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Franciszek Bogowski, Andrzej Wesołowski

THE CONSTRUCTION OF A CERTAIN QUASI-CONFORMAL EXTENSION OF THE FUNCTIONS DEFINED IN THE UNIT DISC, ON THE CLOSED PLANE

Abstract. In this paper a certain method of the construction of a quasi-conformal extension of the functions defined in the unit disc, on the closed plane $\overline{\mathbb{C}}$ is given. The earlier known results are received in particular cases, ([1], [2], [3], [9]).

1. Introduction

Let us denote $\mathbb{C}$—a complex plane, $D = \{z \in \mathbb{C} : |z| < 1\}$—open unit disc,

$$S_f = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

a Schwarzian derivative of a holomorphic function f.

L. V. Ahlfors and G. Weill, [2], proved that if the holomorphic function f satisfies inequality

$$|S_f(z)| \leq k(1-|z|^2)^{-2}, \quad z \in D$$

for a certain constant $k \in (0;2)$, then this function has a quasi-conformal extension on the closed plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

J. Becker, [3], proved that if in D the inequality

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq k,$$

holds,where $h \in < 0;1)$, then f has a quasi-conformal extension on $\overline{\mathbb{C}}$.

In the paper [4] there was proved the following:

THEOREM 1. *Let f, g be the holomorphic functions in D. If f and g are locally uniform in D and if there exists the holomorphic function h such that $\operatorname{Re} h(z) \geq \frac{1}{2}$ for $z \in D$ and the inequality*

$$(1)\quad \left| \frac{h(z)-1}{h(z)}|z|^2 - (1-|z|^2)\left[\frac{zh'(z)}{h(z)} + (1-2\alpha)\frac{zf''(z)}{f'(z)} + 2\alpha\frac{zg''(z)}{g'(z)}\right] - -(1-|z|^2)^2\frac{z}{\bar{z}}h(z)\left[\alpha\left(\alpha-\frac{1}{2}\right)\left(\frac{f''(z)}{f'(z)}-\frac{g''(z)}{g'(z)}\right)^2 + \alpha(S_f - S_g)\right]\right| \le 1$$

holds for an arbitrary $\alpha \in \mathbb{C}$ *and* $z \in D$, *then the function* f *is univalent in* D.

The assumptions given in the above theorem in view of the univalence of the function f appear to be valid too for the quasi-conformal extension of this function on $\overline{\mathbb{C}}$.

A certain modification of Theorem 1 allows to obtain the results given in Theorems 2 and 4.

2. The main result

THEOREM 2. *Let* f *and* g *be the holomorphic functions in* $\overline{D}$. *If this functions are locally univalent in* $\overline{D}$ *and if there exists the function* h *holomorphic in* $\overline{D}$ *such that for a certain fixed* $k \in <0;1)$ *and arbitrary* $\alpha \in \mathbb{C}$ *inequality*

$$(2)\quad \left| \frac{h(z)-1}{h(z)}|z|^2 - (1-|z|^2)\left[\frac{zh'(z)}{h(z)} + (1-2\alpha)\frac{zf''(z)}{f'(z)} + 2\alpha\frac{zg''(z)}{g'(z)}\right] - -\alpha(1-|z|^2)^2\frac{z}{\bar{z}}h(z)\left[\left(\alpha-\frac{1}{2}\right)\left(\frac{f''(z)}{f'(z)}-\frac{g''(z)}{g'(z)}\right)^2 + (S_f(z) - S_g(z))\right]\right| \le k,$$

holds, where $\left|\frac{1}{h(z)} - 1\right| \le k$, $z \in \overline{D}$, *then there exists the function* F *such that* $F|_{\overline{D}} = f$ *and* F *maps in the quasi-conformal manner of the order* $\frac{1+k}{1-k}$ $\overline{\mathbb{C}}$ *on* $\overline{\mathbb{C}}$.

This function has the form

$$(3)\quad F(z) = \begin{cases} f(z) & \textit{for}\quad |z| \le 1, \\ f\left(\frac{1}{\bar{z}}\right) + \dfrac{(z-\frac{1}{\bar{z}})h(\frac{1}{\bar{z}})f'(\frac{1}{\bar{z}})}{1+\alpha(z-\frac{1}{\bar{z}})h(\frac{1}{\bar{z}})\left(\frac{g''(\frac{1}{\bar{z}})}{g'(\frac{1}{\bar{z}})} - \frac{f''(\frac{1}{\bar{z}})}{f'(\frac{1}{\bar{z}})}\right)} & \textit{for}\quad |z| \ge 1. \end{cases}$$

Proof. The function f is univalent in D because it satisfies a "strengthened" sufficient condition of the univalence (1) from Theorem 1.

