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A NOTE ON A CAUCHY-TYPE MEAN VALUE THEOREM

Abstract. We prove that a Cauchy-type mean value theorem [E. Wachnicki, *Une variante du théorème de Cauchy de la valeur moyenne*, Demonstratio Math., 33 (4) (2000), 737–740] is a particular case of Flett's Mean Value Theorem [T. M. Flett, *A mean value theorem*, Math. Gazette 42 (1958), 38–39].

1. Introduction

T. M. Flett [1] proved the following Lagrange-type mean value theorem:

THEOREM 1.1 (The Flett Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f'(a) = f'(b)$. Then, there exists a point $c \in (a,b)$ such that*

$$\frac{f(c)-f(a)}{c-a} = f'(c).$$

In recent years there has been renewed interest in Flett's Mean Value Theorem (see, e.g., [4] and the references therein).

T. Riedel and P. K. Sahoo removed the boundary assumption on the derivatives:

THEOREM 1.2 (T. Riedel and P. K. Sahoo [4]). *Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Then, there exists a point $c \in (a,b)$ such that*

$$f(c)-f(a) = f'(c)(c-a) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2.$$

Recently, Eugeniusz Wachnicki obtained the following Cauchy-type mean value theorem:

THEOREM 1.3 (E. Wachnicki [5]). *Let $f, g\colon [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Suppose that $g'(x) \neq 0$ for $x \in [a,b]$ and*

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}.$$

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Then, there exists a point $\eta \in (a,b)$ such that

$$\frac{f(\eta)-f(a)}{g(\eta)-g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{1.1}$$

2. Main result

We prove that the Cauchy-type Mean Value Theorem 1.3 is nothing else than the Flett Mean Value Theorem 1.1 applied to the function $F = f \circ g^{-1}$, where g^{-1} is the inverse function of g. For definiteness, let us suppose that g is increasing. Since $F'(y) = f'(g^{-1}(y))/g'(g^{-1}(y))$, we obtain

$$F'(g(a)) = \frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)} = F'(g(b)).$$

By the Flett Theorem, there exists a point $c \in (g(a), g(b))$, $c = g(\eta)$, $\eta \in (a,b)$, such that

$$\frac{f(\eta)-f(a)}{g(\eta)-g(a)} = \frac{F(c)-F(g(a))}{c-g(a)} = F'(c) = \frac{f'(g^{-1}(c))}{g'(g^{-1}(c))} = \frac{f'(\eta)}{g'(\eta)}.$$

Similarly, Theorem 4 in [5] is in fact Theorem 1.2 applied to $f \circ g^{-1}$.

References

[1] T. M. Flett, *A mean value theorem*, Math. Gazette 42 (1958), 38–39.

[2] M. Ivan, *A mean value theorem in topological spaces* (in Romanian), Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca) (Elena Popoviciu, ed.), (1982), 145–149.

[3] I. Pawlikowska, *An extension of a theorem of Flett*, Demonstratio Math. 32.2 (1999), 281–286.

[4] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific Publishing Co., NJ, 1998. CMP 1 692 936.

[5] E. Wachnicki, *Une variante du théorème de Cauchy de la valeur moyenne*, Demonstratio Math. 33.4 (2000), 737–740.

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