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Carlo Bardaro, Julian Musielak, Gianluca Vinti

ON NONLINEAR INTEGRAL EQUATIONS IN SOME FUNCTION SPACES

Abstract. There are established some conditions for existence of solutions of a non-linear integral equation $Tf = f + g$, where T is a convolution-type integral operator.

1. Preliminaries

Let (Ω, Σ, μ) be a σ-finite, complete measure space and let $L^0(\Omega)$ denote the space of all Σ-measurable, finite μ-a.e., real valued functions f on Ω, with equality μ-a.e. Let $+ : \Omega \times \Omega \to \Omega$ be a commutative operation on Ω such that $L^0(\Omega)$ is *invariant* with respect to $+$, i.e. $f \in L^0(\Omega)$ implies $f(\cdot + s) \in L^0(\Omega)$ for all $s \in \Omega$.

Let $K : \Omega \times \mathbb{R} \to \mathbb{R}$ be a *Carathéodory function*, i.e. $K(t, u)$ is Σ-measurable with respect to $t \in \Omega$ for every $u \in \mathbb{R}$ and is continuous with respect to u for every $t \in \Omega$. If, moreover, $K(t, 0) = 0$ for all $t \in \Omega$, we call K a *Carathéodory kernel function.*

We are going to investigate the existence of solution of the nonlinear integral equations of type

$$\int_\Omega K(t, f(t+s))d\mu(t) = f(s) + g(s), \tag{1}$$

where $g \in L^0(\Omega)$ is given, generated by the *convolution-type (nonlinear) integral operator* T defined by

$$(Tf)(s) = \int_\Omega K(t, f(t+s))d\mu(t), \qquad s \in \Omega. \tag{2}$$

We shall look for solutions f of this equation belonging to some subspaces $L^0_\rho(\Omega)$ of $L^0(\Omega)$, being $L^0_\rho(\Omega)$ a modular function space. By Dom$T$ we shall

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denote the *domain of the operator* T, i.e. the set of all $f \in L^0(\Omega)$ such that (2) exists for μ-a.e. $s \in \Omega$ and it is a Σ-measurable function of $s \in \Omega$.

If K is a Carathéodory function, it is easily seen that for $f \in L^0(\Omega)$ we have $K(\cdot, f(\cdot + s)) \in L^0(\Omega)$ for all $s \in \Omega$. In order to be able to apply the Fubini-Tonelli theorem, we should know that $K(t, f(t+s))$ is a measurable function on $\Omega \times \Omega$. Let Σ_0 be the smallest σ-algebra of subsets of $\Omega \times \Omega$ containing the sets of type $A \times B$ for $A, B \in \Sigma$, and let Σ_π be any σ-algebra of subsets of $\Omega \times \Omega$ such that $\Sigma_0 \subset \Sigma_\pi$. We denote by μ_0 the product measure on Σ_0, i.e. $\mu_0(A \times B) = \mu(A)\mu(B)$ for $A, B \in \Sigma$, and we denote by μ_π any extension of the measure μ_0 from Σ_0 to Σ_π. We denote by $\mathcal{K}_\pi$ the class of all Carathéodory functions $K : \Omega \times \mathbb{R} \to \mathbb{R}$ such that the function $\widetilde{K} : \Omega \times \Omega \to \mathbb{R}$ defined by $\widetilde{K}(s,t) = K(t, f(t+s))$ is Σ_π-measurable for every $f \in L^0(\Omega)$. Functions $K \in \mathcal{K}_\pi$ will be called *Σ_π-regular Carathéodory functions.*

EXAMPLE 1. Let μ be a σ-finite and complete measure on the σ-algebra Σ of all Lebesgue measurable subsets of $\Omega = \mathbb{R}$ and let μ^2 be the product measure on the σ-algebra Σ^2 of all Lebesgue measurable subsets of $\mathbb{R}^2$. Let $+$ be a commutative operation on $\mathbb{R}$ such that $\mathbb{R}$, $L^0(\mathbb{R})$ are invariant. We shall write $\sigma(s,t) = s + t$, and we suppose that σ is (Σ^2, Σ)-measurable, i.e. if $A \in \Sigma$ then $\sigma^{-1}(A) \in \Sigma^2$. Moreover, we suppose that μ^2 is *σ-absolutely continuous (σ-a.c.)* with respect to μ, i.e. if $A \in \Sigma$ and $\mu(A) = 0$, then $\mu^2(\sigma^{-1}(A)) = 0$ ([13], [11], [3]). Under these assumptions, every Carathéodory function $K : \Omega \times \mathbb{R} \to \mathbb{R}$ is Σ^2-regular, i.e. the function $\widetilde{K}(s,t) = K(t, f(t+s))$ is Σ^2-measurable for every $f \in L^0(\Omega)$ (see [3]).