We may assume, as in Theorem 1, that both functions f and g have a classic normalization in the neighbourhood of the origin, and have the second coefficients equal. Thus

$$f(z) = g(z) + O(z^3), \qquad \frac{f'(z)}{g'(z)} = 1 + O(z^2), \quad \text{for } z \to 0.$$

Let

$$v(z) = \left(\frac{g'(z)}{f'(z)}\right)^{\alpha} = 1 + \beta z^2 + O(z^3) \quad \text{for } z \to 0,\ \alpha \in \mathbb{C},$$

(we assume here this branch of the power $(\cdot)^{\alpha}$, which for $z = 0$ has the value 1) and $u(z) = f(z)v(z)$. For $|z| \geq 1$ we form the function

$$H(z) = \frac{u\left(\frac{1}{\overline{z}}\right) + \left(z - \frac{1}{\overline{z}}\right) h\left(\frac{1}{\overline{z}}\right) u'\left(\frac{1}{\overline{z}}\right)}{v\left(\frac{1}{\overline{z}}\right) + \left(z - \frac{1}{\overline{z}}\right) h\left(\frac{1}{\overline{z}}\right) v'\left(\frac{1}{\overline{z}}\right)}.$$

The simple calculations give

$$v' = \alpha \left(\frac{g'}{f'}\right)^{\alpha-1} \frac{g''f' - g'f''}{(f')^2},$$

$$u' = f'\left(\frac{g'}{f'}\right)^{\alpha} + \alpha f \left(\frac{g'}{f'}\right)^{\alpha-1} \frac{g''f' - g'f''}{(f')^2},$$

where the derivatives are calculated in the point z.

After some transformations using the above formulas we obtain

$$H(z) = f\left(\frac{1}{\overline{z}}\right) + \frac{\left(z - \frac{1}{\overline{z}}\right) h\left(\frac{1}{\overline{z}}\right) f'\left(\frac{1}{\overline{z}}\right)}{1 + \alpha\left(z - \frac{1}{\overline{z}}\right) h\left(\frac{1}{\overline{z}}\right)\left(\frac{g''\left(\frac{1}{\overline{z}}\right)}{g'\left(\frac{1}{\overline{z}}\right)} - \frac{f''\left(\frac{1}{\overline{z}}\right)}{f'\left(\frac{1}{\overline{z}}\right)}\right)} \quad \text{for } |z| \geq 1.$$

It is easy to note that on $H(z) = f(z)$ for $|z| = 1$.

Let

$$F(z) = \begin{cases} f(z) & \text{for } |z| \leq 1, \\ H(z) & \text{for } |z| \geq 1. \end{cases} \tag{4}$$

We will show that the function (4) realize the quasi-conformal mapping of the order $\frac{1+k}{1-k}$ $\overline{\mathbb{C}}$ on $\overline{\mathbb{C}}$.

For $|z| > 1$ and $z \neq \infty$, after calculations we obtain

$$\frac{F'_{\overline{z}}}{F'_{z}} = \frac{1}{\overline{z}^2}\left\{\frac{h-1}{h} - \left(z - \frac{1}{\overline{z}}\right)\left[\frac{h'}{h} + (1-2\alpha)\frac{f''}{f'} + 2\alpha\frac{g''}{g'}\right] - \right.$$

$$\left. -\alpha\left(z - \frac{1}{\overline{z}}\right)^2 h\left[\left(\alpha - \frac{1}{2}\right)\left(\frac{f''}{f'} - \frac{g''}{g'}\right)^2 + S_f - S_g\right]\right\},$$

where h, h', g', g'', S_f, S_g are calculated in the point $1/\overline{z}$.

The modulus of the right side of the $F'_{\overline{z}}/F'_{z}$ is equal to the value of the left side of the relation (2), when we replace z by $1/\overline{z}$.

Hence on the base (2) we have

$$\left|\frac{F'_{\bar z}}{F'_z}\right| = |\mu_F(z)| \le k, \quad z \in \mathbb{C}\setminus\overline{D},$$

where $\mu_F(z)$ is a complex dilatation of the function F.

We will show now, that $F'_z \neq 0$ for $z \in \mathbb{C}\setminus\partial D$. Of course, it holds for $z \in D$, because then, F is univalent in D.

