DEMONSTRATIO MATHEMATICA
Vol. XXXV No 3 2002

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COMPARISON THEOREMS FOR THE OSCILLATION OF HIGHER ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

Abstract. We obtain a necessary and sufficient condition for the oscillation of the higher order neutral delay difference equation

$$\Delta^m(x_n - p_n x_{n-\tau}) + f(n, x_{g_1(n)}, x_{g_2(n)}, \cdots, x_{g_l(n)}) = 0,$$

where $m \geq 1$ is an odd integer. As some application of this result, we estabilish three comparison theorems for the oscillation of the above equation.

1. Introduction

Throughout, we shall use the following notations: $N = \{0, 1, \cdots\}, N(a) = \{a, a+1, \cdots\}$, where $a \in N$, and $N(a,b) = \{a, a+1, \cdots, b\}$, where $b \in N(a)$. Further, for $t \in R$ we define the usual the factorial expression $(t)^{(m)} = \prod_{i=0}^{m-1}(t-i)$ with $(t)^{(0)} = 1$.

In this paper, we consider the higher order neutral delay difference equation

$$\Delta^m(x_n - p_n x_{n-\tau}) + f(n, x_{g_1(n)}, x_{g_2(n)}, \cdots, x_{g_l(n)}) = 0, \tag{1}$$

where m is an odd integer, τ is a positive integer, Δ denotes the forward difference operator i.e., $\Delta x_n = x_{n+1} - x_n$ and $\Delta^i x_n = \Delta(\Delta^{i-1}x_n)$, $i = 1, 2, \cdots, m, \Delta^0 x_n = x_n$. With respect to (1), throughout we shall assume the following

(i) $p : N(K) \to R^+ = [0, \infty), g_i : N(K) \to N, i = 1, 2, \cdots, l$, for some $K \in N$, and $f : N \times R^l \to R$ is continuous with respect to the last l arguments,

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1991 *Mathematics Subject Classification*: 39A10, 34k15.

Key words and phrases: oscillation, comparison theorems, neutral difference equations.

Research supported by National Natural Science Foundation of P.R. China and Research Fund of Hunan Provincial Education Department (01B003).

(ii) $f(n, -u_1, -u_2, \cdots, -u_l) = -f(n, u_1, u_2, \cdots, u_l)$, $u_1 u_i > 0, i = 2, \cdots, l$,

(iii) $u_1 f(n, u_1, u_2, \cdots, u_l) \geq 0$ for $u_1 u_i > 0, i = 2, \cdots, l$,

(iv) $|f(n, u_1, u_2, \cdots, u_l)| \geq |f(n, v_1, v_2, \cdots, v_l)|$, whenever $|v_i| \leq |u_i|$ and $u_i v_i > 0, i = 1, 2, \cdots, l$.

By a solution of (1), we mean a nontrivial sequence $\{x_n\}$ satisfying (1) for $n \in N(\overline{n}_0)$, where $\overline{n}_0$ is some nonnegative integer. A solution $\{x_n\}$ of Eq. (1) is said to be oscillatory if for every $K_1 > \overline{n}_0$ there exists an $n \geq K_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is said to be nonoscillatory.

The neutral delay difference equations arise in a number of important applications including problems in population dynamics when maturation and gestation are included, in "cobweb" models in economics, where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission in lossless transmission lines between circuits in high speed computers. The literature on the oscillations theory of neutral difference equations is growing rapidly (see, for example [2, 8–12]). Zhou and Zhang [10-12] investigated the oscillation and nonoscillation of first order and second order neutral difference equations. The study is a relatively new field and is very interesting in applications.

The motivation for the present work steems from the many comparison theorems in the theory of functional differential equations. We are particularly interested in the work of Erbe, Kong and Zhang [3], Gopalsamy, Lalli and Zhang [4], Yan [6], Zhang, Yu and Wang [7] on linear neutral delay differential equations, and our results are generalizations and extensions of theirs to nonlinear neutral delay difference equations. The plan of the paper is as follows. In Section 2, we shall present some preliminary results, some of which are interesting in their own right. In Section 3, we obtain a necessary and sufficient condition for the oscillation of all solution of (1) which improve and extend the main results in [9]. As some application of this result and the main result in [8], we establish three comparison theorems for the oscillation of (1).

