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GENERAL SOLUTION OF SOME FUNCTIONAL EQUATIONS RELATED TO THE DETERMINANT OF SOME SYMMETRIC MATRICES

Abstract. In this paper, we determine the general solution of the functional equation $f(ux+vy,\, uy+vx) = g(x,y)\,h(u,v)$ where $f,g,h : \mathbb{R}^2 \to \mathbb{R}$ are unknown functions. We also treat the equation $f(ux+vy,\, uy+vx,\, zw) = g(x,y,z)\,h(u,v,w)$ where $f,g,h : \mathbb{R}^3 \to \mathbb{R}$ are unknown functions. Our method is elementary and we do not use any regularity conditions.

1. Introduction

Let us define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = det \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

for all $x, y \in \mathbb{R}$. Then, since

$$det \begin{pmatrix} ux+vy & uy+vx \\ uy+vx & ux+vy \end{pmatrix} = det \begin{pmatrix} x & y \\ y & x \end{pmatrix} \cdot det \begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

we have the functional equation

$$f(ux+vy,\, uy+vx) = f(x,y)\, f(u,v) \tag{1}$$

for all $x, y, u, v \in \mathbb{R}$. Obviously, $f(x,y) = x^2 - y^2$ is a solution of the functional equation (1). In this note we determine all general solutions of the above functional equation (1) and its pexiderized version

$$f(ux+vy,\, uy+vx) = g(x,y)\, h(u,v)$$

without any regularity assumptions. We also treat the functional equation

$$f(ux+vy,\, uy+vx,\, zw) = f(x,y,z)\, f(u,v,w) \tag{2}$$

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and its pexiderized version

$$f(ux+vy,\, uy+vx,\, wz) = g(x,y,z)\, h(u,v,w) \tag{3}$$

for all $x,y,z,u,v,w \in \mathbb{R}$. The functional equation (2) arises in a similar manner by defining a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = det \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix}$$

for all $x,y,z \in \mathbb{R}$. The interested reader should refer to books [1] and [2] for an account on functional equations.

2. The solution of equation (1)

A map $M : \mathbb{R} \to \mathbb{R}$ is said to be multiplicative if and only if $M(xy) = M(x)\, M(y)$ for all $x,y \in \mathbb{R}$.

THEOREM 1. *The function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the functional equation* (1) *for all $x,y,u,v \in \mathbb{R}$ if and only if*

$$f(x,y) = M_1(x+y)\, M_2(x-y) \tag{4}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative maps.

Proof. Suppose f is identically a constant, say $f \equiv c$. Then from (1), we have $c^2 - c = 0$ which implies $c = 0$ or $c = 1$. Hence the identically constant solutions of (1) are $f(x,y) = 0$ and $f(x,y) = 1$ for all $x,y \in \mathbb{R}$. Since multiplicative maps can be identically zero or one, these solutions are included in (4).

From now on we assume that f is not identically constant, that is $f \not\equiv c$, where c is a constant. We define a function $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) = f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \tag{5}$$

for all $x,y \in \mathbb{R}$. Next, using (5) in (1), we see that

$$\begin{aligned} F((x+y)(u+v),\, (x-y)(u-v)) \\ = F(x+y,\, x-y)\, F(u+v,\, u-v) \end{aligned} \tag{6}$$

for all $x,y,u,v \in \mathbb{R}$. Substituting $x_1 = x+y,\ y_1 = x-y,\ x_2 = u+v$ and $y_2 = u-v$ in (6), we have

$$F(x_1x_2, y_1y_2) = F(x_1,y_1)\, F(x_2,y_2) \tag{7}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Setting $y_1 = y_2 = 1$ in (7), we see that

$$F(x_1x_2, 1) = F(x_1,1)\, F(x_2,1) \tag{8}$$

for all $x_1, x_2 \in \mathbb{R}$. Defining $M_2 : \mathbb{R} \to \mathbb{R}$ by

$$M_1(x) = F(x, 1) \tag{9}$$

for all $x \in \mathbb{R}$, we see that (8) reduces to

$$M_1(x_1x_2) = M_1(x_1)\, M_1(x_2) \tag{10}$$

for all $x_1, x_2 \in \mathbb{R}$. Hence $M_1 : \mathbb{R} \to \mathbb{R}$ is a multiplicative map.