REMARK 1. It is obvious that if K is a Σ_π-regular Carathéodory function, then its absolute value $|K|$ defined by $|K|(t,u) = |K(t,u)|$ is also a Σ_π-regular Carathéodory function. Thus, the integral

$$\int_{\Omega \times \Omega} |K(t, f(t+s))| d\mu_\pi(s,t),$$

exists for every $f \in L^0(\Omega)$. By the Fubini-Tonelli theorem, the integral

$$\int_\Omega |K(t, f(t+s))| d\mu(t),$$

exists for a.e. $s \in \Omega$ and is a Σ-measurable function of the variable $s \in \Omega$.

2. Notations and definitions

We are going now to define the modular function spaces $L^0_\rho(\Omega)$ which will play the role of field of solutions of the integral equation. For the sake of convenience, we recall some notions concerning such spaces (see [12]).

A functional $\rho : L^0(\Omega) \to \overline{\mathbb{R}_0^+} = [0, +\infty]$ is called a *modular* on $L^0(\Omega)$ if it satisfies the following conditions:

1) $\rho(0) = 0$,
2) $\rho(-f) = \rho(f)$,
3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for $f, g \in L^0(\Omega), \alpha, \beta \geq 0, \alpha + \beta = 1$.

If moreover ρ satisfies the condition

3') $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ for $f, g \in L^0(\Omega), \alpha, \beta \geq 0, \alpha + \beta = 1$,

then ρ is called a *convex modular* on $L^0(\Omega)$. The linear space $L^0_\rho(\Omega)$ of functions $f \in L^0(\Omega)$ such that $\rho(\lambda f) \to 0$ as $\lambda \to 0^+$ is called the *modular space generated by* ρ. In case of a convex modular ρ, the map $\|\cdot\|_\rho : L^0_\rho(\Omega) \to \mathbb{R}^+_0 = [0, +\infty[$ defined by

$$\|f\|_\rho = \inf\{u > 0 : \rho(f/u) \leq 1\} \tag{3}$$

is a norm on $L^0_\rho(\Omega)$. Consequently $\|f_n - f\|_\rho \to 0$ ($f_n, f \in L^0_\rho(\Omega), n = 1, 2, \ldots$) is equivalent to the condition $\rho(\lambda(f_n - f)) \to 0$ as $n \to \infty$ for every $\lambda > 0$. Besides the norm convergence, there is defined in $L^0_\rho(\Omega)$ the modular convergence (ρ-convergence) by the condition $\rho(\lambda(f_n - f)) \to 0$ as $n \to \infty$ for some $\lambda > 0$; these kinds of convergence are not equivalent, in general.

We will say that $L^0_\rho(\Omega)$ is *measure bounded* (see [14]), if $\|f\|_\rho \to 0$ implies $f_n \to 0$ in measure on Ω. It is obvious that every norm-closed subset of a measure bounded modular space $L^0_\rho(\Omega)$ is also measure bounded.

EXAMPLES 2.

(a) A norm $\rho = \|\cdot\|$ in a normed linear space $X \subset L^0(\Omega)$ is always a convex modular and $\|\cdot\| = \|\cdot\|_\rho$, $L^0_\rho(\Omega) = X$; in this case, norm convergence and modular convergence are equivalent.

(b) Let Φ be the family of all functions $\varphi : \Omega \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that $\varphi(t, u)$ is a Σ-measurable function of t for every $u \geq 0$ and is a nondecreasing, continuous function of u for every $t \in \Omega$, such that $\varphi(t, 0) = 0, \varphi(t, u) > 0$ for $u > 0$ and $\varphi(t, u) \to +\infty$ as $u \to +\infty$ for all $t \in \Omega$. Then ρ defined by the formula

$$\rho(f) = \int_\Omega \varphi(t, |f(t)|)d\mu(t) \tag{4}$$

for $f \in L^0(\Omega)$ is a modular on $L^0(\Omega)$. If $\varphi(t, u)$ is a convex function of $u \geq 0$ for all $t \in \Omega$, then ρ is a convex modular on $L^0(\Omega)$ and in this case the family of such functions φ will be denoted by Φ_c. The modular space $L^0_\rho(\Omega)$ is called a *generalized Orlicz space* or *Musielak-Orlicz space* and is denoted by $L^\varphi(\Omega)$.

