^{\infty} g_j f \circ \varphi_j$. Thus

$$P(f) = P^2(f) = P\Big(\sum_{j=1}^{\infty} g_j f \circ \varphi_j\Big) = \sum_{i=1}^{\infty} g_i \Big(\sum_{j=1}^{\infty} g_j f \circ \varphi_j\Big) \circ \varphi_i$$
$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g_i (g_j \circ \varphi_i)(f \circ \varphi_j \circ \varphi_i).$$

Put $Y_n = \{x \in [0,1] : g_i(g_j \circ \varphi_i)(x) = 0 \text{ for each } i,j > n\}$ since $Y_1 \subset Y_2 \subset \dots$, and $m(\bigcap_{n=1}^{\infty} -Y_n) = m(-\bigcap_{n=1}^{\infty} Y_n) = 0$. Thus $m(-Y_n) \searrow 0$, there exists N, and a set A with $m(A) > 0$, so that

$$P_A P(f) = P_A\Big(\sum_{i=1}^{N} g_i f \circ \varphi_i\Big) = P_A\Big(\sum_{i=1}^{N}\sum_{j=1}^{N} g_i (g_j \circ \varphi_i)(f \circ \varphi_i \circ \varphi_j)\Big).$$

For each i, $1 \le i \le N$, we may choose $\delta_i > 0$ so that $m\big(\{x \in A : |g_i(x)| < \delta_i\}\big) > 0$, where $\sum_{i=1}^{N} \delta_i < m\big(A \cap \bigcup_{i=1}^{n} \operatorname{supp} g_i\big)$. By replacing A with $A \setminus \bigcup_{i=1}^{N}\{x \in A : |g_i(x)| < \delta_i\}$ we may assume that $|g_i| \ge \delta_i$ on $(\operatorname{supp} g_i) \cap A$.

Since $\varphi_i : \operatorname{supp} g_i \to [0,1]$ is measurable, there exists K_i compact subset of $\operatorname{supp} g_i$ such that $m(K_i) > 0$ and $\varphi_i|_{K_i}$ is continuous. Let $K = \bigcup_{i=1}^{N} K_i$, then $P_A P : L_0(K) \to L_0(A)$. By Lemma 3 for each i, $1 \le i \le N$, $\varphi_i = \varphi_j \varphi_k = \varphi_{\psi(i)} \varphi_{\Psi(i)}$ on a set of positive measure, i.e. $\psi(i) = j$, $\Psi(i) = k$.

We shall take a compact set inside that set of positive measure. Thus we have

$$\varphi_1 = \varphi_{\psi(1)}\varphi_{\Psi(1)} = \varphi_{i_1}\varphi_{\Psi(1)} = \varphi_{i_1}\varphi_{i_2}\varphi_{\Psi(\Psi(1))}.$$

Continuing inductively, $\varphi_i = \varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_{N+1}}$ on a compact set of positive measure. But then some φ_i appears more than one time in $\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_{N+1}}$.

So by rearranging indices we may assume $\varphi_1 = \varphi_2\varphi_3\cdots\varphi_k\varphi_1$ for some k, $1 \le k \le N$ on a compact set of positive measure K. It is clear that φ_k is one-to-one on $\varphi_1(K)$. Also $m(\varphi_1(K)) > 0$. Otherwise, if $m(\varphi_1(K)) = 0$, then $m(K) \le m\big(\varphi_1^{-1}(\varphi_1(K))\big) = 0$. Also, $\varphi_1(K) \subset (\operatorname{supp} g_k) \cap A$.

Now let $A_0 = \varphi_1(K)$. Then $\varphi_k : A_0 \to \varphi_k(A_0)$ is a homomorphism (since A_0 is compact, φ_k is continuous and one-to-one). Also, $m(\varphi_k(A_0)) > 0$ by the same argument that $m(\varphi_1(K)) > 0$. Thus, for each $f \in L_0(\varphi_k(A_0))$, $P_{A_0} \circ P(f) = g_k f \circ \varphi_k$.

Hence, $P_{A_0}P|_{L_0(\varphi_k(A_0))} : L_0(\varphi_k(A_0)) \to L_0(A_0)$ is an isomorphism, and its inverse is T defined by

$$T(f) = \frac{1}{g_k \circ \varphi_k^{-1}} f \circ \varphi_k^{-1}.$$

Note that $\varphi_k^{-1} : \varphi_k(A_0) \to A_0$ is also non-singular since

$$\varphi_k^{-1} = \varphi_2\varphi_3\cdots\varphi_{k-1}.$$

At last by Lemma 2 we have

$$X_1 \approx P(L_0) \approx L_0. \ \blacksquare$$

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WARSAW UNIVERSITY OF TECHNOLOGY
INSTITUTE OF MATHEMATICS
Plac Politechniki 1
00-661 WARSAW, POLAND

Mail address of the author:
THE UNIVERSITY OF EL-FATEH
P.O.BOX 13611
TRIPOLI, LIBYA

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