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ENTIRE SOLUTIONS OF A FUNCTIONAL EQUATION OF PEXIDER TYPE

Abstract. We find all entire solutions of the functional equation

$$|f(x+y)|^2 + |g(x-y)|^2 = |h(x+\bar{y})|^2 + |k(x-\bar{y})|^2.$$

The equation

$$|f(x+y)|^2 + |g(x-y)|^2 = |h(x+\bar{y})|^2 + |k(x-\bar{y})|^2, \tag{1}$$

where x, y are complex variables and f, g, h, k are unknown entire functions, was considered by Hiroshi Haruki in [2] in 1988. A few years later Boo Rim Choe studied the more general functional equation

$$|f(x+y)| + |g(x-y)| = |h(x+\bar{y})| + |k(x-\bar{y})| \tag{2}$$

(cf. [1]). The purpose of this paper is to find all solutions of these equations.

THEOREM 1. *The only systems of entire solutions of (1) are the following*

$$(i)\begin{cases} f(z) = az + b \\ g(z) = cz + d \\ h(z) = pz + q \\ k(z) = rz + s, \end{cases}$$

where a, b, c, d, p, q, r, s are arbitrary complex constants satisfying

$$\begin{gathered} |a| = |c| = |p| = |r|, \quad |b|^2 + |d|^2 = |q|^2 + |s|^2, \\ a\bar{b} + c\bar{d} = p\bar{q} + r\bar{s}, \quad a\bar{b} - c\bar{d} = \bar{p}q - \bar{r}s; \end{gathered} \tag{3}$$

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$$(ii)\begin{cases} f(z) = a\exp(\lambda z) + b\exp(-\lambda z) \\ g(z) = c\exp(\lambda z) + d\exp(-\lambda z) \\ h(z) = p\exp(\lambda z) + q\exp(-\lambda z) \\ k(z) = r\exp(\lambda z) + s\exp(-\lambda z), \end{cases}$$

where λ is an arbitrary real constant and a, b, c, d, p, q, r, s are arbitrary complex constants satisfying

(4) $|a| = |p|, \quad |b| = |q|, \quad |c| = |r|, \quad |d| = |s|, \quad a\bar{b} = r\bar{s} \quad c\bar{d} = p\bar{q};$

$$(iii)\begin{cases} f(z) = a\exp(i\lambda z) + b\exp(-i\lambda z) \\ g(z) = c\exp(i\lambda z) + d\exp(-i\lambda z) \\ h(z) = p\exp(i\lambda z) + q\exp(-i\lambda z) \\ k(z) = r\exp(i\lambda z) + s\exp(-i\lambda z), \end{cases}$$

where λ is an arbitrary real constant and a, b, c, d, p, q, r, s are arbitrary complex constants satisfying

(5) $|a| = |r|, \quad |b| = |s|, \quad |c| = |p|, \quad |d| = |q|, \quad a\bar{b} = p\bar{q} \quad c\bar{d} = r\bar{s};$

$$(iv)\begin{cases} f(z) = A\exp(\lambda z) \\ g(z) = B\exp(\mu z) \\ h(z) = C\exp(\lambda z) \\ k(z) = D\exp(\mu z), \end{cases}$$

where λ, μ are arbitrary real constants, A, B, C, D are arbitrary complex constants satisfying the conditions

(6) $$|A| = |C|, \quad |B| = |D|;$$

$$(v)\begin{cases} f(z) = A\exp(i\lambda z) \\ g(z) = B\exp(i\mu z) \\ h(z) = C\exp(i\mu z) \\ k(z) = D\exp(i\lambda z), \end{cases}$$

where λ, μ are arbitrary real constants, A, B, C, D are arbitrary complex constants satisfying the conditions

(7) $$|A| = |D|, \quad |B| = |C|.$$

In paper [2] corresponding Theorem 1 does not contain systems (iv) and (v). Observe that e.g. the system

$$\begin{cases} f(z) = \exp(z) \\ g(z) \equiv 1 \\ h(z) = \exp(z) \\ k(z) \equiv 1 \end{cases}$$

satisfies equation (1) though it has neither the form (*i*) nor (*ii*) nor (*iii*).