It is easily seen that if the function φ satisfies the further condition: for every $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) < +\infty$, we have

$$\delta_A(\varepsilon) := \text{ess}\ \inf_{t \in A}\varphi(t, \varepsilon) > 0,$$

then the Musielak-Orlicz space L^φ is measure bounded.

If φ depends only on the second variable u, then (4) becomes of the form

$$\rho(f) = \int_\Omega \varphi(|f(t)|)d\mu(t),$$

and $L^\varphi(\Omega)$ is called an *Orlicz space*; the family of such functions will be denoted by Φ^0 and by Φ^0_c the convex ones.

It is clear that every Orlicz space is measure bounded.

3. An embedding result

Let $L : \Omega \to \mathbb{R}^+_0$ be a Σ-measurable function such that $L \neq 0$ and $L \in L^1(\Omega)$. Let $\psi : \Omega \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be such that $\psi(t, u)$ is a Σ-measurable function of $t \in \Omega$ for all $u \geq 0$, and a continuous, concave and nondecreasing function of $u \geq 0$ for every $t \in \Omega$, such that $\psi(t, 0) = 0$, $\psi(t, u) > 0$ for $u > 0$, $\psi(t, u) \to \infty$ as $u \to \infty$ for all $t \in \Omega$. The family of all functions ψ satisfying the above assumptions will be denoted by Ψ. For a given function $\psi \in \Psi$, a Σ_π-regular Carathéodory kernel function K will be called $(L, \psi)_0$*-Lipschitz* if $|K(t, u)| \leq L(t)\psi(t, |u|)$ for all $t \in \Omega$ and $u \in \mathbb{R}$.

We shall still need the following assumption on the modular ρ on $L^0(\Omega)$. We say that ρ is *subbounded* with respect to the operation $+$ on Ω, if there are a constant $C \geq 1$ and a nonnegative function $h \in L^\infty(\Omega)$ such that for every $f \in L^0_\rho(\Omega)$ and $s \in \Omega$, there holds the inequality

$$\rho(f(\cdot + s)) \leq \rho(Cf) + h(s).$$

If the above inequality holds with $h = 0$, we call ρ *strongly subbounded* with respect to the operation $+$ on Ω.

The following theorem is obtained by an easy change of the proof of Theorem 1 in [13].

THEOREM 1. *Let ρ be a convex modular on $L^0(\Omega)$, subbounded with respect to the operation $+$ on Ω. Let K be a Σ_π-regular, $(L, \psi)_0$-Lipschitz Carathéodory function with $L : \Omega \to \mathbb{R}^+_0$, $0 \neq L \in L^1(\Omega)$. Finally, let*

$$\int_\Omega L(t)\psi(t, 1)|f(t)|d\mu(t) < +\infty$$

for every $f \in L^0_\rho(\Omega)$. Then there holds

$$L^0_\rho(\Omega) \subset DomT.$$

EXAMPLE 3. Let $\varphi \in \Phi^0_c$ be an N-function, i.e. $\lim_{u\to 0^+} u^{-1}\varphi(u) = 0$ and $\lim_{u\to\infty} u^{-1}\varphi(u) = +\infty$. Then the function $\varphi^* : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ defined by $\varphi^*(u) = \sup_{v>0}[uv - \varphi(v)]$ for $u \geq 0$, called *coniugate to φ* in the sense of Young (see [12]), also belongs to Φ_c^0 and it is an N-function. The modulars

$$\rho(f) = \int_\Omega \varphi(|f(t)|)d\mu(t) \text{ and } \rho^*(f) = \int_\Omega \varphi^*(|f(t)|)d\mu(t)$$

define two Orlicz spaces $L^\varphi(\Omega) = L^0_\rho(\Omega)$ and $L^{\varphi^*}(\Omega) = L^0_{\rho^*}(\Omega)$. It is obvious that when Ω is a locally compact group with respect to the operation $+$, ρ, ρ^* are always subbounded with respect to $+$, since they are invariant with respect $+$.

Now, let K be a Σ_π-regular, $(L,\psi)_0$-Lipschitz Carathéodory kernel function with $L : \Omega \to \mathbb{R}_0^+$ such that $0 \neq L \in L^1(\Omega)$ and $\psi \in \Psi$. Moreover, let $L(\cdot)\psi(\cdot,1) \in L^{\varphi^*}(\Omega)$. It is easily deduced from Theorem 1 that then there holds $L^0_\rho(\Omega) \subset \mathrm{Dom}T$.