For $z \in \mathbb{C}\setminus\overline{D}$ we obtain after some calculations

$$F'_z = \frac{h\left(\frac{1}{\bar z}\right) f'\left(\frac{1}{\bar z}\right)}{\left[1+\alpha\left(z-\frac{1}{\bar z}\right) h\left(\frac{1}{\bar z}\right)\left(\frac{g''\left(\frac{1}{\bar z}\right)}{g'\left(\frac{1}{\bar z}\right)} - \frac{f''\left(\frac{1}{\bar z}\right)}{f'\left(\frac{1}{\bar z}\right)}\right)\right]^2}$$

The numerator of the last expression is different from zero. The denominator would be equal to ∞ only for $z=\infty$. It is impossible since

$$\left(z-\frac{1}{\bar z}\right)\left[\frac{g''\left(\frac{1}{\bar z}\right)}{g'\left(\frac{1}{\bar z}\right)} - \frac{f''\left(\frac{1}{\bar z}\right)}{f'\left(\frac{1}{\bar z}\right)}\right]$$

is bounded in view of the assumed normalization, from which it follows that

$$\frac{g''\left(\frac{1}{\bar z}\right)}{g'\left(\frac{1}{\bar z}\right)} - \frac{f''\left(\frac{1}{\bar z}\right)}{f'\left(\frac{1}{\bar z}\right)} = O\left(\frac{1}{\bar z}\right) \quad \text{for } z\to\infty.$$

First consider such subset $\mathbb{C}\setminus\overline{D}$, in which F, F'_z, $F'_{\bar z}$ are finite. Denote for A set: $A=\{z : z\in\mathbb{C}\setminus\overline{D} \text{ and } M(z)=0\}$, where

$$M(z) = 1+\alpha\left(z-\frac{1}{\bar z}\right) h\left(\frac{1}{\bar z}\right)\left(\frac{g''\left(\frac{1}{\bar z}\right)}{g'\left(\frac{1}{\bar z}\right)} - \frac{f''\left(\frac{1}{\bar z}\right)}{f'\left(\frac{1}{\bar z}\right)}\right), \quad \alpha\in\mathbb{C}.$$

Thus, for $z\in\mathbb{C}\setminus A$ the derivative $F'_z \neq 0$ and is finite and

$$|\mu_F(z)| \le k < 1.$$

Let now $z\in A$. We assume in this case $G(z)=\frac{1}{F(z)}$.

It is easy to note that $|\mu_G(z)| = |\mu_F(z)|$.

We will prove that the derivative $G'_z = \frac{\partial}{\partial z}\left(\frac{1}{F(z)}\right) \neq 0$ and is finite for $z\in A$. For $z\in A$

$$G'_z = -\frac{1}{\left(z-\frac{1}{\bar z}\right)^2 h\left(\frac{1}{\bar z}\right) f'\left(\frac{1}{\bar z}\right)}.$$

Because $z-\frac{1}{\bar z}\neq 0$ (since $z\in A$) $f'\left(\frac{1}{\bar z}\right)\neq 0$ (since f is locally univalent and $h\left(\frac{1}{\bar z}\right)\neq 0$), therefore $|G'_z|<\infty$.

Similarly we can show, that for $z \in A$, $G'_{\overline{z}}$ is finite too, and in this case we have:

$$\frac{G'_{\overline{z}}}{G'_z} = \frac{\alpha\left(z-\frac{1}{\overline{z}}\right)}{\overline{z}^2}\left\{\left[h - h'\left(z-\frac{1}{\overline{z}}\right)\right]\left(\frac{g''}{g'}-\frac{f''}{f'}\right) - \right.$$
$$\left. - \left(z-\frac{1}{\overline{z}}\right)h\left[S_g - S_f + \frac{1}{2}\left(\frac{g''}{g'}\right)^2 - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right]\right\},$$

where h, h', g', g'', f', f'', S_f, S_g are calculated in the point $1/\overline{z}$.

For $z \in A$ the modulus of the right side of the expression $G'_{\overline{z}}/G'_z$ is equal to the value of the left side of the relation (2), when we replace z by $1/\overline{z}$, therefore on the base (2) we have

$$|\mu_F(z)| = |\mu_G(z)| = \frac{|G'_{\overline{z}}|}{|G'_z|} \le k < 1.$$

We will prove now that for each point which is on the unit circle, F is the local homeomorphism (see [5]), which holds the orientation.