In the sequel, when we write a sequential inequality without specifying its domain of validity, we assume that it holds for all sufficiently large positive integer n.

2. Preliminaries

LEMMA 1. [1] *Let $\{y_n\}$ be a sequence of real numbers in N. Let $\{y_n\}$ and $\Delta^m y_n$ be of constant sign with $\Delta^m y_n$ not being identically zero on any subset $N(\overline{n}_1)$ of N. If*

$$y_n \Delta^m y_n \leq 0,$$

then there exists a number m^* in $\{0, 1 \cdots, m-1\}$ with $(-1)^{m-m^*-1} = 1$ and such that

$$y_n \Delta^j y_n > 0, \quad \text{for} \ \ j = 0, 1, 2, \cdots, m^*, \qquad n \geq \overline{n}_2 \geq \overline{n}_1,$$
$$(-1)^{j-m^*} y_n \Delta^j y_n > 0, \quad \text{for} \ \ j = m^* + 1, \cdots, m-1, \qquad n \geq \overline{n}_2 \geq \overline{n}_1.$$

LEMMA 2. *Assume that either there is $\{n_k\} : n_k \to \infty, k \to \infty$ such that $p_{n_k} = 0$ or there exists a positive integer n^* such that $p_n > 0$ for $n > n^*$ and*

$$\sum_{j=1}^{\infty} \Big(\prod_{i=1}^{j} p_{n^*+i\tau} \Big)^{-1} = \infty. \tag{2}$$

Let

$$y_n = x_n - p_n x_{n-\tau}, \tag{3}$$

where $\{x_n\}$ is an eventually positive solution of the difference inequality

$$\Delta^m (x_n - p_n x_{n-\tau}) + f(n, x_{g_1(n)}, x_{g_2(n)}, \cdots, x_{g_l(n)}) \leq 0. \tag{4}$$

Then, we have eventually

$$y_n > 0. \tag{5}$$

Proof. Let $\overline{n}_1$ be a positive integer such that $x_{n-\tau} > 0$ and $x_{g_i(n)} > 0, i = 1, 2, \cdots, l$, for all $n \geq \overline{n}_1$. Then, by (3) and (4), we have

$$\Delta^m y_n \leq -f(n, x_{g_1(n)}, x_{g_2(n)}, \cdots, x_{g_l(n)}) \leq 0, \quad \text{for all} \ \ n \geq \overline{n}_1$$

which yields that the differences of y_n up to order $m-1$ are monotone and either eventually

$$\Delta^{m-1} y_n < 0 \tag{6}$$

or

$$\Delta^{m-1} y_n > 0. \tag{7}$$

We claim that (7) holds. Otherwise, (6) holds, which implies that there exist $a > 0$ and $\overline{n}_2 \geq \overline{n}_1$ such that

$$y_n \leq -a, \quad \text{for all} \ \ n \geq \overline{n}_2.$$

Therefore, we have

$$x_n \leq -a + p_n x_{n-\tau}, \quad \text{for all} \ \ n \geq \overline{n}_2. \tag{8}$$

