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STABILITY OF DIFFERENCE EQUATIONS GENERATED BY QUASILINEAR DIFFERENTIAL FUNCTIONAL PROBLEMS

Abstract. The paper deals with the initial boundary value problem for quasilinear first order partial differential functional equations. A general class of difference methods for the problem is constructed. Theorems on the error estimate of approximate solutions for difference functional equations are presented. The convergence results are proved by means of consistency and stability arguments. Numerical example is given.

1. Introduction

For any metric spaces U and V we denote by $C(U,V)$ the class of all continuous functions defined on U and taking values in V. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. For $x = (x_1, \ldots, x_n) \in R^n$ we put $\|x\| = |x_1| + \ldots + |x_n|$. Let $a > 0$, $\tau_0 \in R_+$, $R_+ = [0, +\infty)$, $\tau = (\tau_1, \ldots, \tau_n) \in R^n_+$ and $b = (b_1, \ldots, b_n) \in R^n$ be given, where $b_i > 0$ for $1 \leqslant i \leqslant n$. Let $c = (c_1, \ldots, c_n) = b + \tau$. Define the sets

$$E = [0,a] \times (-b,b), \qquad D = [-\tau_0, 0] \times [-\tau, \tau],$$

and

$$E_0 = [-\tau_0, 0] \times [-c, c], \qquad \partial_0 E = ([0,a] \times [-c,c]) \setminus E, \qquad E^* = E_0 \cup E \cup \partial_0 E.$$

Given a function $z\colon E^* \to R$ and a point $(t,x) \in \overline{E}$, we consider the function $z_{(t,x)}\colon D \to R$ defined by

$$z_{(t,x)}(s,y) = z(t+s, x+y), \qquad (s,y) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t-\tau_0, t] \times [x-\tau, x+\tau]$ and this restriction is shifted to the set D. For a function $w \in C(D,R)$ we put

$$\|w\|_D = \max\{\, |w(t,x)|\colon \quad (t,x) \in D \,\}.$$

Assume that

$$\varrho: E \times C(D,R) \to R^n, \qquad \varrho = (\varrho_1, \dots, \varrho_n), \qquad f: E \times C(D,R) \to R$$

are given function of the variables (t,x,w). Given a function $\varphi: E_0 \cup \partial_0 E \to R$, we consider the quasilinear differential functional equation

$$\partial_t z(t,x) = \sum_{j=1}^{n} \varrho_j(t,x,z_{(t,x)})\, \partial_{x_j} z(t,x) + f(t,x,z_{(t,x)}) \tag{1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \qquad \text{for } (t,x) \in E_0 \cup \partial_0 E. \tag{2}$$

We consider classical solution of the above problem
In recent years a number of papers concerned with difference methods for first order partial differential equations ([3], [6], [8], [10]) and for functional differential equations ([1], [4], [5], [12]) were published.

A method on difference inequalities and theorems on reccurent inequalities are used in the investigation of stability.

The results presented in [1], [4], [5], [12] are not applicable to problem (1), (2). In the paper we construct a general class of difference methods for this problem. We establish some estimates for the difference between the exact and approximate solutions of the difference functional equations of the Volterra type with initial boundary conditions. These estimates are basic tools in the investigations of the stability of difference methods. We use in the paper these general ideas for finite difference equations which were introduced in [2], [9], [11].

Differential equations with a deviated argument and integral differential problems can be obtained from (1), (2) by a specification of given operators.

2. Difference functional equations

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of positive integers and integers respectively. For $x, \bar{x} \in R^n$, $x = (x_1, \dots, x_n)$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$, we write $x * \bar{x} = (x_1\bar{x}_1, \dots, x_n\bar{x}_n)$. We define a mesh on the set E^* in the following way. Suppose that $h = (h_0, h')$ where $h' = (h_1, \dots, h_n)$ stand for steps of the mesh. Denote by Δ the set of all $h = (h_0, h')$ such that there exist $\tilde{N}_0 \in \mathbb{Z}$ and $N = (N_1, \dots, N_n) \in \mathbb{Z}^n$ with the properties: $N_0 h_0 = \tau_0$ and $N * h' = \tau$. We assume that $\Delta \neq \emptyset$ and that there exist a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Delta$ such that $\lim_{j\to\infty} h^{(j)} = 0$. For $h \in \Delta$ we put $|h| = h_0 + h_1 + \cdots + h_n$. We define nodal points as follows:

$$t^{(i)} = ih_0, \qquad x^{(m)} = m * h', \qquad x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)})$$

where $(i,m) \in \mathbb{Z}^{1+n}$. There exists $N_0 \in \mathbb{N}$ such that $N_0 h_0 \leqslant a < (N_0+1)h_0$. Let

$$R_h^{1+n} = \{(t^{(i)}, x^{(m)}): \quad (i,m) \in \mathbb{Z}^{(1+n)}\}$$

and

$$D_h = D \cap R_h^{1+n}, \qquad E_h = E \cap R_h^{1+n},$$
$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \qquad E_{0.h} = E_0 \cap R_h^{1+n}, \qquad E_h^* = E_h \cup E_{0.h} \cup \partial_0 E_h.$$