Then, let $z_0 = e^{i\theta} \in \partial D$, $\theta \in < 0; 2\pi >$ also

$$O(z_0, r_0) = \{z \in \mathbb{C} : |z - z_0| < r_0,\ r_0 > 0\}$$

and

$$H(z) = f\left(\frac{1}{\overline{z}}\right) + \frac{\left(z-\frac{1}{\overline{z}}\right)h\left(\frac{1}{\overline{z}}\right)f'\left(\frac{1}{\overline{z}}\right)}{1+\alpha\left(z-\frac{1}{\overline{z}}\right)h\left(\frac{1}{\overline{z}}\right)\left(\frac{g''\left(\frac{1}{\overline{z}}\right)}{g'\left(\frac{1}{\overline{z}}\right)} - \frac{f''\left(\frac{1}{\overline{z}}\right)}{f'\left(\frac{1}{\overline{z}}\right)}\right)}, \quad z \in O(z_0, r_0),$$

where r_0 is chosen in such a manner, that in the neighbourhood $O(z_0, r_0)$ we have:

$$f(z_0 + q) = f(z_0) + f'(z_0)q + o(q^2),$$
$$H(z_0 + q_1) = F(z_0) + H'_z(z_0)q_1 + H'_{\overline{z}}(z_0)\overline{q}_1 + o(q_1^2)$$
$$= f(z_0) + f'(z_0)\left[h(z_0)q_1 + e^{2i\theta}(h(z_0)-1)\overline{q}_1\right] + o(q_1^2),$$

where $|q| \le r_0$, $|q_1| \le r_0$ and h is the holomorphic function.

If J and J_1 are respectively Jacobians of the functions f and H, then

$$J(z_0) = |f'(z_0)|^2,$$
$$J_1(z_0) = |H'_z(z_0)|^2 - |H'_{\overline{z}}(z_0)|^2 = |f'(z_0)|^2|h(z_0)|^2\left[1 - \left|\frac{1}{h(z)} - 1\right|^2\right].$$

From the assumptions it follows that $J(z_0) > 0$ and $J_1(z_0) > 0$.

We may assume (see [6], p. 380-381) that $O(z_0, r_0)$ is a neighbourhood chosen so that in the closure of which the function f is univalent, and H is the homeomorphism which holds the orientation.

Let us remind that

$$F(z)=\begin{cases} f(z) & \text{for } \ z\in\overline{D},\\ H(z) & \text{for } \ z\in D^0=\{z\in\overline{\mathbb{C}}:|z|\geq 1\},\end{cases}$$

and that $H(e^{i\psi})=f(e^{i\psi})$, $\psi\in<0;2\pi>$. We will show that there exist such neighbourhood of the point z_0 in which F is the single valued mapping.

From the definition of F and the earlier considerations it follows that this function is univalent in $\overline{O}(z_0,r_0)\cap\overline{D}$ and in $\overline{O}(z_0,r_0)\cap\overline{D}^0$ for each neighbourhood $O(z_0,r_1)$, $r_1\leq r_0$.

Since F is the homeomorphism in $\overline{O}(z_0,r_0)\cap\overline{D}^0$ holding the orientation and $F=f$ on $\overline{O}(z_0,r_0)\cap\partial D$, then such small neighbourhood $\overline{O}(z_0,r_2)$, $r_2\leq r_0$ can be chosen, that in this neighbourhood F is the single valued mapping.

In the opposite case, a certain point

$$z_0+q\in\overline{O}(z_0,r_2)\cap D^0,\ \ |q|=r_2$$

would be mapped by F into the point of the set $F(D\cap\partial\overline{O}(z_0,r_2))$, (where $F(A)$ denotes the image of the set A by the transformation F). The above is impossible because a certain interior point of the set $F(\overline{O}(z_0,r_0)\cap\overline{D}^0)$ would belong to the curve $F(O(z_0,r_0)\cap\partial D)$. It would be contradictory to homeomorphity of F in $\overline{O}(z_0,r_0)\cap\overline{D}^0$.

It has to be shown that F is the local homeomorphism holding the orientation in a certain neighbourhood of the point $z=\infty$.

Let us remind, that from the assumed normalization of the functions f and g it follows, that

$$\frac{f''(w)}{f'(w)}-\frac{g''(w)}{g'(w)}=O(w),\ \ \text{when}\ \ w\to 0.$$

From the above and from the form of the function F defined by the formula (3) for $|z|>1$ the dominator of the fraction on the right side of this formula is bounded in a certain neighbourhood of the point $z=\infty$. Thus $\lim\limits_{z\to\infty}F(z)=\infty$.