If $p_{n_k} = 0$ for $n_k \geq \overline{n}_2$, then we have $x_{n_k} \leq -a$. This is a contradiction. Thus, $p_n > 0$ for $n \geq \overline{n}_2$. For the sake of convenience, we set

$$s(i) = n^* + i\tau, \quad i = 1, 2, \cdots.$$

Now choose a positive integer j such that $s(j) = n^* + j\tau \geq \overline{n}_2$. Then for any positive integer i, by (8) we have

$$\begin{aligned}x_{s(j+i)} &\le p_{s(j+i)}x_{s(j+i-1)} - a \\ &\le p_{s(j+i)}\cdots p_{s(j)}x_{s(j-1)} \\ &- a[1 + p_{s(j+i)} + p_{s(j+i)}p_{s(j+i-1)} + \cdots + p_{s(j+i)}\cdots p_{s(j+1)}] \\ &= \prod_{v=0}^{i} p_{s(j+v)}\left\{x_{s(j-1)} - a\left[\frac{1}{p_{s(j)}} + \frac{1}{p_{s(j)}p_{s(j+1)}} + \cdots + \frac{1}{p_{s(j)}\cdots p_{s(j+i)}}\right]\right\}.\end{aligned}$$

It follows from (2) that $x_{s(j+i)} \le 0$ for sufficiently large i. This is a contradiction and so (7) holds.

Next, we consider the following three possible cases:

Case 1. There is $\overline{m} \in \{2, 3, \cdots, m-1\}$ such that

$$\Delta^{m-\overline{m}}y_n < 0, \quad \Delta^{m-\overline{m}-1}y_n < 0;$$

Case 2. There is $\overline{m} \in \{1, 2, \cdots, m-1\}$ such that

$$\Delta^{m-\overline{m}}y_n > 0, \quad \Delta^{m-\overline{m}-1}y_n > 0;$$

Case 3. $(-1)^{\overline{m}}\Delta^{m-\overline{m}}y_n < 0, \quad \overline{m} = 1, 2, \cdots, m.$

For Case 1, by using a similar method to the above, we can obtain a contradiction and so Case 1 is impossible. For Case 2 we see that eventually $y_n > 0$. For Case 3, since m is an odd integer, it follows that $y_n > 0$ eventually. The proof of Lemma 3 is complete.

LEMMA 3. *Assume that the assumptions of Lemma 2 hold and*

$$\sum_{i=n_0}^{\infty} f(i, d, \cdots, d) = \infty, \quad \text{for some} \quad d > 0. \tag{9}$$

Let $\{x_n\}$ and $\{y_n\}$ be as in Lemma 2. Then we have that $(-1)^i\Delta^i y_n > 0, i = 0, 1, \cdots, m$, eventually.

Proof. By Lemmas 1 and 2, we have $y_n > 0$ eventually and moreover

$$\Delta^j y_n > 0, \quad \text{for} \quad j = 0, 1, \cdots, m^*, \quad n \ge \overline{n}_2,$$

$$(-1)^{j-m^*}\Delta^j y_n > 0, \quad \text{for} \quad j = m^* + 1, \cdots, m-1, \quad n \ge \overline{n}_2.$$

we claim that $m^* = 0$. Otherwise $m^* \ge 2$ and hence $\Delta^j y_n > 0, j = 0, 1, \cdots, m^*$, which implies that there exists a $N_1 \ge \overline{n}_2$ such that $y_n \ge d$ for $n \ge N_1$ and hence $x_n \ge y_n \ge d$, for $n \ge N_1$. Substituting this into (1) we have

$$\Delta^m y_n + f(n, d, \cdots, d) \le 0.$$

Summing it up from N_1 to N for N sufficiently large, we have

$$\sum_{i=N_1}^{N} f(i, d, \cdots, d) \le \Delta^{m-1}y_{N_1} - \Delta^{m-1}y_{N+1} < \Delta^{m-1}y_{N_1}$$

which implies

$$\sum_{i=n_0}^{\infty} f(i,d,\cdots,d) < \infty.$$

This is a contradiction and the proof is complete.