4. Main results

In [13] and [11] there was obtained a theorem on existence and uniqueness of a solution $f \in L^0_\rho(\Omega)$ of the integral equation (1), applying Banach fixed point principle.

Here, we are going to prove an existence theorem for solutions of (1) in $L^0_\rho(\Omega)$, applying Schauder fixed point principle. This principle states in case of a Banach space X with norm $\|\cdot\|$ that if T maps a nonempty, compact, convex subset C_0 of X into itself continuously, then there exists a point $x_0 \in C_0$ such that $Tx_0 = x_0$ (see e.g. [5]).

The following proposition is an immediate consequence of the Schauder fixed point principle:

PROPOSITION 1. *Let X be a Banach space with norm $\|\cdot\|$ and let C_0 be a nonempty, closed, convex subset of X. Let $T : C_0 \to C_0$ be a continuous map of C_0 into itself such that the image TC_0 of C_0 is conditionally compact in X. Then there exists a point $x_0 \in C_0$ such that $Tx_0 = x_0$.*

We shall apply Proposition 1 in case when $X = L^0_\rho(\Omega)$ or $X = E^0_\rho(\Omega)$, where ρ is a convex modular in $L^0(\Omega)$ and $E^0_\rho(\Omega)$ is the set of *finite elements of $L^0_\rho(\Omega)$*, i.e. the set of functions $f \in L^0(\Omega)$ such that $\rho(\lambda f) < +\infty$ for all $\lambda > 0$. We have $E^0_\rho(\Omega) \subset L^0_\rho(\Omega)$ and we provide both spaces with the norm $\|\cdot\|_\rho$, defined by (3). It is well-known that $E^0_\rho(\Omega)$ is a norm-closed subspace of $L^0_\rho(\Omega)$ and so if the space $L^0_\rho(\Omega)$ is complete and measure bounded with respect to the norm $\|\cdot\|_\rho$, then $E^0_\rho(\Omega)$ is also complete and measure bounded with respect to $\|\cdot\|_\rho$. In the sequel we shall need still some further assumptions on a modular ρ on $L^0(\Omega)$. We call ρ to be *monotone* if from $f, g \in L^0(\Omega)$, $|f| \le |g|$ follows $\rho(f) \le \rho(g)$. Let us remark that in case when ρ is a norm in $L^0_\rho(\Omega)$, and ρ is monotone, the normed vector space $L^0_\rho(\Omega)$

is called a *preideal space* (see e.g. [14]). A monotone modular ρ satisfies the equality $\rho(|f|) = \rho(f)$ for every $f \in L^0(\Omega)$. We say that ρ is *J-convex*, if for every $p : \Omega \to \mathbb{R}_0^+$ such that $p \in L^1(\Omega)$ and $\|p\|_1 = 1$ and for every measurable $F : \Omega \times \Omega \to \mathbb{R}_0^+$ there holds the inequality

$$\rho\Big(\int_\Omega p(t)F(\cdot,t)d\mu(t)\Big) \leq \int_\Omega p(t)\rho(F(\cdot,t))d\mu(t),$$

both sides of this inequality being meaningful. (For connections between J-convexity and convexity of a modular see e.g. [4]). Finally, we say that the pair $\{\rho,\psi\}$ with $\psi \in \Psi$ is *c-properly directed*, if there exists a number $c > 0$ such that for every $\lambda > 0$ there exists a number $C_\lambda > 0$ such that $\lambda^{-1}C_\lambda \geq c$ and for each $f \in L^0(\Omega)$, $f \geq 0$, there holds the inequality

$$\rho[C_\lambda \psi(t, |f(\cdot)|)] \leq \rho(\lambda f)$$

for all $t \in \Omega$ up to a fixed set $\Omega_0 \in \Sigma$ of measure zero. If $\psi(t,u) = u$, for every $t \in \Omega$, and $u \geq 0$, then $\{\rho,\psi\}$ is a c-properly directed pair with $c = 1$ and $C_\lambda = \lambda$. There holds the following embedding theorem:

THEOREM 2. *Let the modular ρ on $L^0(\Omega)$ be convex, finite, monotone, J-convex and strongly subbounded with a constant $C \geq 1$ with respect to the operation $+$. Let K be a Σ_π-regular, $(L,\psi)_0$-Lipschitz Carathéodory kernel function, $L : \Omega \to \mathbb{R}_0^+$, with $0 \neq L \in L^1(\Omega)$, $\psi \in \Psi$, $L(\cdot)\psi(\cdot,1) \in L^1(\Omega)$ and $\int_\Omega L(t)\psi(t,1)|f(t)|d\mu(t) < +\infty$ for every $f \in L^0_\rho(\Omega)$. Moreover, let $\{\rho,\psi\}$ be a c-properly directed pair. Denote by X any of the two spaces $L^0_\rho(\Omega), E^0_\rho(\Omega)$ and put $B_\rho(X) = \{f \in X : \|f\|_\rho \leq 1\}$. Let $Tf(s) = \int_\Omega K(t, f(t+s))d\mu(t)$ and $T_1 f = Tf - g$, where $g \in X$ is such that $\|g\|_1 \leq \theta < 1$. Finally let us suppose that $\|L\|_1 \leq cC^{-1}(1-\theta)$. Then T_1 maps $B_\rho(X)$ into itself.*

Proof. First, let us remark that by Theorem 1, $X \subset L^0_\rho(\Omega) \subset \mathrm{Dom}T$. Applying the monotonicity of ρ, the $(L,\psi)_0$-Lipschitz condition and J-convexity of ρ we obtain for arbitrary $\alpha > 0$

$$\begin{aligned}\rho(\alpha Tf) &\leq \rho\Big(\alpha \int_\Omega |K(t, f(t+\cdot))| d\mu(t)\Big)\\ &\leq \rho\Big(\int_\Omega p(t)\alpha\|L\|_1\psi(t, |f(t+\cdot)|)d\mu(t)\Big)\\ &\leq \int_\Omega p(t)\rho[\alpha\|L\|_1\psi(t, |f(t+\cdot)|)]d\mu(t).\end{aligned}$$

Let $\lambda > 0$ be fixed. Since the pair $\{\rho,\psi\}$ is c-properly directed, taking $\alpha > 0$ so small that $\alpha\|L\|_1 \leq C_\lambda$ we obtain

$$\rho[\alpha\|L\|_1\psi(t, |f(t+\cdot)|)] \leq \rho(\lambda|f(t+\cdot)|).$$

Hence, by the strong subboundedness of ρ with a constant $C \geq 1$, we obtain

$$\rho(\alpha Tf) \leq \int_{\Omega} p(t)\rho(\lambda|f(t+\cdot)|)d\mu(t)$$
$$\leq \int_{\Omega} p(t)\rho(\lambda Cf)d\mu(t) = \rho(\lambda Cf).$$

Taking $\alpha = C_\lambda/\|L\|_1$, we thus obtain

$$\rho\left(\frac{C_\lambda}{\|L\|_1}Tf\right) \leq \rho(\lambda Cf)$$

for $\lambda > 0$. But $\lambda^{-1}C_\lambda \geq c$ for $\lambda > 0$, so from the last inequality we obtain

$$\rho\left(\frac{c}{\|L\|_1}\lambda Tf\right) \leq \rho(\lambda Cf)$$

for $\lambda > 0$. If we put $\lambda = 1/u$, we get

$$\rho\left(\frac{cTf}{u\|L\|_1}\right) \leq \rho(u^{-1}Cf)$$

for $u > 0$. Hence we obtain the inequality

$$\|Tf\|_\rho \leq \frac{C}{c}\|L\|_1\|f\|_\rho. \tag{5}$$

Since $\|L\|_1 \leq cC^{-1}(1-\theta)$, we get

$$\|Tf\|_\rho \leq (1-\theta)\|f\|_\rho,$$

if $f \in X$. Now, supposing $f \in B_\rho$, we have $\|f\|_\rho \leq 1$ and so

$$\|T_1 f\|_\rho \leq (1-\theta) + \theta = 1.$$

This shows that $T_1 : B_\rho \to B_\rho$.

It is easily seen that from Proposition 1 and Theorem 2 there follows the following

THEOREM 3. *Let all assumptions of Theorem 2 be satisfied, and let the space $L^0_\rho(\Omega)$ be complete with respect to the norm $\|\cdot\|_\rho$ and measure bounded. Let $X = L^0_\rho(\Omega)$ or $X = E^0_\rho(\Omega)$. Finally, let us suppose that the image $TB_\rho(X)$ of the unit ball $B_\rho(X)$ in X by means of the operator defined by (2) is conditionally compact in X and T is continuous in $B_\rho(X)$. Then the integral equation (1) has a solution $f \in B_\rho(X)$.*

Here we formulate some sufficient conditions which guarantees the continuity of the operator $T : B_\rho(X) \to B_\rho(X)$ with conditionally compact range.