Then, let us take

$$\widetilde{F}(z)=\frac{1}{F(z)}.$$

It is known that $|\mu_{\widetilde{F}}(z)|=|\mu_F(z)|$. From the calculations it follows that

$$\widetilde{F}'_z\Big|_{z=\infty}=\frac{1}{h(0)f'(0)},$$

and

$$\tilde{F}'_{\bar{z}}\Big|_{z=\infty} = \frac{\alpha}{f'(0)}\frac{\partial}{\partial \bar{z}}\left(\frac{g''\left(\frac{1}{\bar{z}}\right)}{g'\left(\frac{1}{\bar{z}}\right)} - \frac{f''\left(\frac{1}{\bar{z}}\right)}{f'\left(\frac{1}{\bar{z}}\right)}\right)\Big|_{z=\infty}.$$

Since

$$\frac{g''\left(\frac{1}{\bar{z}}\right)}{g'\left(\frac{1}{\bar{z}}\right)} - \frac{f''\left(\frac{1}{\bar{z}}\right)}{f'\left(\frac{1}{\bar{z}}\right)} = O\left(\frac{1}{\bar{z}}\right) \quad \text{for } z \to \infty,$$

so

$$\tilde{F}_{\bar{z}}\Big|_{z=\infty} = \frac{-\alpha}{f'(0)}\left(S_f(0) - S_g(0)\right).$$

From the above and from the inequality (2) for $z = 0$ we obtain

$$|\mu_{\tilde{F}}(\infty)| = \left|\frac{\tilde{F}_{\bar{z}}}{\tilde{F}_z}\right|_{z=\infty} = |\mu_F(\infty)| \le k < 1.$$

Because $\tilde{F}_z|_{z\infty} \neq 0$ and is finite, then in $z = \infty$ F is the local homeomorphism holding the orientation.

From the above considerations it follows that the mapping F is the local homeomorphism on $\overline{\mathbb{C}}$ holding the orientation and such that $|\mu_F(z)| \le k < 1$ on $\overline{\mathbb{C}}$. From the definition of F it also follows that the function has the property of the absolute continuity on straight lines, (the property ACL, see [7]).

So F defines the quasi-conformal extension of the order $\frac{1+k}{1-k}$ of the function f on the closed plane $\overline{\mathbb{C}}$.

This finishes the proof of the theorem.

COROLLARY 1. *For $\alpha = 0$ and $h \neq 1$ inequality (2) has the form:*

$$\left|\frac{h(z)-1}{h(z)}|z|^2 - (1-|z|^2)\left[\frac{zh'(z)}{h(z)} + \frac{zf''(z)}{f'(z)}\right]\right| \le k, \tag{5}$$

where $\left|\frac{1}{h(z)} - 1\right| \le k < 1$. The function (3) is given by the formula

$$F(z) = \begin{cases} f(z) & \textit{for } |z| \le 1, \\ f\left(\frac{1}{\bar{z}}\right) + \left(z - \frac{1}{\bar{z}}\right) h\left(\frac{1}{\bar{z}}\right) f'\left(\frac{1}{\bar{z}}\right) & \textit{for } |z| \ge 1. \end{cases} \tag{6}$$

COROLLARY 2. *For $\alpha = 0$ and $h \equiv 1$ inequality (2) has the form:*

$$\left|(1-|z|^2)\frac{zf''(z)}{f'(z)}\right| \le k; \quad \text{(Becker, [3])}. \tag{7}$$

The function (3) *is given by the formula:*

$$F(z) = \begin{cases} f(z) & \text{for } |z| \le 1, \\ f\left(\frac{1}{\bar z}\right) + \left(z - \frac{1}{\bar z}\right) f'\left(\frac{1}{\bar z}\right) & \text{for } |z| \ge 1. \end{cases} \tag{8}$$

COROLLARY 3. *We assume that* $\alpha = 0$ *and* $\frac{h(z)-1}{h(z)} = c, |c| \le k$. *Then* (2) *has the form:*

$$\left| c|z|^2 - (1 - |z|^2)\frac{zf''(z)}{f'(z)} \right| \le k; \quad \text{(Ahlfors, [1])} \tag{9}$$

and the function (3) *is given by the formula:*

$$F(z) = \begin{cases} f(z) & \textit{for } |z| \le 1, \\ f\left(\frac{1}{\bar z}\right) + \frac{1}{1-c}\left(z - \frac{1}{\bar z}\right) f'\left(\frac{1}{\bar z}\right) & \textit{for } |z| \ge 1. \end{cases} \tag{10}$$