In following, let

$$\sigma = \max\{\tau, \overline{n} - \min_{n\geq\overline{n},1\leq i\leq l}\{g_i(n)\}\} \text{ for some } \overline{n} \geq \overline{n}_0.$$

LEMMA 4. *Assume that there exists a $\overline{n} \geq \overline{n}_1$ such that either $p_n \geq 0$ and $g_i(n) < n$, for $n \geq \overline{n}, i = 1,2,\cdots,l$, or $p_n > 0$ for $n \geq \overline{n}$. Further, assume that the inequality*

$$(10) \quad y_n \geq p_n y_{n-\tau} + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)} f(i, y_{g_1(i)},\cdots,y_{g_l(i)})$$

has a positive solution $\{y_n\}_{n=\overline{n}-\sigma}^{\infty}$. Then the corresponding equation

$$(11) \quad x_n = p_n x_{n-\tau} + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)} f(i, x_{g_1(i)},\cdots,x_{g_l(i)})$$

has a positive solution $\{x_n\}_{n=\overline{n}-\sigma}^{\infty}$.

Proof. Define a set of sequences

$$W = \{w = \{w_n\}_{n=\overline{n}-\sigma}^{\infty} : 0 \leq w_n \leq 1 \text{ for } n \geq \overline{n} - \sigma\},$$

and define the mapping S on W as follows

$$(12) \qquad Sw_n = \begin{cases} \dfrac{1}{y_n}\Big[p_n y_{n-\tau} w_{n-\tau} + \dfrac{1}{(m-1)!}\displaystyle\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)} \\ \quad \times f(i, y_{g_1(i)}w_{g_1(i)},\cdots,y_{g_l(i)}w_{g_l(i)})\Big], \quad n \in N(\overline{n}), \\ \dfrac{n-\overline{n}+\sigma}{\sigma} Sw_{\overline{n}} + \left(1 - \dfrac{n-\overline{n}+\sigma}{\sigma}\right), n \in N(\overline{n}-\sigma, \overline{n}). \end{cases}$$

It is easy to see from (10) that $SW \subset W$, and for any $w \in W$, we have $Sw_n > 0$ for $\overline{n} - \sigma \leq n < \overline{n}$.

Define a sequenc $\{w_n^{(j)}\}$ in W as follow: $w_n^{(0)} = 1$, and $w_n^{(j+1)} = Sw_n^{(j)}, j = 0,1,2,\cdots$, $n \geq \overline{n} - \sigma$. From (10), by induction, we have

$$0 \leq w_n^{(j+1)} \leq w_n^{(j)} \leq 1, \quad n \in N(\overline{n}-\sigma), \quad j = 0,1,2,\cdots.$$

Then $\lim_{j\to\infty} w_n^{(j)} = w_n, n \in N(\overline{n}-\sigma)$, exists, and $0 \leq w_n \leq 1$. Further, in view of (12) the following holds

$$w_n = \frac{1}{y_n}\Big[p_n y_{n-\tau} w_{n-\tau} + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}$$
$$\times f(i, y_{g_1(i)}w_{g_1(i)},\cdots,y_{g_l(i)}w_{g_l(i)})\Big], \quad \text{for } n \in N(\overline{n})$$

and

$$w_n = \frac{n-\overline{n}+\sigma}{\sigma} w_{\overline{n}} + 1 - \frac{n-\overline{n}+\sigma}{\sigma} > 0, \quad \text{for} \ \ n \in N(\overline{n}-\sigma, \overline{n}).$$

Set $x_n = w_n y_n$. Then $\{x_n\}$ is a nonnegative solution of (11) and $x_n > 0$ for $\overline{n} - \sigma \le n < \overline{n}$.

Finally it remains to show that $x_n > 0$ for $n \in N(\overline{n} - \sigma)$.

Assume that exists $n' \in N(\overline{n} - \sigma)$ such that $x_n > 0$ for $\overline{n} - \sigma \le n < n'$ and $x_{n'} = 0$. Clearly, $n' \ge \overline{n}$. Thus, by (11) we have

$$0 = x_{n'} = p_{n'} x_{n'-\tau} + \frac{1}{(m-1)!} \sum_{i=n'}^{\infty} (i - n' + m - 1)^{(m-1)} f(i, x_{g_1(i)}, \cdots, x_{g_l(i)})$$

which implies that $p_{n'} = 0$ and $f(i, x_{g_1(i)}, \cdots, x_{g_l(i)}) = 0$. Hence, $p_{n'} = 0$ and $x_{g_i(n)} = 0, n \ge n', i = 1, 2, \cdots, l$. This contradicts the assumptions of Lemma 4. Therefore, $x_n > 0$ for $n \in N(\overline{n} - \sigma)$, and the proof is complete.