Similarly, setting $x_1 = x_2 = 1$ in (7), we have

$$F(1, y_1y_2) = F(1, y_1)\, F(1, y_2) \tag{11}$$

for all $y_1, y_2 \in \mathbb{R}$. Defining $M_2 : \mathbb{R} \to \mathbb{R}$ by

$$M_2(y) = F(1, y) \tag{12}$$

for all $x \in \mathbb{R}$, we see that (11) reduces to

$$M_2(y_1y_2) = M_2(y_1)\, M_2(y_2) \tag{13}$$

for all $y_1, y_2 \in \mathbb{R}$. Hence $M_2 : \mathbb{R} \to \mathbb{R}$ is also a multiplicative map.

Now letting $x_2 = 1 = y_1$ in (7), we obtain

$$F(x_1, y_2) = F(x_1, 1)\, F(1, y_2) \tag{14}$$

for all $x_1, y_2 \in \mathbb{R}$ which yields

$$F(x_1, y_2) = M_1(x_1)\, M_2(y_2) \tag{15}$$

for all $x_1, y_2 \in \mathbb{R}$.

Now using (15) in (5), we have

$$f(x, y) = F(x + y, x - y) = M_1(x + y)\, M_2(x - y) \tag{16}$$

for all $x, y \in \mathbb{R}$, that is the asserted solution (4).

Since the asserted solution given in (4) satisfies the functional equation (1) the proof of the theorem is now complete. ■

The following corollary follows from Theorem 1.

COROLLARY 1. *The continuous or measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the functional equation (1) for all $x, y, u, v \in \mathbb{R}$ if and only if*

$$f(x, y) = 0, \qquad f(x, y) = 1, \quad \text{and} \quad f(x, y) = (x + y)^a (x - y)^b \tag{17}$$

for all $x, y \in \mathbb{R}$, where a and b are arbitrary real constants such that the domain of f is $\mathbb{R}^2$.

Now we give the general solution of the Pexiderized version of the functional equation (1).

THEOREM 2. *The functions $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ satisfy the functional equation*

$$f(ux+vy,\ uy+vx) = g(x,y)\,h(u,v) \tag{18}$$

for all $x, y, u, v \in \mathbb{R}$ if and only if

$$f \equiv 0, \qquad g \equiv 0 \quad \text{and } h \text{ is arbitrary} \tag{19}$$

or

$$f \equiv 0, \qquad h \equiv 0 \quad \text{and } g \text{ is arbitrary} \tag{20}$$

or

$$\begin{aligned} f(x,y) &= \alpha\,\beta\,M_1(x+y)\,M_2(x-y),\\ g(x,y) &= \beta\,M_1(x+y)\,M_2(x-y),\\ h(x,y) &= \alpha\,M_1(x+y)\,M_2(x-y), \end{aligned} \tag{21}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative maps and α, β are nonzero real arbitrary constants.

Proof. Letting $u = v = 1$ in (18), we have for some constant α

$$f(x,y) = \alpha\,g(x,y) \tag{22}$$

for all $x, y \in \mathbb{R}$. Similarly, letting $x = y = 1$ in (18), we get for some constant β

$$f(u,v) = \beta\,h(u,v) \tag{23}$$

for all $u, v \in \mathbb{R}$. If either $\alpha = 0$ or $\beta = 0$, we get

$$f \equiv 0, \qquad g \equiv 0 \quad \text{and } h \text{ is arbitrary}$$

or

$$f \equiv 0, \qquad h \equiv 0 \quad \text{and } g \text{ is arbitrary.}$$

Next, we suppose $\alpha \neq 0 \neq \beta$. Then using (22) and (23) in (18), we have