From now on we will work with a locally compact and σ-compact abelian group $(\Omega, +)$, endowed with its Haar measure μ. In order to do it we need to introduce some notions concerning the operator taken into consideration.

First of all let us remark that if ρ is a norm in $L^0_\rho(\Omega)$ and is monotone, then $L^0_\rho(\Omega)$ is a preideal space (see [14]). We may write

$$(Tf)(s) = \int_\Omega K(t, f(t+s))d\mu(t) = \int_\Omega K(t-s, f(t))d\mu(t), s \in \Omega,$$

and we will put

$$(\widetilde{T}f)(s) = \int_\Omega |K(t-s, f(t))|d\mu(t), s \in \Omega.$$

Moreover it is easy to see that $\widetilde{T} : B_\rho(X) \to B_\rho(X)$.

Now for a sequence of measurable sets $E_n \subset \Omega$, with $E_1 \supseteq E_2, \ldots$, $mes(\cap E_n) = 0$, we will write briefly $E_n \downarrow \emptyset$, while $E_n \uparrow E$ means that $E_n \subseteq E$ and $E \setminus E_n \downarrow \emptyset$. Moreover we will denote by P_{E_n} the projection operator on E_n, i.e. $P_{E_n}(g)(t,s) = \chi_{E_n}(s)g(t,s)$, for any measurable function g defined on $\Omega \times \Omega$. If the function g depends on s only, then simply $P_{E_n}(g)(s) = \chi_{E_n}(s)g(s)$.

Now, we say that the operator $T : B_\rho(X) \to B_\rho(X)$ is *uniformly regular* (see [14]) if there holds the following conditions:

(a) for any sequence $E_n \subseteq \Omega, E_n \downarrow \emptyset$,

$$\lim_{n \to +\infty} \sup_{f \in B_\rho(\Omega)} \left\| \int_{E_n} |K(t - \cdot, f(t))|d\mu(t) \right\|_\rho = 0$$

and

$$\lim_{n \to +\infty} \sup_{f \in B_\rho(\Omega), \|f\|_\infty \le 1} \|P_{E_n}\widetilde{T}f\|_\rho = 0;$$

(b) there exists sets $\Omega_k \uparrow \Omega$ such that for every $\varepsilon > 0$ and for any sequence $Q_n \downarrow \emptyset$ with $Q_n \subseteq \Omega_k \times \Omega_k$ (for some fixed k), there holds:

$$\lim_{n \to +\infty} \sup_{f \in B_\rho(\Omega), \|f\|_\infty \le 1} mes\{s \in \Omega : \int_\Omega |P_{Q_n}K(t-s, f(t))|dt \ge \varepsilon\} = 0.$$

Then, from [14] we may formulate the following theorem:

THEOREM 4. *Let X be measure bounded and let the operator $T : B_\rho(X) \to B_\rho(X)$ be uniformly regular. Then T is continuous with conditionally compact range.*

REMARKS.

1. In [9] there is a description of complete continuity of integral operators of Urysohn type in L^p-spaces. In particular, corresponding results can be obtained for convolution integral operators.
2. For Musielak-Orlicz spaces, there are known some necessary and sufficient conditions in order that a set $\mathcal{A} \subset X$ be conditionally compact in X, where $X = L^0_\rho(\Omega)$ or $X = E^0_\rho(\Omega)$, (see e.g. [8], [7], [12]).

3. In case of a general modular ρ on $L^0(\Omega)$ there are not known any necessary and sufficient conditions in order that a set $\mathcal{A} \subset X$ be conditionally compact in X, where $X = L^0_\rho(\Omega)$ or $X = E^0_\rho(\Omega)$.

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Carlo Bardaro
DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÁ DEGLI STUDI
Via Vanvitelli, 1
06123 PERUGIA, ITALY
Phone: (075) 5855034
Fax: (075) 5855024-5853822
E-mail: bardaro@unipg.it

Julian Musielak
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
Matejki 48/49
60-769 POZNAŃ, POLAND
Phone: (61) 8666615
Fax: (61) 8662992

Gianluca Vinti
DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÁ DEGLI STUDI
Via Vanvitelli, 1
06123 PERUGIA, ITALY
Phone: (075) 5855032
Fax: (075) 5855024-5853822
E-mail: mategian@unipg.it

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