For a function $z: E_h^* \to R$ we write $z^{(i,m)} = z(t^{(i)}, x^{(m)})$. For the above z and for a point $(t^{(i)}, x^{(m)}) \in E_h$ we define the function $z_{[i,m]}: D_h \to R$ by the formula

$$z_{[i,m]}(s,y) = z(t^{(i)} + s, x^{(m)} + y), \qquad (s,y) \in D_h.$$

The function $z_{[i,m]}$ is the restriction of z to the set $([t^{(i)} - \tau_0, t^{(i)}] \times [x^{(m)} - \tau, x^{(m)} + \tau]) \cap R_h^{1+n}$ and this restriction is shifted to the set D_h.
For a function $w: D_h \to R$ we put

$$\|w\|_h = \max\{\, |w^{(i,m)}|: \; (t^{(i)}, x^{(m)}) \in D_h \,\}.$$

Let $e_j = (0,\ldots,0,1,0,\ldots,0) \in R^n$ with 1 standing on j-th place and let $z: E_h^* \to R$. We shall consider difference operators δ_0, $\delta = (\delta_1,\ldots,\delta_n)$ defined in the following way

$$\delta_0 z^{(i,m)} = \frac{1}{h_0}[\, z^{(i+1,m)} - A z^{(i,m)}\,],$$

where

$$A z^{(i,m)} = \frac{1}{2n}\sum_{j=1}^{n}(\, z^{(i,m+e_j)} + z^{(i,m-e_j)}\,)$$

and

$$\delta_j z^{(i,m)} = \frac{1}{2h_j}[z^{(i,m+e_j)} - z^{(i,m-e_j)}], \qquad 1 \leqslant j \leqslant n.$$

Let

$$E_h' = \{\, (t^{(i)}, x^{(m)}) \in E_h: \; (t^{(i)} + h_0, x^{(m)}) \in E_h \,\}$$

and now denote by as $\mathfrak{F}(D_h, R)$ denote the set of all functions $w: D_h \to R$. Suppose that

$$\varrho_h = (\varrho_{h.1},\ldots,\varrho_{h.n}): E_h' \times \mathfrak{F}(D_h,R) \to R,$$
$$f_h: E_h' \times \mathfrak{F}(D_h,R) \to R, \qquad \varphi_h: E_{0.h} \cup \partial_0 E_h \to R$$

are given functions. Let the operator F_h be defined by

$$(3) \quad F_h[z]^{(i,m)} = \sum_{j=1}^{n} \varrho_{h.j}(\, t^{(i)}, x^{(m)}, z_{[i,m]})\, \delta_j z^{(i,m)} + f_h(\, t^{(i)}, x^{(m)}, z_{[i,m]}).$$

We will approximate solutions of problem (1), (2) by means of solutions of the difference equation

$$\delta_0\, z^{(i,m)} = F_h[z]^{(i,m)} \tag{4}$$

with the initial boundary condition

$$z^{(i,m)} = \varphi_h^{(i,m)} \qquad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{5}$$

There exists exactly one solution $u_h\colon E^* \to R$ of problem (4), (5). We need to know what is the relation between the solution u_h of (4), (5) and a function $v_h\colon E_h \to R$ satisfying the condition

$$|\delta_0 v_h^{(i,m)} - F_h[v_h]^{(i,m)}| \leqslant \alpha(h) \qquad \text{on } E'_h \tag{6}$$

and

$$|v_h^{(i,m)} - \varphi_h^{(i,m)}| \leqslant \alpha_0(h) \qquad \text{on } E_{0.h} \cup \partial_0 E_h, \tag{7}$$

where

$$\alpha, \alpha_0\colon \Delta \to R_+ \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0, \quad \lim_{h\to 0} \alpha(h) = 0.$$

The function v_h satisfying the above relation is considered as an approximate solution of problem (4), (5).

Assumption H$[\varrho_h, f_h]$. *Suppose that the function $\varrho_h\colon E'_h \times \mathfrak{F}(D_h, R) \to R^n$ and $f_h\colon E'_h \times \mathfrak{F}(D_h, R) \to R$ satisfy the following Lipshitz condition*

$$\|\varrho_h(t^{(i)}, x^{(m)}, w) - \varrho_h(t^{(i)}, x^{(m)}, \bar{w})\| \leqslant L\|w - \bar{w}\|_h,$$
$$|f_h(t^{(i)}, x^{(m)}, w) - f_h(t^{(i)}, x^{(m)}, \bar{w})| \leqslant L\|w - \bar{w}\|_h,$$

where $L \in R_+$.

Theorem 1. *Suppose that Assummption H$[\varrho_h, f_h]$ is satisfied and*

1) $h \in \Delta$ and

$$\frac{1}{n} - \frac{h_0}{h_j}|\varrho_{h\cdot j}(t, x, w)| \geqslant 0 \qquad \text{on } E'_h \times \mathfrak{F}(D_h, R), \quad 1 \leqslant j \leqslant n, \tag{8}$$

2) $u_h\colon E^*_h \to R$ is the solution of problem (4), (5) and the function $v_h\colon E^*_h \to R$ satisfies relations (6), (7) and there is $c_0 \in R_+$ such that

$$|\delta_j v_h^{(i,m)}| \leqslant c_0 \qquad \text{on } E_h \text{ for } 1 \leqslant j \leqslant n.$$

Under these assumptions we have

$$|u_h^{(i,m)} - v_h^{(i,m)}| \leqslant \alpha_0(h) e^{\tilde{L}a} + \alpha(h)\frac{e^{\tilde{L}a} - 1}{\tilde{L}} \qquad \text{on } E_h \tag{9}$$

if $L > 0$ and

$$|u_h^{(i,m)} - v_h^{(i,m)}| \leqslant \alpha_0(h) + a\alpha(h) \qquad \text{on } E_h \tag{10}$$

if $L = 0$, where $\tilde{L} = L(1 + c_0)$.