To prove Theorem 1 we shall need some lemmas which can be found in Hiroshi Haruki's paper [2].

LEMMA 1 (see Lemma 5 in [2]).

(*i*) *The only system of entire solutions of the equation*

$$|f(x+y)| = |h(x+\bar{y})|$$

is

$$\begin{cases} f(z) = A\exp(\lambda z) \\ h(z) = B\exp(\lambda z), \end{cases}$$

where λ *is an arbitrary real constant and* A, B *are arbitrary complex constants satisfying* $|A| = |B|$.

(*ii*) *The only system of entire solution of the equation*

$$|f(x+y)| = |k(x-\bar{y})|$$

is

$$\begin{cases} f(z) = A\exp(i\lambda z) \\ k(z) = B\exp(i\lambda z), \end{cases}$$

where λ *is an arbitrary real constant and* A, B *are arbitrary complex constants satisfying* $|A| = |B|$.

LEMMA 2 (see Lemma 6 in [2]). *The only systems of entire solutions of the equation*

$$|f(x+y)|^2 + |g(x-y)|^2 = |h(x+\bar{y})|^2 \tag{8}$$

are

$$(a) \begin{cases} f(z) \equiv p \\ g(z) \equiv q \\ h(z) \equiv r, \end{cases}$$

where p, q, r *are arbitrary complex constants satisfying*

$$|p|^2 + |q|^2 = |r|^2, \tag{9}$$

$$(b) \begin{cases} f(z) = A\exp(\lambda z) \\ g(z) \equiv 0 \\ h(z) = B\exp(\lambda z) \end{cases}$$

and

$$(c)\begin{cases} f(z) \equiv 0 \\ g(z) = C\exp(i\mu z) \\ h(z) = D\exp(i\mu z), \end{cases}$$

where λ, μ are arbitrary real constants and A, B, C, D are arbitrary complex constants satisfying

$$|A| = |B| \quad \text{and} \quad |C| = |D|. \tag{10}$$

LEMMA 3 (see [2], p. 9). *If f, g, h, k is a system of entire solutions of equation (1), then there exist real constants $\alpha, \beta, \gamma, \delta$ such that*

$$f''(z) = \alpha f(z), \quad g''(z) = \beta g(z), \quad h''(z) = \gamma h(z), \quad k''(z) = \delta k(z). \tag{11}$$

LEMMA 4 (see [2], p. 11). *Let f, g, h, k be a system of entire solutions of equation (1). Then the following equations are satisfied:*

$$\alpha|f(x+y)|^2 + \beta|g(x-y)|^2 = \gamma|h(x+\bar{y})|^2 + \delta|k(x-\bar{y})|^2, \tag{12}$$

$$(\beta-\alpha)|g(x-y)|^2 = (\gamma-\alpha)|h(x+\bar{y})|^2 + (\delta-\alpha)|k(x-\bar{y})|^2, \tag{13}$$

$$(\gamma-\alpha)(\gamma-\beta)|h(x+\bar{y})|^2 + (\delta-\alpha)(\delta-\beta)|k(x-\bar{y})|^2 = 0, \tag{14}$$

$$(\delta-\alpha)(\delta-\beta)(\delta-\gamma)|k(x-\bar{y})|^2 = 0, \tag{15}$$

where $\alpha, \beta, \gamma, \delta$ are the real constants which appear in Lemma 3.

LEMMA 5. *Let f, g, h, k be a system of entire solutions of equation (1). Then the following equation is satisfied:*

$$(\gamma-\alpha)(\delta-\gamma)|h(x+\bar{y})|^2 - (\delta-\beta)(\beta-\alpha)|g(x-y)|^2 = 0, \tag{16}$$

where $\alpha, \beta, \gamma, \delta$ are the real constants which appear in Lemma 3.