LEMMA 5. *Assume that* $1 \le m^* \le m - 1, c > 0$ *and that the inequality*

$$\text{(13)} \quad y_n \ge c + p_n y_{n-\tau} + \frac{1}{(m^*-1)!(m-m^*-1)!} \sum_{i=\overline{n}}^{n-1} (n-i+1)^{(m^*-1)} \times \sum_{j=i}^{\infty} (j-i+m-m^*-1)^{(m-m^*-1)} f(j, y_{g_1(j)}, \cdots, y_{g_l(j)})$$

has a positive solution $\{y_n\}_{n=\overline{n}-\sigma}^{\infty}$. *Then the corresponding equation*

$$\text{(14)} \quad x_n = c + p_n x_{n-\tau} + \frac{1}{(m^*-1)!(m-m^*-1)!} \sum_{i=\overline{n}}^{n-1} (n-i+1)^{(m^*-1)} \times \sum_{j=i}^{\infty} (j-i+m-m^*-1)^{(m-m^*-1)} f(j, x_{g_1(j)}, \cdots, x_{g_l(j)})$$

also has a positive solution $\{x_n\}_{n=\overline{n}-\sigma}^{\infty}$.

The proof of Lemma 5 is similar to the proof of Lemma 4 and hence we omit it here.

3. Main results

THEOREM 1. *Assume that either there is* $\{n_k\} : n_k \to \infty$ *as* $k \to \infty$ *such that* $p_{n_k} = 0$ *and* $g_i(n) < n$, *for* $n \ge \overline{n}, i = 1, 2, \cdots, l$, *or there exists a positive integer* n^* *such that* $p_n > 0$ *for* $n \ge n^*$ *and that* (2) *holds. Then every solution of Eq.* (1) *is oscillatory if and only if the corresponding inequality* (4) *has no eventually positive solutions.*

Proof. The sufficiency is obvious. To prove the necessity, we assume that (4) has an eventually positive solution $\{x_n\}$. Set $y_n = x_n - p_n x_{n-\tau}$. By Lemma 2, $y_n > 0$ eventually. According to Lemma 1, there exists an even number m^* such that $m^* \in N(0, m-1)$, and that

$$\begin{aligned} \Delta^i y_n > 0, &\quad \text{for} \quad i = 0, 1, \cdots, m^*, \\ (-1)^i \Delta^i y_n > 0, &\quad \text{for} \quad i = m^* + 1, \cdots, m-1. \end{aligned} \tag{15}$$

If $m^* = 0$, summing up (4) from n to ∞ m times, we have

$$y_n \geq \frac{1}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} f(j, x_{g_1(j)}, \cdots, x_{g_l(j)}).$$

That is

$$x_n \geq p_n x_{n-\tau} + \frac{1}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} f(j, x_{g_1(j)}, \cdots, x_{g_l(j)}).$$

By Lemma 4, the corresponding equation

$$z_n = p_n z_{n-\tau} + \frac{1}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} f(j, z_{g_1(j)}, \cdots, z_{g_l(j)})$$

also has a positive solution $\{z_n\}$. Clearly, $\{z_n\}$ is an eventually positive solution of (1), contradicting the assumption.