Proof. Let $\Gamma: E'_h \to R$ and $\Gamma_{0\cdot h}: E_{0\cdot h} \cup \partial_0 E_h \to R$ be functions defined by the relations

(11) $$\delta_0 v_h^{(i,m)} = F_h[v_h]^{(i,m)} + \Gamma_h^{(i,m)} \qquad \text{on } E'_h$$

and

(12) $$v_h^{(i,m)} = \varphi_h^{(i,m)} + \Gamma_{0\cdot h}^{(i,m)} \qquad \text{on } E_{0\cdot h} \cup \partial_0 E_h.$$

Then

(13) $$\begin{aligned} |\,\Gamma_h^{(i,m)}\,| &\leqslant \alpha(h) \qquad \text{on } E'_h, \\ |\,\Gamma_{0\cdot h}^{(i,m)}\,| &\leqslant \alpha_0(h) \qquad \text{on } E_{0\cdot h} \cup \partial_0 E_h \end{aligned}$$

and the function $w_h = u_h - v_h$ satisfies the difference functional equation

(14) $$\begin{aligned} \delta_0 w_h^{(i,m)} &= \sum_{j=1}^{n} \varrho_{h\cdot j}(\, t^{(i)}, x^{(m)}, (u_h)_{[i,m]}\,)\, \delta_j w_h^{(i,m)} + \\ &+ \sum_{j=1}^{n} \Big[\, \varrho_{h\cdot j}(\, t^{(i)}, x^{(m)}, (u_h)_{[i,m]}\,) - \varrho_{h\cdot j}(\, t^{(i)}, x^{(m)}, (v_h)_{[i,m]}\,) \Big] \delta_j v_h^{(i,m)} + \\ &+ f_h(\, t^{(i)}, x^{(m)}, (u_h)_{[i,m]}\,) - f_h(\, t^{(i)}, x^{(m)}, (v_h)_{[i,m]}\,) - \Gamma_h^{(i,m)}. \end{aligned}$$

Write

(15) $$P^{(i,m)}[z] = (\, t^{(i)}, x^{(m)}, z_{[i,m]}\,)$$

and put

(16) $$\begin{aligned} \Lambda_h^{(i,m)} = &\sum_{j=1}^{n} \Big[\, \varrho_{h\cdot j}(\, P^{(i,m)}[u_h]\,) - \varrho_{h\cdot j}(\, P^{(i,m)}[v_h]\,) \Big]\, \delta_j v_h^{(i,m)} + \\ &+ f_h(\, P^{(i,m)}[u_h]\,) - f_h(\, P^{(i,m)}[v_h]\,) - \Gamma_h^{(i,m)}. \end{aligned}$$

From (14) it follows that the function w_h satisfies the recursive equation

(17) $$\begin{aligned} w_h^{(i+1,m)} = &\frac{1}{2} \sum_{j=1}^{n} w_h^{(i,m+e_j)} \Big[\, \frac{1}{n} + \frac{h_0}{h_j} \varrho_{h\cdot j}(\, P^{(i,m)}[u_h]\,) \Big] + \\ &+ \frac{1}{2} \sum_{j=1}^{n} w_h^{(i,m-e_j)} \Big[\, \frac{1}{n} - \frac{h_0}{h_j} \varrho_{h\cdot j}(\, P^{(i,m)}[u_h]\,) \Big] + h_0 \Lambda_h^{(i,m)}. \end{aligned}$$

Denote by

(18) $$\omega_h^{(i)} = \max\{\, |w_h^{(j,m)}|: \quad (t^{(j)}, x^{(m)} \in E_h^* \cap ([-\tau_0, t^{(i)}] \times R^n)\,\}, \qquad 0 \leqslant i \leqslant N_0.$$

The term Λ_h can be estimated as follows

(19) $$|\Lambda_h^{(i,m)}| \leqslant \omega_h^{(i)} L\,(1 + c_0) + \alpha(h) \qquad \text{on } E'_h.$$

From (7), (17) and (19) we conclude that the function ω_h satisfies the reccursive inequality

$$\omega_h^{(i+1)} \leqslant \omega_h^{(i)}(1+\tilde{L}h_0)+h_0\alpha(h), \qquad 0 \leqslant i \leqslant N_0-1, \tag{20}$$

and

$$\omega_h^{(0)} \leqslant \alpha_0(h). \tag{21}$$

Consider the difference equation

$$\eta^{(i+1)} = \eta^{(i)}(1+\tilde{L}h_0)+h_0\alpha(h), \qquad 0 \leqslant i \leqslant N_0-1, \tag{22}$$

with the initial condition

$$\eta^{(0)} = \alpha_0(h) \tag{23}$$

and its solution

$$\begin{aligned} &\eta_h^{(0)} = \alpha_0(h), \\ &\eta_h^{(i)} = \alpha_0(h)(1+\tilde{L}h_0)^i + h_0\alpha(h)\sum_{j=0}^{i-1}(1+\tilde{L}h_0)^j, \qquad 1 \leqslant i \leqslant N_0. \end{aligned} \tag{24}$$

From (20), (21) it follows that

$$\omega_h^{(i)} \leqslant \eta_h^{(i)}, \qquad 0 \leqslant i \leqslant N_0.$$

This gives (9), (10) and Theorem 1 is proved. ∎

We shall consider now a difference functional problem (4), (5) where $F_h = (F_{h\cdot 1}, \ldots, F_{h\cdot n})$ is given by (3) and the difference operator δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ are calculated in the following way:

$$\delta_0 z^{(i,m)} = \frac{1}{h_0}[z^{(i+1,m)} - z^{(i,m)}], \tag{25}$$

$$\delta_j z^{(i,m)} = \frac{1}{h_j}[z^{(i,m+e_j)} - z^{(i,m)}] \qquad \text{if} \quad \varrho_{h\cdot j}(t^{(i)}, x^{(m)}, z_{[i,m]}) \geqslant 0, \tag{26}$$

$$\delta_j z^{(i,m)} = \frac{1}{h_j}[z^{(i,m)} - z^{(i,m-e_j)}] \qquad \text{if} \quad \varrho_{h\cdot j}(t^{(i)}, x^{(m)}, z_{[i,m]}) \leqslant 0. \tag{27}$$

It can be easily seen that the problem (4), (5) with difference operators defined by (2)-(2) has exactly one solution $u_h\colon E_h^* \to R$.