COROLLARY 4. *For* $\alpha = \frac{1}{2}$ *we have, respectively,*

$$\left| \frac{h(z) - 1}{h(z)}|z|^2 - (1 - |z|^2)\left(\frac{zh'(z)}{h(z)} + \frac{zg''(z)}{g'(z)}\right) - \frac{1}{2}(1 - |z|^2)^2 \frac{z}{\bar z} h(z)(S_f(z) - S_g(z)) \right| \le k < 1, \tag{11}$$

where $\left|\frac{1}{h(z)} - 1\right| \le k$ *and*

$$F(z) = \begin{cases} f(z) & \textit{for } |z| \le 1, \\ f\left(\frac{1}{\bar z}\right) + \dfrac{\left(z - \frac{1}{\bar z}\right) h\left(\frac{1}{\bar z}\right) f'\left(\frac{1}{\bar z}\right)}{1 + \frac{1}{2}\left(z - \frac{1}{\bar z}\right) h\left(\frac{1}{\bar z}\right)\left(\frac{g''\left(\frac{1}{\bar z}\right)}{g'\left(\frac{1}{\bar z}\right)} - \frac{f''\left(\frac{1}{\bar z}\right)}{f'\left(\frac{1}{\bar z}\right)}\right)} & \textit{for } |z| \ge 1. \end{cases} \tag{12}$$

This result was obtained by Wesołowski, [9].

COROLLARY 5. *For* $\alpha = \frac{1}{2}$, $h(z) \equiv 1$, $g(z) \equiv z$ *The inequality* (2) *and the function* (3) *have the form:*

$$|S_f(z)| \le \frac{2k}{(1 - |z|^2)^2}, \quad k \in < 0, 1 >, \quad \text{(Ahlfors, Weill [2])} \tag{13}$$

$$F(z) = \begin{cases} f(z) & \textit{for } |z| \le 1, \\ f\left(\frac{1}{\bar z}\right) + \dfrac{\left(z - \frac{1}{\bar z}\right) f'\left(\frac{1}{\bar z}\right)}{1 - \frac{1}{2}\left(z - \frac{1}{\bar z}\right)\frac{f''\left(\frac{1}{\bar z}\right)}{f'\left(\frac{1}{\bar z}\right)}} & \textit{for } |z| \ge 1. \end{cases} \tag{14}$$

THEOREM 3. *Let* f, g *be the holomorphic functions and locally univalent in* D.

If for a certain function h holomorphic in D and for a certain fixed $k \in (0;1)$ *inequalities:*

$$\left|\frac{1}{h(z)} - 1\right| \le k;$$

$$\left|\frac{h(z)-1}{h(z)}|z|^2 - (1-|z|^2)\left[\frac{zh'(z)}{h(z)} + (1-2\alpha)\frac{zf''(z)}{f'(z)} + 2\alpha\frac{zg''(z)}{g'(z)}\right]\right| \le k;$$

$$\left|\alpha(1-|z|^2)^2\frac{z}{\bar z}h(z)\left[\left(\alpha-\frac{1}{2}\right)\left(\frac{f''(z)}{f'(z)} - \frac{g''(z)}{g'(z)}\right)^2 + (S_f(z) - S_g(z))\right] - \right.$$

$$\left. - \left[\frac{h(z)-1}{h(z)}|z|^2 - (1-|z|^2)\left[\frac{zh'(z)}{h(z)} + (1-2\alpha)\frac{zf''(z)}{f'(z)} + 2\alpha\frac{zg''(z)}{g'(z)}\right]\right]\right| \le k,$$

$$z \in D,$$

hold, then the function f has the quasiconformal extension of the order $\frac{1+k}{1-k}$ *on* $\overline{\mathbb{C}}$.

For the clarity of the proof it is comfortably to take $h(z) = \frac{1}{1-c(z)}$ and to give Theorem 3 in the equivalent form