The proof of the above Lemma is similar to that of Lemma 4 (see [2], pp. 11–12).

Proof of Theorem 1. It is not difficult to check that systems (i),(ii),(iii) with conditions (3),(4),(5), respectively, satisfy equation (1). We observe that systems (iv) and (v) with conditions (6) and (7), respectively, on A, B, C, D, where λ, μ are real numbers, also satisfy this equation. Thus it is sufficient to show that every system of entire solutions of (1) is one of the form (i)-(v).

Suppose that entire functions f, g, h, k satisfy equation (1). Let $\alpha, \beta, \gamma, \delta$ be real constants such that linear differential equations (11) hold. We distinguish three cases:

(I) $\alpha = \beta = \gamma = \delta = 0$;
(II) $\alpha = \beta = \gamma = \delta \neq 0$;

(III) $\alpha, \beta, \gamma, \delta$ are not all equal (see [2], p. 8).

Case(I). By Lemma 3 f, g, h, k are linear functions of the form (i) in the theorem. It is easy to verify that conditions (3) are satisfied (cf. [2], p. 10).

Case(II). In this case also by Lemma 3 the functions f, g, h, k are of form (ii) or (iii) in our theorem and constants a, b, c, d, p, q, r, s satisfy conditions (4) or (5), respectively.

In the case (III) we shall use equation (15). Thus we have four cases: $\alpha = \delta$ or $\beta = \delta$ or $\gamma = \delta$ or $k(z) \equiv 0$.

Case A. The case when $\alpha = \delta$. By (14) we get

$$(\gamma - \alpha)(\gamma - \beta)|h(x + \bar{y})|^2 = 0.$$

Hence we have

$$\alpha = \gamma \quad \text{or} \quad \beta = \gamma \quad \text{or} \quad h(z) \equiv 0.$$

In the sequel we consider three subcases:

Case A1: $\alpha = \gamma$. In this case $\alpha = \gamma = \delta$. With respect to (III), $\alpha \neq \beta$. By (13), $g(z) \equiv 0$, so with respect to (1) replacing y by $\bar{y}$ we get

$$|h(x + y)|^2 + |k(x - y)|^2 = |f(x + \bar{y})|^2.$$

Lemma 2 leads to the following systems of solutions:

$$(i) \begin{cases} f(z) \equiv r \\ g(z) \equiv 0 \\ h(z) \equiv p \\ k(z) \equiv q, \end{cases}$$

where p, q, r are complex constants satysfying (9);

$$(iv) \begin{cases} f(z) = A \exp(\lambda z) \\ g(z) \equiv 0 \\ h(z) = B \exp(\lambda z) \\ k(z) \equiv 0 \end{cases}$$

and

$$(v) \begin{cases} f(z) = C \exp(i\mu z) \\ g(z) \equiv 0 \\ h(z) \equiv 0 \\ k(z) = D \exp(i\mu z), \end{cases}$$

where λ, μ are real constants and A, B, C, D are complex constants such that $|A| = |B|$ and $|C| = |D|$.

Case A2: $\beta = \gamma$. This case is not considered in paper [2]. Since $\alpha = \delta, \beta = \gamma$ and $\alpha \neq \beta$, replacing y by $-y$ we obtain by (16)

$$|g(x+y)| = |h(x-\bar{y})|.$$

Now we can apply Lemma 1(ii). Hence

$$g(z) = A\exp(i\lambda z)$$
$$h(z) = B\exp(i\lambda z),$$

where λ is an real constant and A, B are complex constants such that $|A| = |B|$. On the other hand

$$|f(x+y)| = |k(x-\bar{y})|$$

in virtue of (1). Again by the same Lemma

$$f(z) = C\exp(i\mu z)$$
$$k(z) = D\exp(i\mu z),$$

where μ is a real constant and C, D are complex constants such that $|C| = |D|$. Thus in this case we get a system of solutions of (1) in the form (v).