Now we give an estimate between the exact and approximate solution of the above problem.

THEOREM 2. *Suppose that Assumption $H[\varrho_h, f_h]$ is satisfied and*

1) $h \in \Delta$ and

$$1-h_0\sum_{j=1}^{n}\frac{1}{h_j}|\varrho_{h\cdot j}(t,x,w)| \geqslant 0 \qquad \text{on } E_h' \times \mathfrak{F}(D_h, R), \quad 1 \leqslant j \leqslant n, \tag{28}$$

2) $u_h: E_h^* \to R$ is the solution of the problem (4), (5) with δ_0 and δ given by (2)-(2) and the function $v_h: E_h^* \to R$ satisfies relations (6), (7) and there is $c_0 \in R_+$ such that

$$|\,\delta_j v_h^{(i,m)}\,| \leqslant c_0 \qquad \text{on } E_h,\ \ 1 \leqslant j \leqslant n.$$

Under these assumptions we have

$$|\,u_h^{(i,m)} - v_h^{(i,m)}\,| \leqslant \alpha_0(h)\, e^{\tilde{L}a} + \alpha(h)\frac{e^{\tilde{L}a}-1}{\tilde{L}} \qquad \text{on } E_h \tag{29}$$

if $L > 0$ and

$$|\,u_h^{(i,m)} - v_h^{(i,m)}\,| \leqslant \alpha_0(h) + a\alpha(h) \qquad \text{on } E_h \tag{30}$$

if $L = 0$, where $\tilde{L} = L(1 + c_0)$.

Proof. Let $\Gamma_h: E_h' \to R$ and $\Gamma_{0.h}: E_{0.h} \cup \partial_0 E_h \to R$ be the functions defined by (11) and (12), respectively. Then the estimate (13) holds and the function $w_h = u_h - v_h$ satisfies the difference functional equation

$$\begin{aligned} w_h^{(i+1,m)} &= w_h^{(i,m)} + h_0 \sum_{j=1}^{n} \varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,)\,\delta_j w_h^{(i,m)} \\ &+ h_0 \sum_{j=1}^{n} [\,\varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,) - \varrho_{h\cdot j}(\,P^{(i,m)}[v_h]\,)\,]\,\delta_j v_h^{(i,m)} \\ &+ h_0[\,f_h(\,P^{(i,m)}[u_h]\,) - f_h(\,P^{(i,m)}[v_h]\,)\,] - h_0 \Gamma_h^{(i,m)}, \qquad (t^{(i)}, x^{(m)}) \in E_h', \end{aligned}$$

where $P^{(i,m)}[z]$ is given by (15). Write

$$I_+^{(i,m)} = \{\,j:\ 1 \leqslant j \leqslant n,\quad \varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,) \geqslant 0\,\}$$
$$I_-^{(i,m)} = \{1, \ldots, n\} \setminus I_+^{(i,m)}$$

and suppose that Λ_h is defined by (16). Then we have

$$\begin{aligned} w_h^{(i+1,m)} &= h_0\, \Lambda_h^{(i,m)} \\ &+ w_h^{(i,m)}\Big[\,1 - h_0 \sum_{j \in I_+^{(i,m)}} \frac{1}{h_j}\varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,) + h_0 \sum_{j \in I_-^{(i,m)}} \frac{1}{h_j}\varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,)\Big] \\ &+ h_0 \sum_{j \in I_+^{(i,m)}} \frac{1}{h_j}\varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,)\,w_h^{(i,m+e_j)} \\ &- h_0 \sum_{j \in I_-^{(i,m)}} \frac{1}{h_j}\varrho_{h\cdot j}(\,P^{(i,m)}[u_h]\,)\,w_h^{(i,m-e_j)} \qquad (t^{(i)}, x^{(m)}) \in E_h'. \end{aligned}$$

From (13), (19), (28), it follows that the function ω_h defined by (18) satisfies the recursive inequality (20) and the initial estimate (21) holds. Then we get the estimate

$$\omega_h{}^{(i)} \leqslant \eta_h^{(i)} \qquad \text{for } 0 \leqslant i \leqslant N_0,$$

where $\eta_h^{(i)}$ is the solution of (22), (23). Now we get (29), (30) from (24) and Theorem 2 is proved. ∎

3. Difference methods for mixed problem

We will need the following operator $T_h\colon \mathfrak{F}(D_h,R) \to C(D,R)$. Let

$$S_+ = \{\, \xi = (\xi_1,\dots,\xi_n)\colon \quad \xi_j = \{0,1\}, \quad \text{for } 0 \leqslant j \leqslant n \,\}.$$

Suppose that $w \in \mathfrak{F}(D_h,R)$. For every $(t,x) \in D$ there is $(t^{(i)},x^{(m)}) \in D_h$ such that $(t^{(i+1)},x^{(m+1)}) \in D_h$, where $m+1 = (m_1+1,\dots,m_n+1)$ and $t^{(i)} \leqslant t \leqslant t^{(i+1)}$, $x^{(m)} \leqslant x \leqslant x^{(m+1)}$. Then we put

$$(T_h\, w)(t,x) = \frac{t-t^{(i)}}{h_0} \sum_{\xi\in S_+} w^{(i+1,m+\xi)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi} \Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\xi}$$
$$+ \Big(1-\frac{t-t^{(i)}}{h_0}\Big) \sum_{\xi\in S_+} w^{(i,m+\xi)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi} \Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\xi},$$

where

$$\Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi} = \prod_{j=1}^{n} \Big(\frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{\xi_j},$$

$$\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\xi} = \prod_{j=1}^{n} \Big(1-\frac{x_j - x_j^{(m_j)}}{h_j'}\Big)^{1-\xi_j}$$

and we put $0^0 = 1$ in the above formulas.