If $2 \leq m^* \leq m-1$, then summing up (4) from n to ∞ $m - m^*$ times, we have

$$\begin{aligned} &\Delta^{m^*} y_n \geq \\ &\geq \frac{1}{(m - m^* - 1)!} \sum_{j=n}^{\infty} (j - n + m - m^* - 1)^{(m-m^*-1)} f(j, x_{g_1(j)}, \cdots, x_{g_l(j)}). \end{aligned} \tag{16}$$

Then summing up (16) from $\overline{n}$ to $n-1$ m^* times we have

$$\begin{aligned} y_n \geq y_{\overline{n}} &+ \frac{1}{(m^*-1)!(m-m^*-1)!} \sum_{i=\overline{n}}^{n-1} (n - i + 1)^{(m^*-1)} \\ &\times \sum_{j=i}^{\infty} (j - i + m - m^* - 1)^{(m-m^*-1)} f(j, x_{g_1(i)}, \cdots, x_{g_l(j)}), \quad n \geq \overline{n}, \end{aligned}$$

where $\overline{n}$ is sufficiently large such that $x_{\overline{n}} > 0$. Thus we have, for $n \geq \overline{n}$,

$$\begin{aligned} x_n \geq y_{\overline{n}} + p_n x_{n-\tau} &+ \frac{1}{(m^*-1)!(m-m^*-1)!} \sum_{i=\overline{n}}^{n-1} (n - i + 1)^{(m^*-1)} \\ &\times \sum_{j=i}^{\infty} (j - i + m - m^* - 1)^{(m-m^*-1)} f(j, x_{g_1(i)}, \cdots, x_{g_l(j)}), \quad n \geq \overline{n}. \end{aligned}$$

By Lemma 5, the corresponding equation

$$z_n = y_{\overline{n}} + p_n z_{n-\tau} + \frac{1}{(m^*-1)!(m-m^*-1)!}\sum_{i=\overline{n}}^{n-1}(n-i+1)^{(m^*-1)}$$
$$\times \sum_{j=i}^{\infty}(j-i+m-m^*-1)^{(m-m^*-1)} f(j, z_{g_1(j)}, \cdots, z_{g_l(j)}), \quad n \geq \overline{n}$$

has a positive solution $\{z_n\}$. Clearly, $\{z_n\}$ is a positive solution of (1). This contradiction completes the proof.

When $\tau = 1$ or $p_n \equiv 0$, Theorem 1 reduces the necessary and sufficient condition for the oscillation of all solutions of the even or odd order difference equation with deviating arguments.

COROLLARY 1. *Every solution of the equation*

$$\Delta^{m+1} x_{n-1} + f(n, x_{g_1(n)}, \cdots, x_{g_l(n)}) = 0 \tag{17}$$

is oscillatory if and only if the corresponding inequality

$$\Delta^{m+1} x_{n-1} + f(n, x_{g_1(n)}, \cdots, x_{g_l(n)}) \leq 0 \tag{18}$$

has no eventually positive solutions.

COROLLARY 2. *Every solution of the equation*

$$\Delta^{m} x_{n} + f(n, x_{g_1(n)}, \cdots, x_{g_l(n)}) = 0 \tag{19}$$

is oscillatory if and only if the corresponding inequality

$$\Delta^{m} x_{n} + f(n, x_{g_1(n)}, \cdots, x_{g_l(n)}) \leq 0 \tag{20}$$

has no eventualy positive solutions.

THEOREM 2. *Assume that the assumptions of Lemma 2 hold. Further, assume that $g_i(n) \leq h_i(n), i = 1, 2, \cdots, l$ and that (9) holds. Then every solution of the equation*

$$\Delta^{m} x_{n} + f(n, x_{h_1(n)}, \cdots, x_{h_l(n)}) = 0 \tag{21}$$

is oscillatory implies the same for Eq. (1).

Proof. Assume the contrary, and let $\{x_n\}$ be an eventually positive solution of (1). Let $y_n = x_n - p_n x_{n-\tau}$. Then by Lemmas 2 and 3, $y_n > 0, \Delta y_n < 0$ eventually. In view of (iv) and since $x_n \geq y_n$, we have

$$\begin{aligned} 0 &= \Delta^m y_n + f(n, x_{g_1(n)}, \cdots, x_{g_l(n)}) \\ &\geq \Delta^m y_n + f(n, y_{g_1(n)}, \cdots, y_{g_l(n)}) \\ &\geq \Delta^m y_n + f(n, y_{h_1(n)}, \cdots, y_{h_l(n)}). \end{aligned}$$

By Corollary 2, (21) has an eventually positive solution. This is a contradiction and complete the proof of Theorem 2.