Case A3. $h(z) \equiv 0$. If we replace y by $-y$ in (1) then we get

$$|g(x+y)|^2 + |f(x-y)|^2 = |k(x+\bar{y})|^2.$$

Lemma 2 yields the following systems of solutions of (1):

$$(i)\begin{cases} f(z) \equiv q \\ g(z) \equiv p \\ h(z) \equiv 0 \\ k(z) \equiv r, \end{cases}$$

where p, q, r are complex constants satisfying (9);

$$(iv)\begin{cases} f(z) \equiv 0 \\ g(z) = A\exp(\lambda z) \\ h(z) \equiv 0 \\ k(z) = B\exp(\lambda z) \end{cases}$$

and

$$(v)\begin{cases} f(z) = C\exp(i\mu z) \\ g(z) \equiv 0 \\ h(z) \equiv 0 \\ k(z) = D\exp(i\mu z), \end{cases}$$

where λ, μ are real constants and A, B, C, D are complex constants such that $|A| = |B|$ and $|C| = |D|$.

Case B. The case when $\beta = \delta$. Similarly as in Case A we have by (14)

$$(\gamma - \alpha)(\gamma - \beta)|h(x + \bar{y})|^2 = 0.$$

We shall study three subcases:

Case B1: $\alpha = \gamma$. Of course $\alpha \neq \beta$. Hence by (13) we derive

$$|g(x + y) = |k(x + \bar{y})|, \tag{17}$$

whence by Lemma 1(i)

$$g(z) = A\exp(\lambda z)$$

$$k(z) = B\exp(\lambda z),$$

where λ is a real constant and A, B are complex constants such that $|A| = |B|$. On the other hand (17) and (1) yield

$$|f(x + y)| = |h(x + \bar{y})|.$$

Again by Lemma 1(i)

$$f(z) = C\exp(\mu z)$$

$$h(z) = D\exp(\mu z),$$

where μ is a real constant and C, D are complex constants such that $|C| = |D|$. So in this case we infer the system of the form (iv).

Case B2: $\beta = \gamma$. In this case $\alpha \neq \beta$ since $\alpha = \beta = \gamma = \delta$ does not hold. Dividing (13) by $\beta - \alpha$ and replacing y by $-\bar{y}$ we obtain

$$|k(x + y)|^2 + |h(x - y)|^2 = |g(x + \bar{y})|^2.$$

Lemma 2 provides the following systems of solutions of (1)

$$(i) \begin{cases} f(z) \equiv 0 \\ g(z) \equiv r \\ h(z) \equiv q \\ k(z) \equiv p, \end{cases}$$

where p, q, r are complex constants satisfying (9);

$$(iv) \begin{cases} f(z) \equiv 0 \\ g(z) = A\exp(\lambda z) \\ h(z) \equiv 0 \\ k(z) = B\exp(\lambda z) \end{cases}$$

and

$$(v)\begin{cases} f(z) \equiv 0 \\ g(z) = C\exp(i\mu z) \\ h(z) = D\exp(i\mu z) \\ k(z) \equiv 0, \end{cases}$$

where λ, μ are real constants and A, B, C, D are complex constants such that $|A| = |B|$ and $|C| = |D|$.

Case B3: $h(z) \equiv 0$. This case is the same as Case A3.

Case C. The case when $\gamma = \delta$. Using (16) we have

$$(\delta - \beta)(\beta - \alpha)|g(x - y)|^2 = 0.$$

Thus we shall study three subcases:

Case C1. The case when $\alpha = \beta$. Since $\alpha \neq \delta$, by (14) we obtain

$$|h(x + \bar{y})|^2 + |k(x - \bar{y})|^2 = 0,$$

whence $h(z) \equiv 0$ and $k(z) \equiv 0$. By (1) we have also $f(z) \equiv 0$ and $g(z) \equiv 0$.