LEMMA 3. *Suppose that the function $w\colon D \to R$ is of class C^2 and denote by w_h the restriction of w to the set D_h. Let $\tilde{C} \in R_+$ be such a constant that*

$$|\,\partial_{tt}w(t,x)|,\ |\,\partial_{tx_j}w(t,x)|,\ |\,\partial_{x_jx_k}w(t,x)| \leqslant \tilde{C} \quad \text{on } D$$

where $j,k = 1,\dots,n$. Then

$$\|\, T_h w_h - w \,\|_D \leqslant \tilde{C}\Big[\, h_0^2 + 2h_0 \sum_{j=1}^{n} h_j + \sum_{j,k=1}^{n} h_j h_k \Big].$$

The proof of lemma (3) is given in [5], Chapter 5.

ASSUMPTION $\mathrm{H}[\varrho, f]$. *Suppose that the function $\varrho\colon E' \times C(D,R) \to R^n$ and $f\colon E \times C(D,R) \to R$ are continous and there is $L \in R_+$ such that*

$$\|\, \varrho(t,x,w) - \varrho(t,x,\bar{w})\| \leqslant L\|w-\bar{w}\|_D,$$

$$|\, f(t,x,w) - f(t,x,\bar{w})| \leqslant L\|w-\bar{w}\|_D$$

on $E \times C(D,R)$.

Now we consider functional differential problem (1), (2) and the difference equation

$$\delta_0 z^{(i,m)} = \sum_{j=1}^{n} \varrho_j(\, t^{(i)},\, x^{(m)},\, T_h z_{[i,m]}\,)\, \delta_j z^{(i,m)} + f(\, t^{(i)},\, x^{(m)},\, T_h z_{[i,m]}\,) \tag{31}$$

with the initial boundary condition (5).

THEOREM 4. *Suppose that Assumption* $H[\varrho, f]$ *is satisfied and*

1) $h \in \Delta$ *and*

$$\frac{1}{n} - \frac{h_0}{h_j}|\,\varrho_j(t,x,w)| \geqslant 0 \qquad \text{on } E \times C(D,R) \quad \text{for } 1 \leqslant j \leqslant n$$

and there is $M = (M_1,\ldots,M_n) \in R^n_+$ *such that* $h' \leqslant M h_0$,

2) the function $u_h\colon E^*_h \to R$ *is a solution of the problem* (31), (5),

3) $v\colon E^* \to R$ *is a solution of* (1), (2) *and* v_h *is the restriction of* v *to* E^*_h,

4) $v|_{\overline{E}}$ *is of class* C^2 *and* $c_0 \in R+$ *is such a constant that*

$$|\,\partial_{x_j} v(t,x)\,| \leqslant c_0 \qquad \text{on } \overline{E},\ 1 \leqslant j \leqslant n$$

5) there is $\alpha_0\colon \Delta \to R_+$ *such that*

$$\begin{aligned} &|\,\varphi_h^{(i,m)} - \varphi^{(i,m)}\,| \leqslant \alpha_0(h) \qquad \text{on } E_{0.h} \cup \partial_0 E_h, \\ &\lim_{h\to 0} \alpha_0(h) = 0. \end{aligned} \tag{32}$$

Then there are $A,\, B \in R_+$ *such that*

$$|\,u_h^{(i,m)} - v_h^{(i,m)}\,| \leqslant \alpha_0(h)\, e^{\tilde{L}a} + \left(Ah_0 + Bh_0^2\right) \frac{e^{\tilde{L}a}-1}{\tilde{L}} \qquad \text{on } E_h \tag{33}$$

if $L > 0$ *and*

$$|\,u_h^{(i,m)} - v_h^{(i,m)}\,| \leqslant \alpha_0(h) + a\left(Ah_0 + Bh_0^2\right) \qquad \text{on } E_h \tag{34}$$

if $L = 0$ *where* $\tilde{L} = L(1+c_0)$.

Proof. We shall apply Theorem 1 to prove the above assertion. Write

$$\begin{aligned} \psi_h^{(i,m)} = \delta_0 v_h^{(i,m)} &- \sum_{j=1}^{n} \varrho_j(\, t^{(i)},\, x^{(m)},\, T_h(v_h)_{[i,m]}\,)\, \delta_j v_h^{(i,m)} \\ &- f(\, t^{(i)},\, x^{(m)},\, T_h(v_h)_{[i,m]}\,). \end{aligned} \tag{35}$$

We see at once that

$$\psi_h^{(i,m)} = \delta_0 v_h^{(i,m)} - \partial_t v^{(i,m)} +$$
$$+\sum_{j=1}^{n}\Big[\varrho_j(t^{(i)}, x^{(m)}, v_{(t^{(i)},x^{(m)})}) - \varrho_j(t^{(i)}, x^{(m)}, T_h(v_h)_{[i,m]})\Big]\delta_j v_h^{(i,m)} +$$
$$+\sum_{j=1}^{n}\varrho_j(t^{(i)}, x^{(m)}, v_{(t^{(i)},x^{(m)})})[\partial_{x_j} v^{(i,m)} - \delta_j v_h^{(i,m)}] +$$
$$+ f(t^{(i)}, x^{(m)}, v_{(t^{(i)},x^{(m)})}) - f(t^{(i)}, x^{(m)}, T_h(v_h)_{[i,m]}).$$