We now compare Eq. (1) with the equation

$$\Delta^m(x_n - P_n x_{n-\tau}) + F(n, x_{g_1(n)}, \cdots, x_{g_l(n)}) = 0, \tag{22}$$

where $P : N(K) \to R^+ = [0, \infty), F : N \times R^l \to R$ is continuous with respect to the last l arguments satisfying

(I) $F(n, -u_1, -u_2, \cdots, -u_l) = -F(n, u_1, u_2, \cdots, u_l)$, for $u_1 u_i > 0$, $i = 2, \cdots, l$.

(II) $u_1 F(n, u_1, \cdots, u_l) \geq 0$, for $u_1 u_i > 0, i = 2, \cdots, l$.

THEOREM 3. *Assume that either there is $\{n_k\}$: $n_k \to \infty$ as $k \to \infty$ such that $P_{n_k} = 0$ and $g_i(n) < n$, for $n \geq \overline{n}, i = 1, 2, \cdots, l$, or there exists a positive integer n^* such that $P_n > 0$ for $n \geq n^*$ and*

$$\sum_{j=1}^{\infty} \Big(\prod_{i=1}^{j} P_{n^*+i\tau} \Big)^{-1} = \infty. \tag{23}$$

Further, assume that $P_n \geq p_n$ and that

$$|F(n, u_1, u_2, \cdots, u_l)| \geq |f(n, u_1, u_2, \cdots, u_l)|, \quad \text{for } u_1 u_i > 0, i = 2, \cdots, l. \tag{24}$$

Then every solution of Eq. (1) is oscillatory implies the same for Eq. (22).

Proof. Assume the contrary, and let $\{x_n\}$ be an eventually positive solution of (22). Let $y_n = x_n - P_n x_{n-\tau}$. As in the proof of Theorem 1 we see that $y_n > 0$ for large n and (15) holds. If $m^* = 0$. Summing (22) from n to ∞ m times we have

$$y_n \geq y_\infty + \frac{1}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} F(j, x_{g_1(j)}, \cdots, x_{g_l(j)})$$

and so

$$x_n \geq y_\infty + P_n x_{n-\tau} + \frac{1}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} F(j, x_{g_1(j)}, \cdots, x_{g_l(j)})$$
$$\geq p_n x_{n-\tau} + \frac{1}{(m-1)!} \sum_{j=n}^{\infty} (j - n + m - 1)^{(m-1)} f(j, x_{g_1(j)}, \cdots, x_{g_l(j)}),$$

where $y_\infty = \lim_{n\to\infty} y_n \geq 0$. Using a similar method as in the proof of Theorem 1, one can see that Eq. (1) has also an eventually positive solution which contradicts the assumption.

If $m^* > 0$, as shown in the proof of Theorem 1, we obtain that for $n \geq \overline{n}$

$$y_n \geq y_{\overline{n}} + \frac{1}{(m^* - 1)!(m - m^* - 1)!} \sum_{i=N}^{n-1} (n - i + 1)^{(m^*-1)}$$

$$\times \sum_{j=i}^{\infty}(j-i+m-m^*-1)^{(m-m^*-1)}F(j,x_{g_1(j)},\cdots,x_{g_l(j)}).$$