Case C2. The case when $\beta = \delta$. Since also $\gamma = \delta$, $\alpha \neq \delta$ we infer by (13) $f(z) \equiv 0$. This case has already been considered in Case B2.

Case C3. The case when $g(z) \equiv 0$. This case is the same as Case A1.

Case D. The case when $k(z) \equiv 0$. Equation (1) leads to the new one

$$|f(x + y)|^2 + |g(x - y)|^2 = |h(x + \bar{y})|^2.$$

Lemma 2 states that the functions f, g, h, k may create one of the following systems of solutions:

$$(i)\begin{cases} f(z) \equiv p \\ g(z) \equiv q \\ h(z) \equiv r \\ k(z) \equiv 0, \end{cases}$$

where p, q, r are complex constants satisfying (9);

$$(iv)\begin{cases} f(z) = A\exp(\lambda z) \\ g(z) \equiv 0 \\ h(z) = B\exp(\lambda z) \\ k(z) \equiv 0; \end{cases}$$

$$(v)\begin{cases} f(z) \equiv 0 \\ g(z) = C\exp(i\mu z) \\ h(z) = D\exp(i\mu z) \\ k(z) \equiv 0, \end{cases}$$

where λ, μ are some real constants and A, B, C, D are some complex constants satisfying $|A| = |B|$ and $|C| = |D|$. ∎

The example from the above:

$$\begin{cases} f(z) = \exp(z) \\ g(z) \equiv 1 \\ h(z) = \exp(z) \\ k(z) \equiv 1 \end{cases}$$

shows that Theorem 1 of paper [1] does not contain all entire solutions of equation (2). To obtain all such solutions we have to add systems (iv) and (v). More exactly, Theorem 1 in paper [1] should read as follows.

THEOREM 2. *The only systems of entire solutions of equation* (2) *are the following*

$$(i) \begin{cases} f(z) = (az + b)^2 \\ g(z) = (cz + d)^2 \\ h(z) = (pz + q)^2 \\ k(z) = (rz + s)^2, \end{cases}$$

where a, b, c, d, p, q, r, s *are arbitrary complex constants satisfying* (3);

$$(ii) \begin{cases} f(z) = [a\exp(\lambda z) + b\exp(-\lambda z)]^2 \\ g(z) = [c\exp(\lambda z) + d\exp(-\lambda z)]^2 \\ h(z) = [p\exp(\lambda z) + q\exp(-\lambda z)]^2 \\ k(z) = [r\exp(\lambda z) + s\exp(-\lambda z)]^2, \end{cases}$$

where λ *is an arbitrary real constant and* a, b, c, d, p, q, r, s *are arbitrary complex constants satisfying* (4);

$$(iii) \begin{cases} f(z) = [a\exp(\lambda iz) + b\exp(-\lambda iz)]^2 \\ g(z) = [c\exp(\lambda iz) + d\exp(-\lambda iz)]^2 \\ h(z) = [p\exp(\lambda iz) + q\exp(-\lambda iz)]^2 \\ k(z) = [r\exp(\lambda iz) + s\exp(-\lambda iz)]^2, \end{cases}$$

where λ *is an arbitrary real constant and* a, b, c, d, p, q, r, s *are arbitrary complex constants satisfying* (5);

$$(iv) \begin{cases} f(z) = A\exp(\lambda z) \\ g(z) = B\exp(\mu z) \\ h(z) = C\exp(\lambda z) \\ k(z) = D\exp(\mu z), \end{cases}$$

where λ, μ *are arbitrary real constants,* A, B, C, D *are arbitrary complex constants satisfying conditions* (6);

$$(v)\begin{cases} f(z) = A\exp(i\lambda z) \\ g(z) = B\exp(i\mu z) \\ h(z) = C\exp(i\mu z) \\ k(z) = D\exp(i\lambda z), \end{cases}$$

where λ, μ *are arbitrary real constants,* A, B, C, D *are arbitrary complex constants satisfying conditions* (7).

References

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