There are $C,\, d \in R_+$ such that

$$|\partial_{tt} v(t,x)|,\ |\partial_{tx_j} v(t,x)|,\ |\partial_{x_j x_k} v(t,x)| \leqslant C \quad \text{on } D,$$

where $j, k = 1, \dots, n$ and

$$|\varrho_j(t,x,v_{(t,x)})| \leqslant d \quad \text{on } E \qquad \text{for } 1 \leqslant j \leqslant n.$$

An easy calculation shows that

$$|\partial_t v^{(i,m)} - \delta_0 v_h^{(i,m)}| \leqslant \frac{h_0}{2} C \Big(1 + \frac{1}{n}\sum_{j=1}^{n} M_j^2\Big)$$

and

$$|\partial_{x_j} v^{(i,m)} - \delta_j v_h^{(i,m)}| \leqslant \frac{h_0}{2} C M_j \qquad \text{for } 1 \leqslant j \leqslant n.$$

According to the above estimates, we have

$$|\psi_h^{(i,m)}| \leqslant A h_0 + B h_0^2 \quad \text{on } E_h',$$

where

$$A = \frac{1}{2} C\Big[1 + \frac{1}{n}\sum_{j=1}^{n} M_j^2 + d\sum_{j=1}^{n} M_j\Big],$$
$$B = L(1+c_0)C\Big[1 + 2\sum_{j=1}^{n} M_j + \sum_{j,k=1} M_j M_k\Big].$$

Then all assumptions of Theorem 1 are satisfied and assertions (33), (34) follow from (9), (10). ∎

Now we consider the functional differential problem (1), (2) and the difference functional problem consisting of (31) where δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ are defined by (25)–(27) and initial boundary condition (5). We start with a lemma on the interpolating operator T_h.

LEMMA 5. *Suppose that the function $w: D \to R$ is of class C^1 and w_h is the restriction of w to the set D_h. Let C_0 be such a constant that*

(36) $\quad |\partial_t w(t,x)|, \leqslant C_0, \quad |\partial_{x_j} w(t,x)| \leqslant C_0 \quad$ *for* $1 \leqslant i \leqslant n, \quad (t,x) \in D.$

Then

$$\|T_h w_h - w\|_D \leqslant C_0\|h\|, \tag{37}$$

where $\|h\| = h_0 + h_1 + \cdots + h_n$.

Proof. Let $(t,x) \in D$. Then there is $(t^{(i)}, x^{(m)}) \in D_h$ such that $(t^{(i+1)}, x^{(m+1)}) \in D$ and $t^{(i)} \leqslant t \leqslant t^{(i+1)}$, $x^{(m)} \leqslant x \leqslant x^{(m+1)}$. One can find such θ, $\tilde{\theta} \in D$ such that

$$\begin{aligned} w(t,x) - T_h w_h(t,x) &= w(t,x) \\ &- \frac{t-t^{(i)}}{h_0} \sum_{\xi\in S_+} \Big[w(t,x) + \partial_t w(\theta)(t^{(i+1)}-t) + \sum_{j=1}^{n} \partial_{x_j} w(\theta)(x_j^{(m_j+\xi_j)} - x_j)\Big] \\ &\qquad \times \Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi}\Big(1 - \frac{x-x^{(m)}}{h'}\Big)^{1-\xi} \\ &- \Big(1 - \frac{t-t^{(i)}}{h_0}\Big) \sum_{\xi\in S_+} \Big[w(t,x) - \partial_t w(\tilde{\theta})(t^{(i+1)}-t) + \sum_{j=1}^{n} \partial_{x_j} w(\tilde{\theta})(x_j^{(m_j+\xi_j)} - x_j)\Big] \\ &\qquad \times \Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi}\Big(1 - \frac{x-x^{(m)}}{h'}\Big)^{1-\xi}. \end{aligned}$$

For $x^{(m)} \leqslant x \leqslant x^{(m+1)}$ we have

$$\sum_{\xi\in S_+} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi}\Big(1 - \frac{x-x^{(m)}}{h'}\Big)^{1-\xi} = 1.$$

Then from (36) we get (37). ∎

THEOREM 6. *Suppose that Assummption $H[\varrho, f]$ is satisfied and*

1) $h \in \Delta$ and

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\varrho_j(t,x,w)| \geqslant 0 \qquad \text{on } E \times C(D,R)$$

and there is $M = (M_1, \ldots, M_n) \in R_+^n$ such that $h' \leqslant M h_0$,

2) the function $u_h\colon E_h^* \to R$ is a solution of the problem (31), (5) *with δ_0 and δ given by* (25)–(27),

3) $v\colon E^* \to R$ is a solution of (1), (2),

4) the function $v|_{\overline{E}}$ is of class C^1 and $c_0 \in R_+$ is such a constant that

$$|\partial_t v(t,x)|\,, \quad |\partial_{x_j} v(t,x)| \leqslant c_0 \qquad \text{on } \overline{E}, \quad 1 \leqslant j \leqslant n,$$

5) there is $\alpha_0\colon \Delta \to R_+$ such that condition (32) *is satisfied.*

Then there is $\alpha\colon \Delta \to R_+$ such that

$$|u_h^{(i,m)} - v_h^{(i,m)}| \leqslant \alpha_0(h)\, e^{\tilde{L}a} + \alpha(h) \frac{e^{\tilde{L}a} - 1}{\tilde{L}} \qquad \text{on } E_h \tag{38}$$

if $L > 0$ *and*

$$(39)\qquad |u_h^{(i,m)} - v_h^{(i,m)}| \leqslant \alpha_0(h) + a\alpha(h) \qquad \text{on } E_h$$

if $L = 0$, *where* $\tilde{L} = L(1 + c_0)$ *and* $\lim_{h\to 0} \alpha(h) = 0$.