$$\frac{f(ux+vy,\ uy+vx)}{\alpha\,\beta} = \frac{f(x,y)}{\alpha\,\beta}\,\frac{f(u,v)}{\alpha\,\beta} \tag{24}$$

where $x, y, u, v \in \mathbb{R}$. From Theorem 1, we have

$$f(x,y) = \alpha\,\beta\,M_1(x+y)\,M_2(x-y) \tag{25}$$

for all $x, y \in \mathbb{R}$. From (22), (23) and (25),

$$g(x,y) = \beta\,M_1(x+y)\,M_2(x-y)$$

and

$$h(x,y) = \alpha\,M_1(x+y)\,M_2(x-y)$$

and the proof is now complete. ∎

3. The solution of equation (2)

THEOREM 3. *The function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies the functional equation (2) for all $x, y, z, u, v, w \in \mathbb{R}$ if and only if*

$$f(x,y,z) = M_1(x+y)\, M_2(x-y)\, M_3(z), \tag{26}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative maps.

Proof. If f is identically a constant function, then similar to the proof of Theorem 1, we have $f(x,y,x) = 0$ or $f(x,y,z) = 1$ for all $x, y, z \in \mathbb{R}$. These solutions are included in (26).

Next we assume f is not identically a constant function. Define a function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x,y,z) = f\left(\frac{x+y}{2}, \frac{x-y}{2}, z\right). \tag{27}$$

As in the proof of Theorem 1, using (27) in (2) and then substituting $x_1 = x+y$, $y_1 = x-y$, $x_2 = u+v$ and $y_2 = u-v$, we have

$$F(x_1x_2,\, y_1y_2,\, zw) = F(x_1,\, y_1,\, z)\, F(x_2,\, y_2,\, w). \tag{28}$$

Using $z = w = 1$ in (28) and then using a similar argument as in Theorem 1, we get

$$F(x,y,1) = M_1(x)\, M_2(y), \tag{29}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions. Letting $x_1 = x_2 = y_1 = y_2 = 1$ in (28), we get

$$F(1,1,zw) = F(1,1,z)\, F(1,1,w) \tag{30}$$

for all $z, w \in \mathbb{R}$. Hence

$$F(1,1,z) = M_3(z) \tag{31}$$

where $M_3 : \mathbb{R} \to \mathbb{R}$ is a multiplicative function. Next, letting $x_2 = y_2 = z = 1$ in (28), we get

$$F(x_1, y_1, w) = F(x_1, y_1, 1)\, F(1,1,w). \tag{32}$$

Hence by (29), (31) and (32), we have

$$F(x_1, y_1, w) = M_1(x_1)\, M_2(y_1) M_3(w). \tag{33}$$

Using (33) together with (27), we have the asserted solution (26) and the proof of the theorem is now complete. ∎

By using Theorem 3, one can easily prove the following theorem similar to the proof of Theorem 2.

THEOREM 4. *The functions $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ satisfy the functional equation*

$$f(ux+vy,\, uy+vx,\, zw) = g(x,y,z)\, h(u,v,w) \tag{34}$$

for all $x, y, z, u, v, w \in \mathbb{R}$ *if and only if*

$$f \equiv 0, \qquad g \equiv 0 \quad \text{and } h \text{ is arbitrary} \tag{35}$$

or

$$f \equiv 0, \qquad h \equiv 0 \quad \text{and } g \text{ is arbitrary} \tag{36}$$

or

$$\begin{aligned} f(x,y,z) &= \alpha\,\beta\, M_1(x+y)\, M_2(x-y),\, M_3(z),\\ g(x,y,z) &= \beta\, M_1(x+y)\, M_2(x-y)\, M_3(z),\\ h(x,y,z) &= \alpha\, M_1(x+y)\, M_2(x-y)\, M_3(z) \end{aligned} \tag{37}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ *are multiplicative maps and* α, β *are nonzero real arbitrary constants.*

References

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