THEOREM 4. *Let f, g be the holomorphic functions and locally univalent in D.*

If for a certain function c(z) holomorphic in D and a the certain fixed $k \in (0;1)$ *inequalities*

$$|c(z)| \le k,$$

$$\left|c(z)|z|^2 - (1-|z|^2)\left[\frac{zc'(z)}{1-c(z)} + (1-2\alpha)\frac{zf''(z)}{f'(z)} + 2\alpha\frac{zg''(z)}{g'(z)}\right]\right| \le k; \tag{15}$$

$$\left|\alpha(1-|z|^2)^2\frac{z}{\bar z}\left[\left(\alpha-\frac{1}{2}\right)\left(\frac{f''(z)}{f'(z)} - \frac{g''(z)}{g'(z)}\right)^2 + (S_f(z) - S_g(z))\right] - \right. \tag{16}$$

$$\left. - \left[c(z)|z|^2 - (1-|z|^2)\left(\frac{zc'(z)}{1-c(z)} + (1-2\alpha)\frac{zf''(z)}{f'(z)} + 2\alpha\frac{zg''(z)}{g'(z)}\right)\right](1-c(z)\right| -$$

$$\le k|1-c(z)|,$$

are satisfied, then the function f has the quasi-conformal extension of the order $\frac{1+k}{1-k}$ *on* $\overline{\mathbb{C}}$.

P r o o f. Because there are no assumptions that f, g and c are holomorphic in $\overline{D}$, first consider the following functions:

$$f_r(z) = \frac{1}{r}f(rz), \quad g_r(z) = \frac{1}{r}g(rz), \quad c_r = c(rz), \quad 0 < r < 1, \ z \in \overline{D}. \tag{17}$$

The assumptions of Theorem 2 for the above functions have the form (we take $h(z)=\frac{1}{1-c(z)}$ and we divide the both sides of the inequality (2) by $(1-|z|^2)^2$):

$$|c_r(z)|\le k,$$

$$\left|\alpha\frac{z}{\overline{z}}\left[\left(\alpha-\frac{1}{2}\right)\left(\frac{f_r''(z)}{f_r'(z)}-\frac{g_r''(z)}{g_r'(z)}\right)^2+S_{f_r}(z)-S_{g_r}(z)\right]-\right.$$
$$\left.-\frac{1-c_r(z)}{(1-|z|^2)^2}\left[c_r(z)|z|^2-(1-|z|^2)\left[\frac{zc_r'(z)}{1-c_r(z)}+(1-2\alpha)\frac{zf_r''(z)}{f_r'(z)}+2\alpha\frac{zg_r''(z)}{g_r'(z)}\right]\right]\right|$$
$$\le\frac{k|1-c_r(z)|}{(1-|z|^2)^2},\quad z\in\overline{D}. \tag{18}$$

For the function (17), after simple calculations we obtain:

$$\frac{zf_r''(z)}{f_r'(z)}=\frac{zrf''(rz)}{f'(rz)},\qquad\frac{zg_r''(z)}{g_r'(z)}=\frac{zrg''(rz)}{g'(rz)},$$
$$c_r'=rc'(rz),$$
$$S_{f_r}(z)=r^2S_f(rz),\qquad S_{g_r}(z)=r^2S_g(rz).$$

Replaicing in the inequalities (15) and (16) z by rz and taking into account the above dependences we have:

$$|c_r(z)||\le k,$$

$$\left|\frac{c_r(z)r^2|z|^2}{(1-r^2|z|^2)}-\left[\frac{zc_r'(z)}{1-c_r(z)}+(1-2\alpha)\frac{zf_r''(z)}{f_r'(z)}+2\alpha\frac{zg_r''(z)}{g_r'(z)}\right]\right|$$
$$\le\frac{k}{(1-r^2|z|^2)}, \tag{19}$$

$$\left|\alpha\frac{z}{\overline{z}}\left[\left(\alpha-\frac{1}{2}\right)\left(\frac{f_r''(z)}{f_r'(z)}-\frac{g_r''(z)}{g_r'(z)}\right)+S_{f_r}(z)-S_{g_r}(z)\right]-\frac{r^2(1-c_r(z))}{(1-r^2|z|^2)^2}\right.$$
$$\left.\times\left[c_r(z)r^2|z|^2-(1-r^2|z|^2)\left[\frac{zc_r'(z)}{1-c_r(z)}+(1-2\alpha)\frac{zf_r''(z)}{f_r'(z)}+2\alpha\frac{zg_r''(z)}{g_r'(z)}\right]\right]\right|$$
$$\le\frac{kr^2|1-c_r(z)|}{(1-r^2|z|^2)^2},\quad z\in\overline{D}. \tag{20}$$