Proof. We apply Theorem 2 to prove the above assertion. Let $\psi\colon E'_h \to R$ be a function given by (35). It follows from Assumption $H[\varrho, f]$ and from Lemma (5) that there is a function $\alpha\colon \Delta \to R_+$ such that

$$|\psi_h^{(i,m)}| \leqslant \alpha(h) \qquad \text{on } E'_h \qquad \text{and} \qquad \lim_{h\to 0} \alpha(h) = 0.$$

Then the assumptions of Theorem 2 are satisfied and assertion (38), (39) follow from (29), (30). ∎

4. Numerical example

For $n = 2$ we put

$$E = [0,1] \times [-2,2] \times [-2,2],$$
$$D = \{0\} \times [-1,1] \times [-1,1], \qquad D_0 = [-1,1] \times [-1,1].$$

Denote by z the unknown function of the variables (t,x,y) and consider the differential integral equation

$$(40)\quad \partial_t z(t,x,y)$$
$$= \left[-1 + \frac{1}{1 + \left(1 + z(t,x,y) - \int_{D_0} z(\frac{t}{2}, x+r, y+s)\,dr\,ds\right)^2}\right]\partial_x z(t,x,y)$$
$$+ \left[1 - \frac{1}{1 + \left(1 - 4z(t,x,y) + \int_{D_0} z(\frac{t}{2}, x+r, y+s)\,dr\,ds\right)^2}\right]\partial_y z(t,x,y)$$
$$- z(t, x+1, y-1) + z(t, x-1, y+1) + 2t(x + y - xy) + \frac{5}{2}t^2(x-y)$$

with the initial boundary condition

$$(41)\qquad z(t,x,y) = t^2(x + y - xy) \quad \text{for } (t,x,y) \in E_0 \cup \partial_0 E,$$

where

$$E_0 = \{0\} \times [-3,3] \times [-3,3],$$
$$\partial_0 E = [0,1] \times \Big[([-3,3] \times [-3,3]) \setminus ([-3,3] \times [-3,3])\Big].$$

The difference method for the problem is of the form

$$(42)\quad \delta_t z^{(i,j,k)} = \left[-1 + \frac{1}{1 + \left(1 + z^{(i,j,k)} - I_h^{(i,j,k)}\right)^2}\right]\delta_1 z^{(i,j,k)}$$

$$+\left[1-\frac{1}{1+\left(1-4z^{(i,j,k)}+\tilde{I}_h^{(i,j,k)}\right)^2}\right]\delta_2 z^{(i,j,k)}$$
$$-T_h z_{[i,j,k]}(0,1,-1)+T_h z_{[i,j,k]}(0,-1,1)$$
$$+2t^{(i)}(x^{(j)}+y^{(k)}-x^{(j)}y^{(k)})+\frac{5}{2}(t^{(i)})^2(x^{(j)}-y^{(k)})$$

and

(43) $$z^{(i,j,k)}=2t^{(i)}(x^{(j)}+y^{(k)}-x^{(j)}y^{(k)}) \text{ for } (t^{(i)},x^{(j)},y^{(k)})\in E_0\cup\partial_0 E,$$

where

$$\delta_0 z^{(i,j,k)}=\frac{1}{h_0}[z^{(i+1,j,k)}-z^{(i,j,k)}],$$
$$\delta_1 z^{(i,j,k)}=\frac{1}{h_1}[z^{(i,j,k)}-z^{(i,j-1,k)}],$$
$$\delta_2 z^{(i,j,k)}=\frac{1}{h_2}[z^{(i,j,k+1)}-z^{(i,j,k)}].$$

and

$$I_h^{(i,j,k)}=\int_{D_0}T_h z_{[\frac{i}{2},j,k]}(r,s)\,dr\,ds,\qquad \tilde{I}_h^{(i,j,k)}=\int_{D_0}T_h z_{[i,j,k]}(r,s)\,dr\,ds.$$

The operator T_h, $h=(h_0,h_1,h_2)$ is defined in Section 3.

The function $v(t,x,y)=t^2(x+y-xy)$ is the solution of (40), (41). Let $u_h\colon E_h^*\to R$ be the solution of (42), (43) and $\varepsilon=u_h-v$. The values $\varepsilon(0.2,x^{(j)},y^{(k)})$, $\varepsilon(0.4,x^{(j)},y^{(k)})$, $\varepsilon(0.6,x^{(j)},y^{(k)})$, $\varepsilon(0.8,x^{(j)},y^{(k)})$ and $u_h(0.2,x^{(j)},y^{(k)})$, $u_h(0.4,x^{(j)},y^{(k)})$, $u_h(0.6,x^{(j)},y^{(k)})$, $u_h(0.8,x^{(j)},y^{(k)})$ are listed in the tables for $h_0=0.005$, $h_1=0.05$ and $h_2=0.05$,