Hence for $n \geq \overline{n}$

$$x_n \geq y_{\overline{n}} + P_n x_{n-\tau} + \frac{1}{(m^*-1)!(m-m^*-1)!}\sum_{i=N}^{n-1}(n-i+1)^{(m^*-1)}$$

$$\times \sum_{j=i}^{\infty}(j-i+m-m^*-1)^{(m-m^*-1)}F(j,x_{g_1(j)},\cdots,x_{g_l(j)})$$

$$\geq y_{\overline{n}} + p_n x_{n-\tau} + \frac{1}{(m^*-1)!(m-m^*-1)!}\sum_{i=N}^{n-1}(n-i+1)^{(m^*-1)}$$

$$\times \sum_{j=i}^{\infty}(j-i+m-m^*-1)^{(m-m^*-1)}f(j,x_{g_1(j)},\cdots,x_{g_l(j)}).$$

The rest of the proof is the same as that of Theorem 1. One can see that Eq. (1) has also an eventually positive solution which contradicts the assumption and competes with the proof of Theorem 3.

Now we consider the case where $p_n \equiv 1$ and $g_i(n) = n - \sigma_i, \sigma_i \in R^+$, $i = 1, 2, \cdots, l$.

LEMMA 6. *Every solution of the neutral difference equation*

$$\Delta^m(x_n - x_{n-\tau}) + f(n, x_{n-\sigma_1}, \cdots, x_{n-\sigma_l}) = 0 \tag{25}$$

is oscillatory if and only if every solution of the even order non-neutral diffrence equation

$$\Delta^{m+1}x_{n-1} + \frac{1}{\tau}f(n, x_n, \cdots, x_n) = 0 \tag{26}$$

is oscillatory.

Lemma 6 with $l = 1$ has been proved by Zhang and Yang in [8]. For the case where $l > 1$, the results of Lemma 6 can be proved, using a slight modification of that in Theorem 2.1 of [8], and thus, the proof is omitted. By Theorem 3 and Lemma 6, we can obtain the following result.

THEOREM 4. *Assume that $p_n \geq 1$ and that there exists a n^* such that* (2) *holds. Then every solution of Eq.* (26) *is oscillatory implies the same for Eq.* (1).

COROLLARY 3. *Let $\sigma_i (i = 1, 2\cdots, l)$ be any nonnegative integers. Then the oscillation of the following two equations*

$$\Delta^{m+1}x_{n-1} + f(n, x_{n-\sigma_1}, \cdots, x_{n-\sigma_l}) = 0$$

and

$$\Delta^{m+1}x_{n-1} + f(n, x_n, \cdots, x_n) = 0$$

is equivalent.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic Publishers, Dordrecht, 1997.

[3] L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[4] K. Gopalsamy, B. S. Lalli and B. G. Zhang, *Oscillation in odd order neutral differential equations*, Czech. Math. J. 42 (1992), 313–323.

[5] P. J. Y. Wong and R. P. Agarwal, *Comparison theorems for the oscillation of higher order difference equations with deviating arguments*, Math. Comput. Modelling 24 (1996), 39–48.

[6] J. Yan, *Comparison theorems for the oscillation of higher order neutral equations and applications*, Scientia Sinica, Ser A12 (1990), 1256–1266.

[7] B. G. Zhang, J. S. Yu and Z. C. Wang, *Oscillations of higher order neutral differential equations*, Rocky Mountain J. Math. 25 (1995), 557–568.

[8] B. G. Zhang and B. Yang, *Oscillation in higher order nonlinear difference equations* (in Chinese), Chinese Ann. Math. 20 (1999), 71–80.

[9] G. Zhang and Y. Gao, *Positive solutions of higher order nonlinear difference equations* (in Chinese), J. Systems Sci. Math. Sci. 19 (1999), 157–161.

[10] Yong Zhou, *Oscillation of neutral difference equations*, Kyungpook Math. J. 39 (1999), 283–291.

[11] Yong Zhou and B. G. Zhang, *The semicycles of solutions of neutral difference equations*, Appl. Math. Lett. 13 (2000), 59–66.

[12] Yong Zhou and B. G. Zhang, *Classification and existences of nonoscillatory solutions of second-order neutral delay difference equation*, J. Anal. Appl. 20 (2001), 223–234.

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Received July 11st., 2001; revised version February 2nd, 2002.