The inequalities (18) and (20) show us that the variable w

$$w=\alpha\cdot\frac{z}{\overline{z}}\left[\left(\alpha-\frac{1}{2}\right)\left(\frac{f_r''(z)}{f_r'(z)}-\frac{g_r''(z)}{g_r'(z)}\right)+S_{f_r}(z)-S_{g_r}(z)\right],$$

is in the discs:

$$|w - M(z)| \le R_1(z), \tag{21}$$

$$|w - N(z)| \le R_2(z), \tag{22}$$

respectively, where

$$M(z) = \frac{1 - c_r(z)}{1 - |z|^2)^2}\left[c_r(z)|z|^2 - (1 - |z|^2)B(z)\right], \tag{23}$$

$$N(z) = \frac{r^2(1 - c_r(z))}{1 - r^2|z|^2)^2}\left[c_r(z)r^2|z|^2 - (1 - r^2|z|^2)B(z)\right], \tag{24}$$

$$B(z) = \frac{zc_r'(z)}{1 - c_r(z)} + (1 - 2\alpha)\frac{zf_r''(z)}{f_r'(z)} + 2\alpha\frac{zg_r''(z)}{g_r'(z)} \tag{25}$$

and

$$R_1(z) = \frac{k|1 - c_r(z)|}{(1 - |z|^2)^2}, \qquad R_2(z) = \frac{kr^2|1 - c_r(z)|}{(1 - r^2|z|^2)^2}.$$

If we show, that the disc defined by the inequality (22) is included in the disc defined by the inequality (21), then we will indicate that the function (17) satisfies the assumptions of Theorem 2.

Consider the modulus of the difference of the centres of this discs. After certain calculations taking under considerations (23), (24) and (25) we obtain:

$$\begin{aligned}|M(z) - N(z)| &= |1 - c_r(z)|\left(\frac{1}{1 - |z|^2} - \frac{r^2}{1 - r^2|z|^2}\right)\\ &\times\left|B(z) - c_r(z)\frac{r^2|z|^2}{1 - r^2|z|^2} - c_r(z)\frac{|z|^2}{1 - |z|^2}\right|\\ &\le |1 - c_r(z)|\left(\frac{1}{1 - |z|^2} - \frac{r^2}{1 - r^2|z|^2}\right)\\ &\times\left[\left|B(z) - c_r(z)\frac{r^2|z|^2}{1 - r^2|z|^2}\right| + |c_r(z)|\frac{|z|^2}{1 - |z|^2}\right].\end{aligned}$$

Using the inequality (19) we obtain:

$$\begin{aligned}|M(z) - N(z)| &\le |1 - c_r(z)|\left(\frac{1}{1 - |z|^2} - \frac{r^2}{1 - r^2|z|^2}\right)\left(\frac{k}{1 - r^2|z|^2} + \frac{k|z|^2}{1 - |z|^2}\right)\\ &= k|1 - c_r(z)|\left(\frac{1}{(1 - |z|^2)^2} - \frac{r^2}{(1 - r^2|z|^2)^2}\right) = R_1(z) - R_2(z).\end{aligned}$$

It follows from the above that the function (17) satisfies the assumptions of Theorem 2 and so $f_r(z)$ has the quasi-conformal extension of the order $\frac{1+k}{1-k}$ $\overline{\mathbb{C}}$ on $\overline{\mathbb{C}}$.

The function f as the limit $f_r(z)$ for $r \to 1$ has also the quasi-conformal extension $\overline{\mathbb{C}}$ on $\overline{\mathbb{C}}$ of the same order. (see [8], p. 135, property 1.2.6).

The function F such that $F|_D = f$ and F maps in the quasi-conformal manner of the order $\frac{1+k}{1-k}$ $\overline{\mathbb{C}}$ on $\overline{\mathbb{C}}$, has in this case the form:

$$F(z) = \begin{cases} f(z) & \text{for } |z| < 1, \\ f\left(\frac{1}{\bar{z}}\right) + \dfrac{\left(z - \frac{1}{\bar{z}}\right) f'\left(\frac{1}{\bar{z}}\right)}{1 - c\left(\frac{1}{\bar{z}}\right) + \alpha\left(z - \frac{1}{\bar{z}}\right)\left(\frac{g''\left(\frac{1}{\bar{z}}\right)}{g'\left(\frac{1}{\bar{z}}\right)} - \frac{f''\left(\frac{1}{\bar{z}}\right)}{f'\left(\frac{1}{\bar{z}}\right)}\right)} & \text{for } |z| > 1. \end{cases}$$

Use appropriate functions f, g, c and the parameter α, then the previously known results can be obtained (see [7]).

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DEPARTMENT OF APPLIED MATHEMATHICS
MARIAE CURIE SKLODOWSKA UNIVERSITY
pl. M. Curie-Sklodowskiej 5
20-031 LUBLIN, POLAND

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