TABLE $t=0.2$

$x^{(j)}=-0.5$	$y^{(k)}=-0.5$	$u_h=-0.0488$	$\varepsilon=1.236\,10^{-3}$
$x^{(j)}=-0.5$	$y^{(k)}=0.0$	$u_h=-0.0195$	$\varepsilon=4.685\,10^{-4}$
$x^{(j)}=-0.5$	$y^{(k)}=0.5$	$u_h=0.0097$	$\varepsilon=2.995\,10^{-4}$
$x^{(j)}=0.0$	$y^{(k)}=-0.5$	$u_h=-0.0195$	$\varepsilon=5.178\,10^{-4}$
$x^{(j)}=0.0$	$y^{(k)}=0.0$	$u_h=-0.0000$	$\varepsilon=3.075\,10^{-6}$
$x^{(j)}=0.0$	$y^{(k)}=0.5$	$u_h=0.0195$	$\varepsilon=5.246\,10^{-4}$
$x^{(j)}=0.5$	$y^{(k)}=-0.5$	$u_h=0.0098$	$\varepsilon=2.035\,10^{-4}$
$x^{(j)}=0.5$	$y^{(k)}=0.0$	$u_h=0.0195$	$\varepsilon=4.770\,10^{-4}$
$x^{(j)}=0.5$	$y^{(k)}=0.5$	$u_h=0.0292$	$\varepsilon=7.508\,10^{-4}$

TABLE $t = 0.4$

$x^{(j)} = -0.5$	$y^{(k)} = -0.5$	$u_h = -0.1978$	$\varepsilon = 2.242\,10^{-3}$
$x^{(j)} = -0.5$	$y^{(k)} = 0.0$	$u_h = -0.0792$	$\varepsilon = 7.971\,10^{-4}$
$x^{(j)} = -0.5$	$y^{(k)} = 0.5$	$u_h = 0.0393$	$\varepsilon = 6.672\,10^{-4}$
$x^{(j)} = 0.0$	$y^{(k)} = -0.5$	$u_h = -0.0790$	$\varepsilon = 9.994\,10^{-4}$
$x^{(j)} = 0.0$	$y^{(k)} = 0.0$	$u_h = -0.0000$	$\varepsilon = 2.458\,10^{-5}$
$x^{(j)} = 0.0$	$y^{(k)} = 0.5$	$u_h = 0.0789$	$\varepsilon = 1.062\,10^{-3}$
$x^{(j)} = 0.5$	$y^{(k)} = -0.5$	$u_h = 0.0397$	$\varepsilon = 3.127\,10^{-4}$
$x^{(j)} = 0.5$	$y^{(k)} = 0.0$	$u_h = 0.0791$	$\varepsilon = 8.911\,10^{-4}$
$x^{(j)} = 0.5$	$y^{(k)} = 0.5$	$u_h = 0.1185$	$\varepsilon = 1.476\,10^{-3}$

TABLE $t = 0.6$

$x^{(j)} = -0.5$	$y^{(k)} = -0.5$	$u_h = -0.4477$	$\varepsilon = 2.320\,10^{-3}$
$x^{(j)} = -0.5$	$y^{(k)} = 0.0$	$u_h = -0.1792$	$\varepsilon = 7.545\,10^{-4}$
$x^{(j)} = -0.5$	$y^{(k)} = 0.5$	$u_h = 0.0891$	$\varepsilon = 9.075\,10^{-4}$
$x^{(j)} = 0.0$	$y^{(k)} = -0.5$	$u_h = -0.1788$	$\varepsilon = 1.198\,10^{-3}$
$x^{(j)} = 0.0$	$y^{(k)} = 0.0$	$u_h = -0.0001$	$\varepsilon = 7.429\,10^{-5}$
$x^{(j)} = 0.0$	$y^{(k)} = 0.5$	$u_h = 0.1786$	$\varepsilon = 1.410\,10^{-3}$
$x^{(j)} = 0.5$	$y^{(k)} = -0.5$	$u_h = 0.0897$	$\varepsilon = 3.098\,10^{-4}$
$x^{(j)} = 0.5$	$y^{(k)} = 0.0$	$u_h = 0.1788$	$\varepsilon = 1.151\,10^{-3}$
$x^{(j)} = 0.5$	$y^{(k)} = 0.5$	$u_h = 0.2680$	$\varepsilon = 2.021\,10^{-3}$

TABLE $t = 0.8$

$x^{(j)} = -0.5$	$y^{(k)} = -0.5$	$u_h = -0.7996$	$\varepsilon = 4.268\,10^{-4}$
$x^{(j)} = -0.5$	$y^{(k)} = 0.0$	$u_h = -0.3200$	$\varepsilon = 3.387\,10^{-5}$
$x^{(j)} = -0.5$	$y^{(k)} = 0.5$	$u_h = 0.1594$	$\varepsilon = 6.050\,10^{-4}$
$x^{(j)} = 0.0$	$y^{(k)} = -0.5$	$u_h = -0.3193$	$\varepsilon = 7.100\,10^{-4}$
$x^{(j)} = 0.0$	$y^{(k)} = 0.0$	$u_h = -0.0001$	$\varepsilon = 1.356\,10^{-4}$
$x^{(j)} = 0.0$	$y^{(k)} = 0.5$	$u_h = 0.3189$	$\varepsilon = 1.135\,10^{-3}$
$x^{(j)} = 0.5$	$y^{(k)} = -0.5$	$u_h = 0.1598$	$\varepsilon = 1.535\,10^{-4}$
$x^{(j)} = 0.5$	$y^{(k)} = 0.0$	$u_h = 0.3189$	$\varepsilon = 1.056\,10^{-3}$
$x^{(j)} = 0.5$	$y^{(k)} = 0.5$	$u_h = 0.4780$	$\varepsilon = 2.008\,10^{-3}$

The computation was performed by the IBM PC computer.

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